Dynamics of a Singularly Perturbed Quadratic Family

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June 28, 2019

$$x \to x^2 + c + \frac{\beta}{x^d}$$

Abstract

Some of the dynamics of the family $x \to x^2 + c + \frac{\beta}{x^d}$ is described. Different behaviors occur as the parameters are varied. These transitions are called bifurcations. This singularly perturbed quadratic family is primarily treated as a real system, but is also viewed as a complex system. The main families studied in this paper either have d fixed at one or two, and real parameters β and c varying.

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1 Introduction

Dynamical systems is the study of how systems evolve in time. The focus of dynamical systems is on long-term behavior: whether the system in question settles down to equilibrium, keeps repeating in cycles, or does something more complicated. Differential equations are called continuous dynamical systems. Discrete dynamical systems are studied in this paper. The system is only tracked at discrete times. The ideas of dynamics have been used in various subjects, including classical mechanics, chemical kinetics, population biology, etc. Viewed from the perspective of dynamics, these subjects can be studied in a common framework [**?**]. Chaos and fractals are special parts of this grander subject called dynamics. Chaos is one of many surprising dynamical phenomena. The surprise is that chaos and fractals can occur in simple systems, like the families studied here. Simple numerical experiments lead to stunning mathematical images which no one has ever seen before. A sample of textbooks in dynamics, in increasing order of level of difficulty, is Devaney 1992 [**?**], Strogatz 1994 [**?**], and Bonatti 2000 [**?**].

Discrete dynamical systems are described by recurrence relations of the form $x_{n+1} = f(x_n)$. The family we study is a special family of rational maps of $\Re$:

$$x \to f_{\beta,c,d}(x) \equiv x^2 + c + \frac{\beta}{x^d}$$

This paper primarily investigates the dynamics of this family, with two real parameters, β and c, with d fixed at $d = 1$ or $d = 2$. The following questions will be considered, (1) What are the dynamical behaviors of these families with varying parameter values? (2) What are the similarities and differences between maps with different parameter values? (3) What bifurcations occur in these families?

See the following references for related studies: [**?**, **?**, **?**, **?**, **?**, **?**]

2 Basic Dynamical Properties and Examples

2.1 Preliminary

For background, some terminology and typical behaviors of dynamical systems will be presented. See a reference such as [**?**] for more details. The *orbit* of x_0 under the map F is the sequence $(x_0, x_1, x_2, \ldots)$ determined by the recurrence relation $x_{n+1} = F(x_n)$. Denote the nth iterate of x_0 under F by $F^n(x_0)$. The orbit depends on an initial condition x_0, and the the iteration function F. The goal of dynamical systems is to determine the fate of all orbits, and their dependence on initial conditions and parameters in the iteration function.

2.2 Linear Example: $L(x) = ax + b$

Iteration means to repeat the application of the function over and over. In other words, to iterate a function is to evaluate the function repeatedly, using the output from the previous evaluation as the input for the following evaluation.

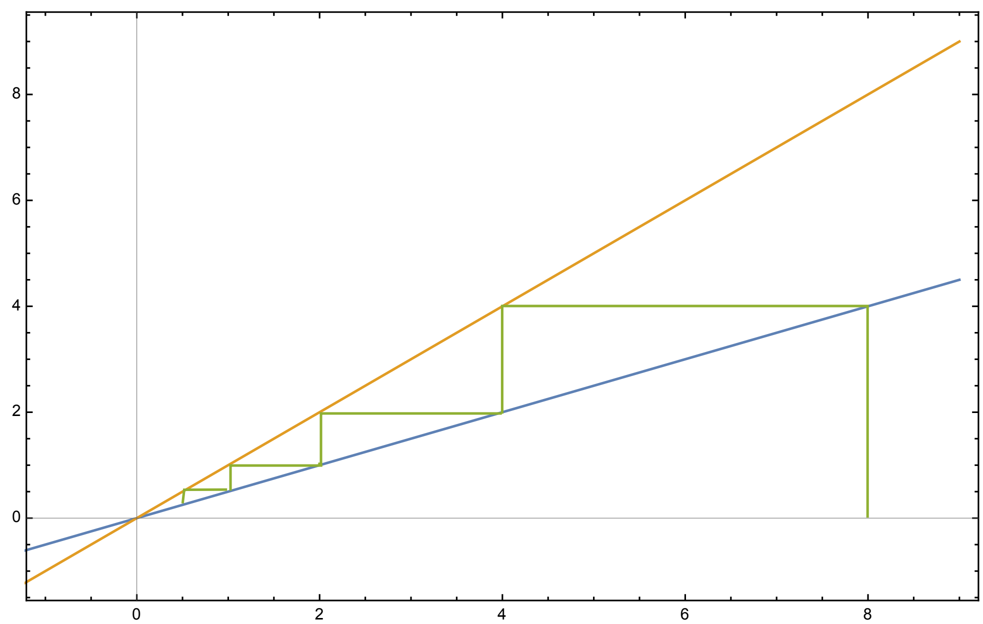


Figure 1: An iteration for $L(x) = x/2$ starting from $x = 8$

Figure **??** is an example to show the graphical iteration for a linear function. For the function $L(x) = x/2$, if the initial value is 8, then after one iteration the output will be 4. Then using $x = 4$ as the input will get the next output $x = 2$. Similiarly, the following outputs will be $x = 1, 1/2$, etc. The behavior of the orbit of $x_0 = 8$ is to approach 0.

$$8 \longrightarrow 4 \longrightarrow 2 \longrightarrow 1 \longrightarrow 1/2 \longrightarrow \ldots \longrightarrow 0$$

In dynamical systems, the reference line, $f(x) = x$, is used to help graphically track the behavior of an orbit. During each iteration, the output at one step would be the input for the next iteration. For example, x_0 is the initial condition for the iteration function f and after one iteration the output would be $x_1 = f(x_0)$. Travelling horiontally from (x_0, x_1) to the point (x_1, x_1) allows us to now use the "height" x_1 as a horizontal distance. We can now "plug in" x_1 to obtain the point (x_1, x_2). And we repeat. Moreover, if this method is applied to other initial conditions, the corresponding orbits will behave similiarly with $x_0 = 8$. Therefore, the behavior of the function $L(x) = x/2$ for all initial conditions can be seen to approach 0.

2.3 Quadratic Example: $Q_c(x) = x^2 + c$

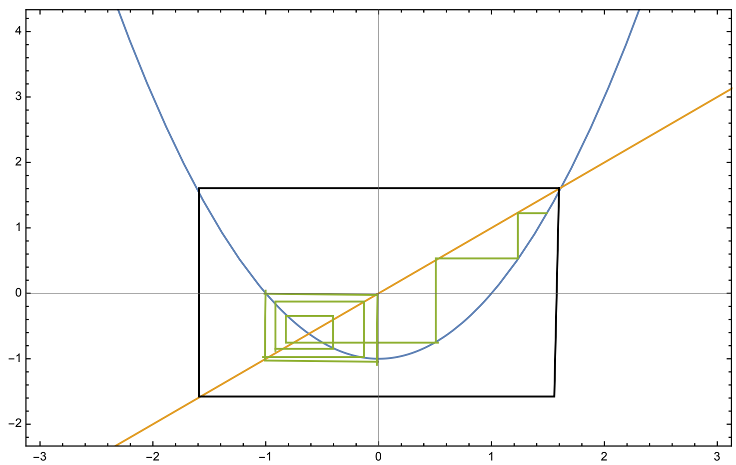
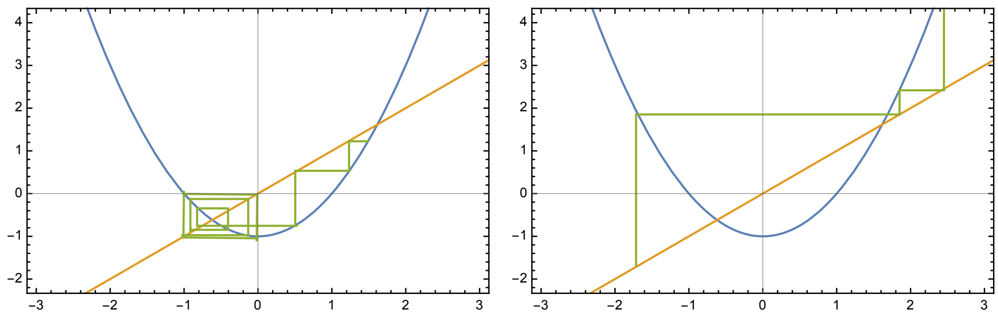


Figure 2: Bounded Box for $Q_c(x) = x^2 + (-1)$. Orbits starting in the interval determined by the black "box" stay in that interval; orbits starting outside the interval escape.

For a much more interesting example, consider the quadratic family, defined by the iteration function $Q_c(x) \equiv x^2 + c$. This function with $c = -1$ is illustrated in Fig. **??**. The orbit displayed approaches an orbit which alternates between $x = 0$ and $x = 1$. This orbit is called a period-2 orbit or a 2-cycle.

Figure **??** shows there are (at least) two different behaviors for the same c parameter value. Figure **??** shows there exists an orbit which approaches a period-2 orbit in the system. Figure (**??**) shows a different initial condition whose orbit will go to infinty. Therefore a single map could have multiple behaviors.



(a) For $x_0 \approx 1.6$,the orbit will approach the period-2 orbit $\{0, -1\}$.

(b) For $x_0 = -1.85$, the orbit will go to infinty.

Figure 3: Iteration for $Q_{(-1)}(x) = x^2 - 1$. Different initial conditions can have different fates.

2.4 Periodic Points & their Stability

As a starting point for dynamical analysis, orbits for a dynamical system can usually be divided into those which stay bounded, and those which do not. Bounded orbits can be further classified with many different types of behaviors. A periodic orbit is one of the most important types of bounded orbit in a dynamical system. A periodic point will satisfy the following conditions.

$$F^n(x) = x \tag{1}$$

When n=2, this is a period-2 orbit:

$$x_0 \longrightarrow x_1 \longrightarrow x_0 \longrightarrow x_1 \longrightarrow x_0 \longrightarrow \dots$$

When n=1, this is a period-1 orbit, which is also called a fixed point. A fixed point will never change under iteration since it satisfies

$$F(x) = x. \tag{2}$$

Therefore, the behavior of the fixed point will be a constant sequence.

$$x_0 \longrightarrow x_0 \longrightarrow x_0 \longrightarrow \dots \longrightarrow x_0$$

The stability of a fixed point is determined by the derivative at the fixed point. A fixed-point x_0 is an attracting fixed point for F if $|F'(x_0)| < 1$. A fixed point x_0 is a repelling fixed point, if $|F'(x_0)| > 1$. If $|F'(x_0)| = 1$, then the fixed point x_0 is (linearly) neutral or an indifferent fixed point. If $|F'(x_0)| = 0$, then the fixed point x_0 is a super-attracting fixed point. More generally, a period-n point p is attracting (repelling) for F

if $|(F^n)'(p)| < 1$ (> 1). If $x_0, x_1, \ldots x_{n-1}$ all lie on an n-cycle for F(x), then the chain rule [?] implies

$$(F^n)'(x_0) = (F^n)'(x_1) = \ldots = (F^n)'(x_{n-1})$$

Sometimes, a prefixed point is also important in a system. The following sequence is a pre-2-fixed point which will go to the fixed point x_2 after two iterations.

$$x_0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \ldots \longrightarrow x_2$$

2.5 Critical Points

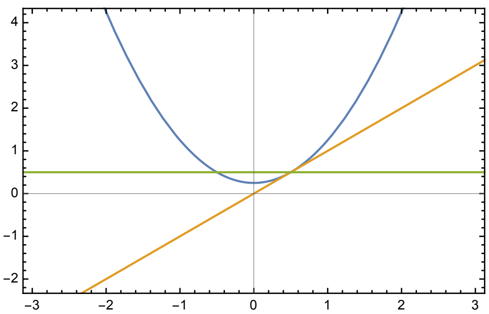
A critical point of an iteration function F is a point where the derivative of the iteration function F is 0. Often the behavior of a critical orbit (an orbit starting at a critical point) has consequences for many other orbits for that map. When a critical point is a fixed point, then this fixed point is a superattracting fixed point. Alternatively, this point is a fixed point with slope 0.

2.6 Bifurcations in the Quadratic Family: $Q_c(x) = x^2 + c$

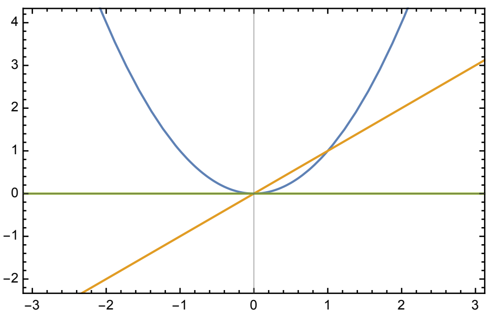
Bifurcations are changes in the behavior of a system as parameters in the iteration function are varied. The following table shows four classic bifurcations in the quadratic family.

Bifurcations			
Index	Figure	Properties	Defining Equations
(1)	(Figure **??**)	saddle-node fixed point	$\begin{cases} Q_c(x) = x \\ Q_c'(x) = 1 \end{cases}$
(2)	(Figure **??**)	super-attracting fixed point	$\begin{cases} Q_c(x) = x \\ Q_c'(x) = 0 \end{cases}$
(3)	(Figure **??**)	period-doubling fixed point	$\begin{cases} Q_c(x) = x \\ Q_c'(x) = -1 \end{cases}$
(4)	(Figure **??**)	critical orbit is fixed after n iterates	$\begin{cases} Q_c'(x) = 0 \\ Q_c(Q_c^n(x)) = Q_c^n(x) \end{cases}$

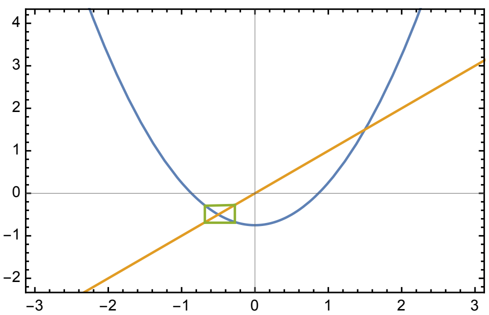
In this family, the most prominent occurences of case (4) are for $(x, c, n) = (0, 0, 0)$ (super-attracting fixed point) and $(0, -2, 2)$ $(0 \mapsto -2 \mapsto 2)$. Figure **??** below illustrates graphs of the iteration function at these four key bifurcation parameter values.



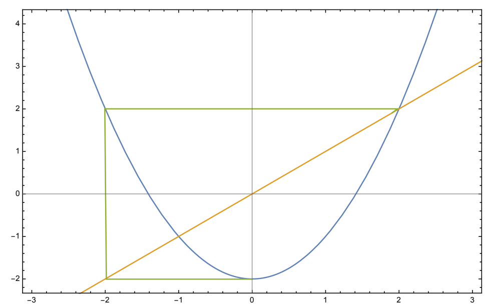
(a) $Q_c(x) = x^2 + (1/4)$: saddle-node



(b) $Q_c(x) = x^2$: super-attracting fixed point



(c) $Q_c(x) = x^2 + (-3/4)$: period-doubling fixed-point: the fixed point is "weakly attracting" although the green orbit (incorrectly) looks to be period-2.



(d) $Q_c(x) = x^2 + (-2)$: pre-fixed critical orbit

Figure 4: Graphs of $Q_c \equiv x^2 + c$ at some bifurcation values of c.

2.7 Orbit Diagram

An orbit diagram is useful for analyzing fates of a dynamical system as a single parameter is varied. It is one of the most instructive and intricate images in all of dynamical systems. In the orbit diagram for Q_c, the asymptotic orbit of the critical point $x = 0$ is plotted in the (x, c) plane. For each fixed parameter value, the critical orbit is computed. The first 150 iterations are discarded, and the next 150 are plotted, illustrating the eventual behavior, or fate, of the critical orbit. If this critical orbit lands outside an escape radius prior to 150 iterations, nothing is plotted for that c value in the orbit diagram.

Most of the graphs shown in this project, including oribit diagrams, are created using Mathematica 10. We used the "ListPlot" command to create the orbit diagram in Fig. **??**.

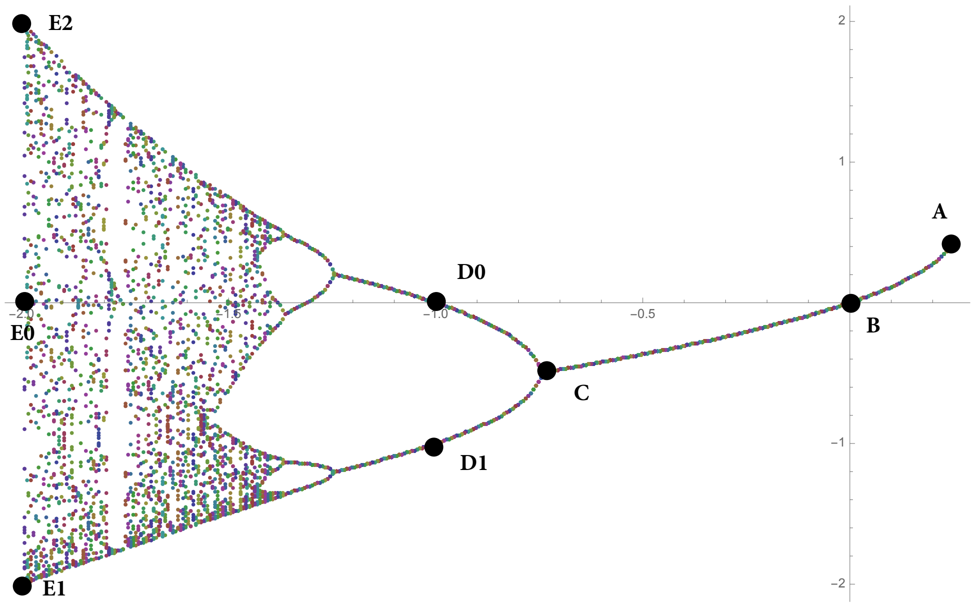


Figure 5: An orbit diagram for $Q_c(x) = x^2 + c$. Plot of x vs c.

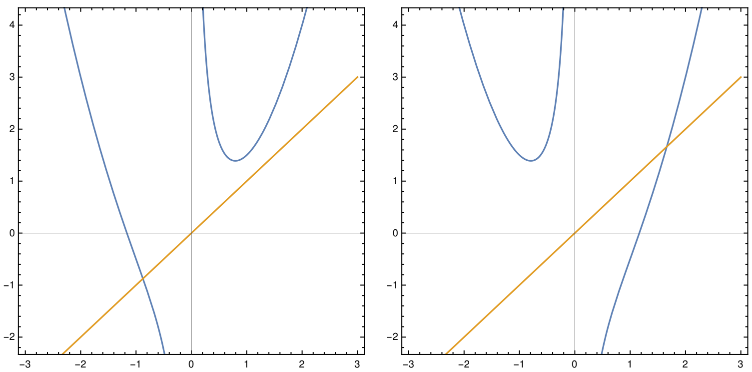
Some Key Bifurcations in Orbit Diagram		
Figure **??** Label	Properties	Figure **??** Label
(A)	saddle-node	(Figure **??**)
(B)	super-attracting fixed point	(Figure **??**)
(C)	period-doubling	(Figure **??**)
(D)	super-attracting period-2-orbit	(Figure **??**)
(E)	critical point is prefixed	(Figure **??**)

Note that the fate of the critical orbit is often shared by many other orbits. For example, when $c = -1$, $Q_{-1}(x) = x^2 - 1$ will have a super-attracting period-2 orbit between $x = 0$ and $x = -1$. This orbit is labelled by points $D0$ and $D1$ in Fig. **??**. Figure **??** shows that all bounded orbits except the fixed points and their preimages will also approach this period-2 orbit.

For $c = -2$, the critical point (E0 in Figure **??**) is a prefixed point, landing on the fixed point E2 in two iterations: $E0 \to E1 \to E2 \to E2 \to \ldots$. This "prefixed point" is also called a "homoclinic point" since there is a sequence of backward images of the critical point E0 approaching the repelling fixed point E2, as well as the second forward iterate landing on E2. It is known that the dynamics restricted to the invariant interval $[E1, E2]$ is chaotic [**?**]. Orbits starting outside this interval all escape.

3 Singular Perturbation with $d = 1$: $f_{\beta,c}(x) \equiv x^2 + c + \frac{\beta}{x}$

We now consider a "singular perturbation" of $x^2 + c$ by adding β/x. The graphs have two qualitative shapes, depending on the sign of β. Note that there is a unique critical point for each case. The critical point is given by $x_{crit} = (\beta/2)^{1/3}$. For $\beta = \pm 1$, $x_{crit} = (\pm 1/2)^{1/3} \approx \pm 0.7937$

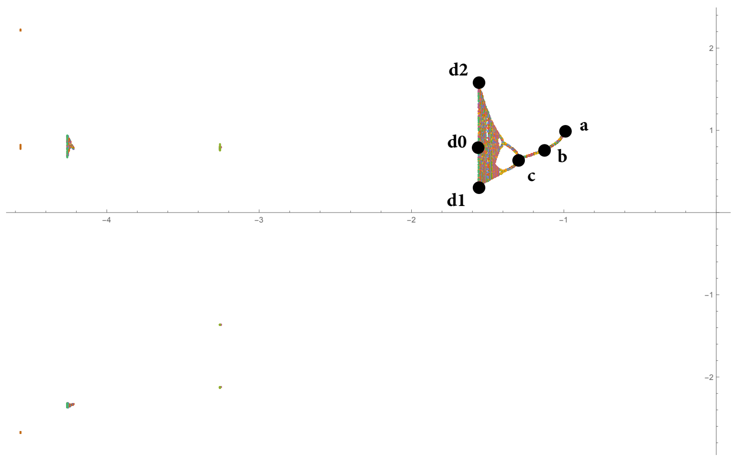


(a) $f_{\beta,c}(x) = x^2 + 0.5 + \frac{1}{x}$ (b) $f_{\beta,c}(x) = x^2 + 0.5 + \frac{(-1)}{x}$

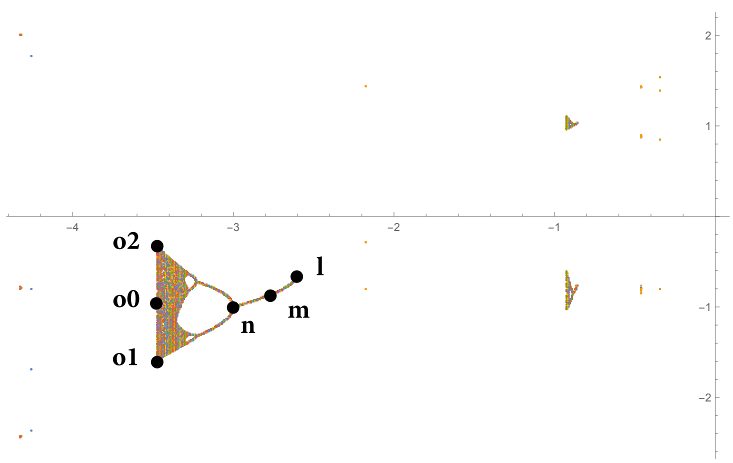
Figure 6: Examples of graphs of $f_{\beta,c}(x) = x^2 + c + \frac{\beta}{x}$

3.1 Orbit diagrams

We start our investigation by illustrating two orbit diagrams in the parameter c for our family of interest, one for $\beta = +1$, and one for $\beta = -1$. In both cases the fate of the unique critical orbit is displayed.



(a) An Orbit Diagram for c with $\beta = 1$; x versus c.



(b) An Orbit Diagram for c with $\beta = -1$; x versus c.

Figure 7: The Orbit Diagrams for c with $\beta = \pm 1$

The primary feature in each of the two orbit diagrams in Fig. **??** appears to be a smaller topological copy of Fig. **??**, the orbit diagram illustrating the "period-doubling route to chaos" for the standard quadratic family $Q_c = x^2 + c$, which exists for $(c, x) \in [-2.0, 0.25] \times [-2.0, 2.0]$. For $\beta = +1$, the topological copy exists approximately for $(c, x) \in [-1.55, -0.9] \times [.25, 1.65]$, and for $\beta = -1$, the topological copy exists approximately for $(c, x) \in [-3.5, -2.6] \times [-1.65, -0.25]$. But both orbit diagrams have additional interesting features which exist outside these primary regions. Many of these features are additional smaller topological copies of the period-doubling route to chaos for other periods. We do not completely describe these additional features in this paper. (See Oman [?] for a much more complete description of the bifurcations for the famlily: $x^2 + c + 0.001/x^2$,which is a different one-parameter cut through our two-parameter family.)

The orbit diagrams in Fig. **??** are in the (x, c) plane. The labelled points have correponding figures that appear later in the paper. Specifically, Fig. **??** shows some bifurcation curves in the (β, c) parameter plane, and Figs. **??** and **??** show graphical iteration pictures in the (x_n, x_{n+1}) space, for fixed values of β and c. The following chart shows correspondences between these figures.

Labels for corresponding figures.		
Label for Figures **??** and **??**	Figure **??** or **??**	Description
(a)	**??**	fixed-point saddle-node
(b)	**??**	super-attracting fixed point
(c)	**??**	fixed-point period doubling
(d)	**??**	critical orbit is fixed after two iterations
(l)	**??**	fixed-point saddle-node
(m)	**??**	super-attracting fixed point
(n)	**??**	fixed-point period doubling
(o)	**??**	critical orbit is fixed after two iterates
Note: Figure **??** illustrates graphs at several bifurcation points for $\beta = +1$, and Figure **??** illustrates $\beta = -1$.		

3.2 Parameter plane

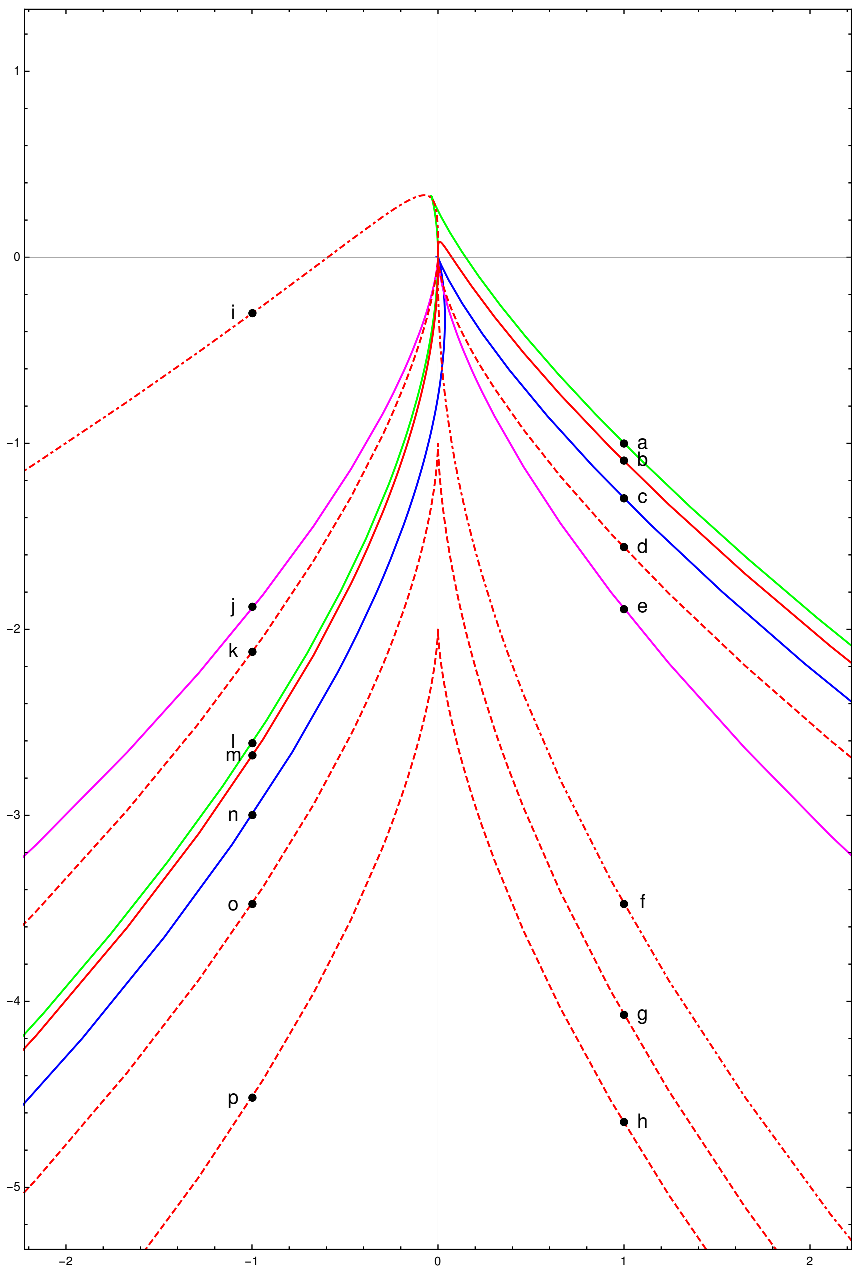


Figure 8: Some bifurcation curves in the (β, c) parameter plane for $x^2 + c + \beta/x$. Greeen: saddle-node, Blue: period-doubling, Red: superattracting fixed point (critical point fixed), Red dot-dashed: critical value fixed; Red dashed: critical orbit fixed after two iterates; Magenta: critial orbit pre-pole. See also the tables in sections **??** and **??**.

In this section, we will focus on bifurcation curves for

$$f_{\beta,c}(x) = x^2 + c + \frac{\beta}{x} \tag{3}$$

in the (β, c) parameter plane. The defining equations for a few key bifurcations have already been given in section **??**. For example, the fixed-point saddle-node is determined by the pair of equations:

$$\begin{cases} f_{\beta,c}(x) = x \\ f'_{\beta,c}(x) = 1 \end{cases} \tag{4}$$

This pair of equations has parametric solution (for $x \neq 0$):

$$\begin{cases} \beta = -x^2 + 2x^3 \\ c = 2x - 3x^2 \end{cases} \tag{5}$$

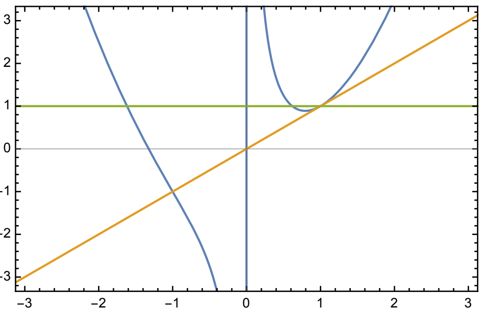
Other parametric solutions to the defining equations are similarly obtained. By graphing these bifurcation curves (Fig. **??**), we divide the plane into multiple regions. Parameter values in these regions do not necessarily have dynamically equivalent (conjugate) systems, because we have computed only bifurcations related to fixed points. Maps corresponding to different regions, however, are inequivalent. Compare Fig. **??** with Figs. **??**, **??** and **??** for more complete understanding.

Parametric representations of some Key Bifurcation Sets in the (β, c) plane		
Properties	Defining Equations	Solution
super-attracting fixed point	$\begin{cases} f_{\beta,c}(x) = x \\ f'_{\beta,c}(x) = 0 \end{cases}$	$\begin{cases} \beta = 2x^3 \\ c = x - 3x^2 \end{cases}$
fixed-point saddle-node	$\begin{cases} f_{\beta,c}(x) = x \\ f'_{\beta,c}(x) = 1 \end{cases}$	$\begin{cases} \beta = -x^2 + 2x^3 \\ c = 2x - 3x^2 \end{cases}$
fixed-point period doubling	$\begin{cases} f_{\beta,c}(x) = x \\ f'_{\beta,c}(x) = -1 \end{cases}$	$\begin{cases} \beta = 2x^3 \\ c = -3x^2 \end{cases}$
critical value is 0 (pre-pole)	$\begin{cases} f_{\beta,c}(x) = 0 \\ f'_{\beta,c}(x) = 0 \end{cases}$	$\begin{cases} \beta = 2x^3 \\ c = -3x^2 \end{cases}$
critical orbit fixed after one iterate	$\begin{cases} f^2_{\beta,c}(x) = f_{\beta,c}(x) \\ f'_{\beta,c}(x) = 0 \end{cases}$	$\begin{cases} \beta = -x^2 + 2x^3 \\ c = -2x - 3x^2 \end{cases}$
critical orbit fixed after two iterates	$\begin{cases} f^2_{\beta,c}(x) = f^2_{\beta,c}(x) \\ f'_{\beta,c}(x) = 0 \end{cases}$	$\begin{cases} f^3_{\beta,c}(x) = f^2_{\beta,c}(x) \\ \beta = 2x^3 \end{cases}$ explicit solution for c too complicated to display

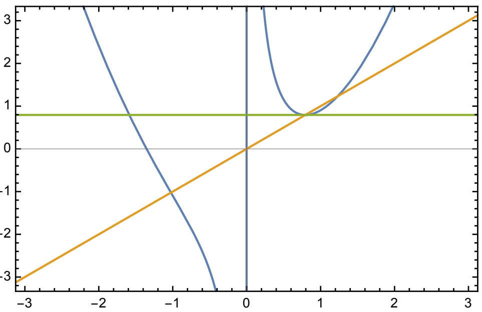
3.3 Bifurcations for $\beta = 1$ with c varying

Figure **??** shows graphical iteration for the bifurcations points labelled in Fig. **??** along $\beta = 1$.

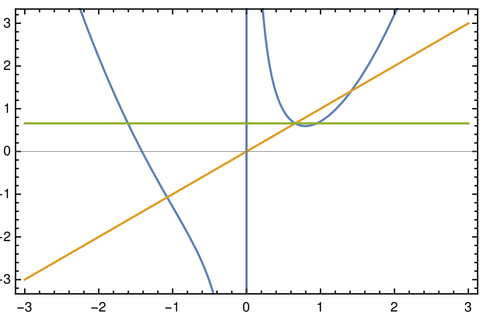
Some Fixed-point Bifurcations for $\beta = 1$			
Figure **??** Label	Figure **??** Label	Descriptions	# of fixed pts
??(a)	**??**	saddle-node	2
??(b)	**??**	super-attracting fixed point	3
??(c)	**??**	period doubling	3
??(d)	**??**	critical orbit is fixed after two iterations	3
??(e)	**??**	critical orbit lands on 0 (prepole)	3
??(f)	**??**	critical value is fixed	3
??(g)	**??**	critical orbit is fixed after two iterations	3
??(h)	**??**	critical orbit is fixed after two iterations	3



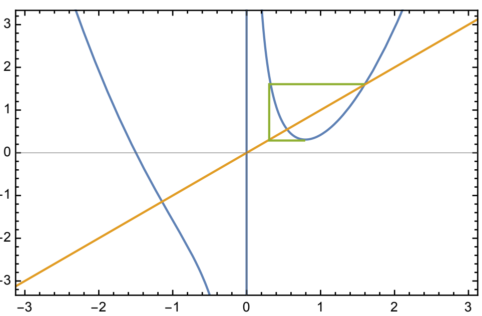
(a) $f_{\beta,c}(x) = x^2 + (-1) + \frac{1}{x}$: saddle-node



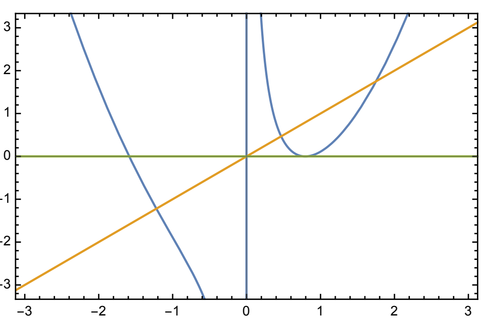
(b) $f_{\beta,c}(x) = x^2 + (-1.09618) + \frac{1}{x}$: super-attracting fixed point



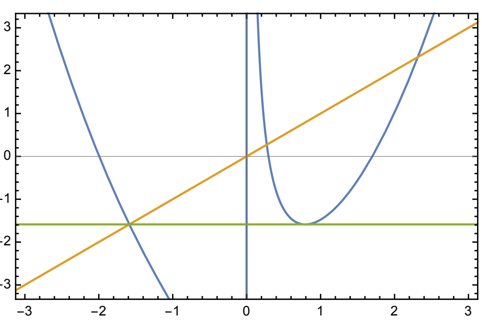
(c) $f_{\beta,c}(x) = x^2 + (-1.29612) + \frac{1}{x}$: period-doubling



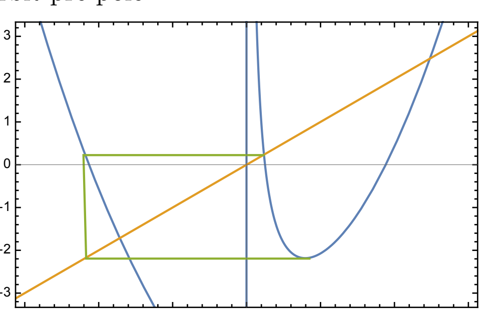
(d) $f_{\beta,c}(x) = x^2 + (-1.58) + \frac{1}{x}$: critical orbit pre-fixed



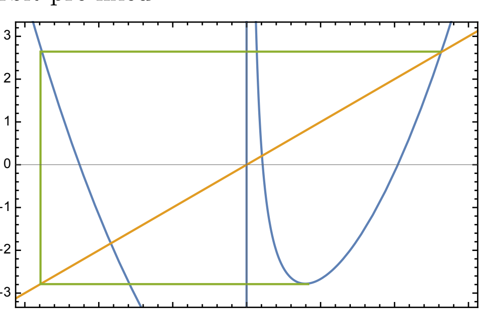
(e) $f_{\beta,c}(x) = x^2 + (-1.8899) + \frac{1}{x}$: critical orbit pre-pole



(f) $f_{\beta,c}(x) = x^2 + (-3.47728) + \frac{1}{x}$: critical orbit pre-fixed



(g) $f_{\beta,c}(x) = x^2 + (-4.08) + \frac{1}{x}$: critical orbit pre-fixed



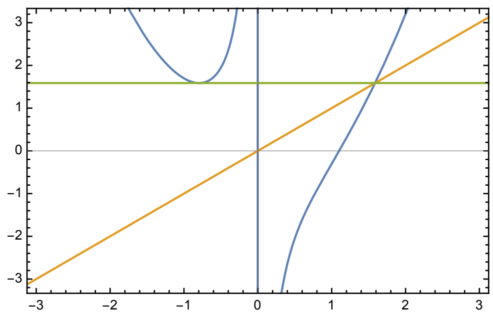
(h) $f_{\beta,c}(x) = x^2 + (-4.67) + \frac{1}{x}$: critical orbit pre-fixed

Figure 9: Cobweb graphs at bifurcation c-values for $x^2 + c + 1/x$

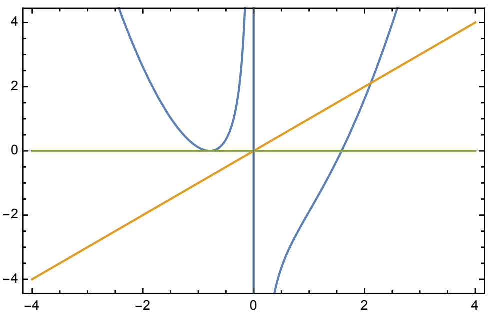
3.4 Bifurcations for $\beta = -1$ with c varying

Figure **??** shows graphical iteration for the bifurcations points labelled in Fig. **??** along $\beta = -1$.

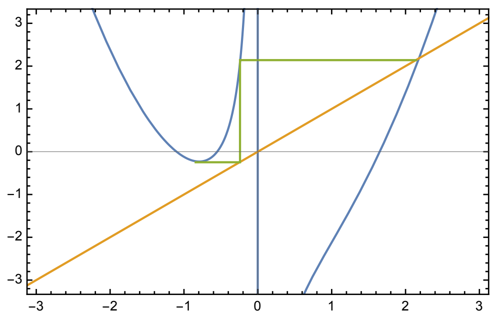
Some Fixed-point Bifurcations for $\beta = -1$		
Figure **??** & **??** Labels	Descriptions	# of fixed points
(i)	the critical value is a fixed point	1
(j)	the critical point prepole	1
(k)	the critical orbit fixed after two iterations	1
(l)	saddle-node	2
(m)	super-attracting fixed-point	3
(n)	period doubling	3
(o)	the critical point fixed after two iterations	3
(p)	the critical point fixed after two iterations	3



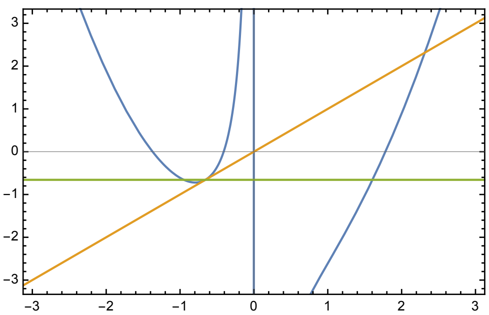
(i) $f_{\beta,c}(x) = x^2 + (-0.30248) + \frac{-1}{x}$: critical orbit pre-fixed



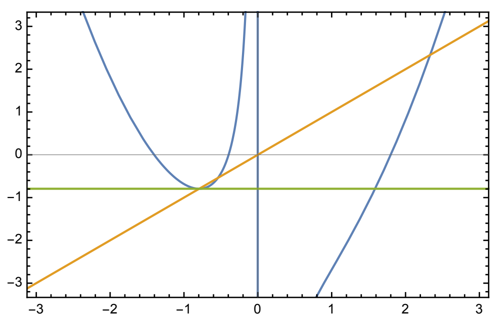
(j) $f_{\beta,c}(x) = x^2 + (-1.8899) + \frac{-1}{x}$: critical orbit pre-pole



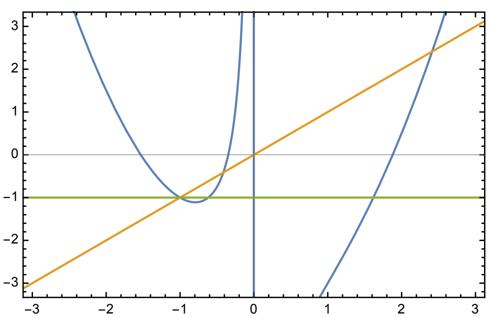
(k) $f_{\beta,c}(x) = x^2 + (-2.12) + \frac{-1}{x}$: critical orbit pre-fixed



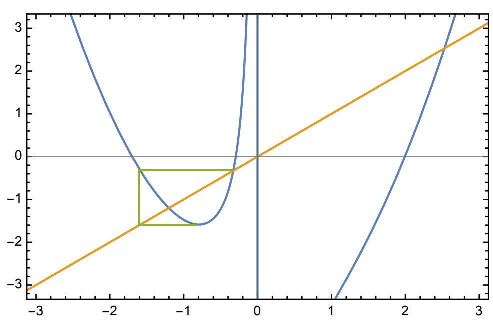
(l) $f_{\beta,c}(x) = x^2 + (-2.61071) + \frac{-1}{x}$: saddle-node



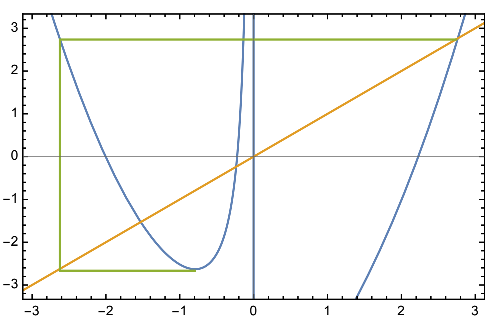
(m) $f_{\beta,c}(x) = x^2 + (-2.68358) + \frac{-1}{x}$: super-attracting



(n) $f_{\beta,c}(x) = x^2 + (-3) + \frac{-1}{x}$: period-doubling



(o) $f_{\beta,c}(x) = x^2 + (-3.48) + \frac{-1}{x}$: critical orbit pre-fixed



(p) $f_{\beta,c}(x) = x^2 + (-4.52) + \frac{-1}{x}$:critical orbit pre-fixed

Figure 10: Cobweb graphs at bifurcation c-values for $x^2 + c - 1/x$

3.5 Codimension-two Bifurcation Points

There are several codimension-two bifurcation points along the codimension-one bifurcation curves in Fig. **??**. We have identified eight special points q through x in the enlargement of Fig. **??**.

- q Cusp point on the saddle-node curve. Well-known saddle-node with a higher-order degeneracy in its normal form.
- r A crossing of two unrelated bifurcations: a saddle-node curve, and a curve along which a critical value is a fixed point.
- s A critical value is a saddle-node fixed point. We call this a CP2SN point. This codimension-two point is not a well-known bifurcation, but has a simple model which we present below.
- t A crossing of a period-doubling curve with a curve along which the critical orbit is fixed after two iterates. The respective curves generically cross the period-doubling curve with a non-zero angle since the two bifurcations are independent.
- u A crossing of a period-doubling curve with a curve along which the critical value of the one critical point is zero (and therefore a pre-pole).
- v Similar in type to point s. It has a critical value that is also a period-doubling point. We call it a CP2PD point. See a model for this point below.
- w Period-doubling at an inflection point. This does not appear to be a true codimension-two bifurcation point, although it has an interesting interpretation for this specific family. It seems at first to be analogous to the cusp point, a saddle-node with a higher order degeneracy. However, the higher order degeneracy for a period-doubling bifurcation is determined by a combination of the second *and* third derivatives at the period-doubling point [**?**]. This combination is not zero at point w. The period-doubling curve passes smoothly through w. The point does appear to have the property of being a local maximum value of β along the period-doubling curve.
- x This is a "degenerate period-doubling" point [**?**] where the combination of second and third derivatives that defines a higher-order degeneracy:$-f'''(p)-3/2(f''(p))^2 = 0$. Curiously, this bifurcation happens at exactly the same parameters as the critical value landing on a period-doubling point, although the conditions for this point are purely local at the fixed point, and the conditions for point v involve both the period-doubling point and a critical point which is located elsewhere in the phase space. These two points would not be coincident in a generic family. If bifurcation curves for period-two points were added to Fig. **??**, then a period-two saddle-node curve would emanate from this point, tangent to the period-doubling curve.

Corresponding graphs are shown in Fig. **??**.

3.5.1 Model of a CP2SN point

We present a model for the codimension-two bifurcation in which a critical value is also a saddle-node fixed point. The label 'CP2SN' is chosen since a critical point maps to a saddle-node point. The model is $x_{n+1} = f_{a,b}(x_n)$, where f has the local form of a saddle-node (arbitrarily positioned at $x = 0$), and the local form of a critical point (arbitrarily placed at $x = -1$), which maps to the saddle-node when the "unfolding parameters" (a, b) are at $(0, 0)$. That is, $f_{a,b}(x) = x + x^2 + a$ near $x = 0$, and $(x + 1)^2 + b$ near $x = -1$. For this model, the saddle-node curve is the b axis ($a = 0$), and the critical value is a fixed point when $a = -b^2$. These two bifurcation curves are tangent. This is consistent with the tangency of the two curves in Fig. **??** at point s.

3.5.2 Model of a CP2PD point

We present a model for the codimension-two bifurcation in which a critical value is also a period-doubling fixed point. The label 'CP2PD' is chosen since a critical point maps to a period-doubling point. Its model has a fixed point (placed at zero) which undegoes a period-doubling, and a critical point (arbitrarily placed at $x = 1$) with critical value zero. After adding the unfolding terms, the form of the model bifurcation $f_{(a,b)}$ is $-x \pm x^3 + ax$ near $x = 0$, and $-(x - 1)^2 + b$ near $x = 1$. The period-doubling curve is $a = 0$, and the critical value is fixed if $b = 0$. Thus the model is topologically consistent with the computed curves which cross at point v with a non-zero angle in Fig. **??**.

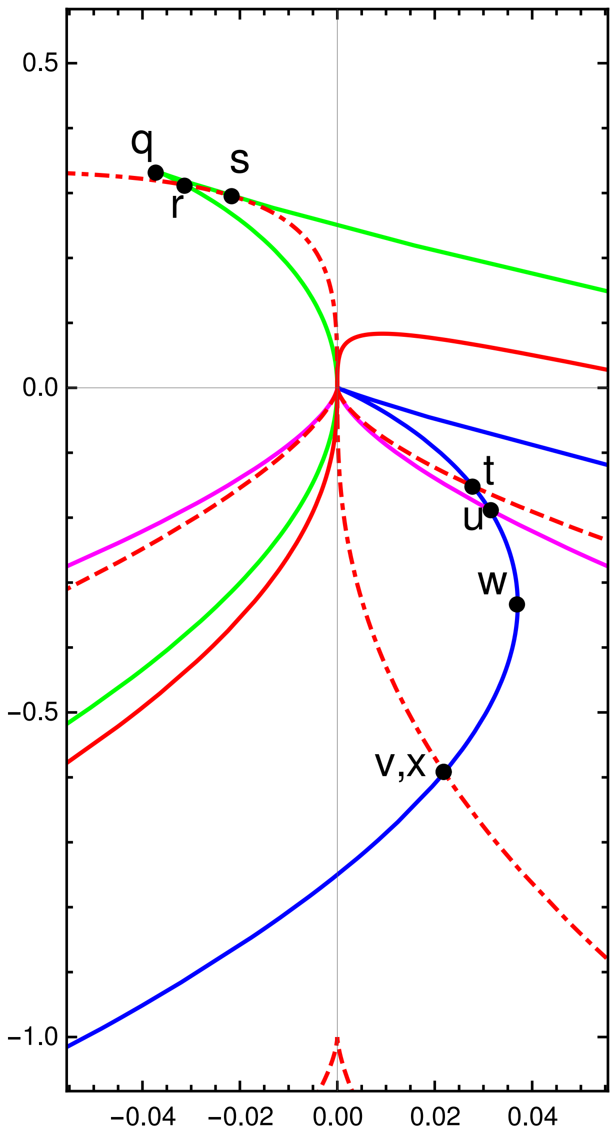
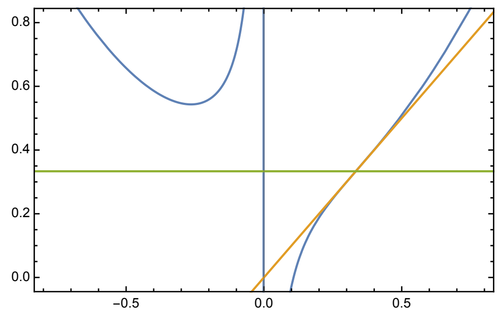


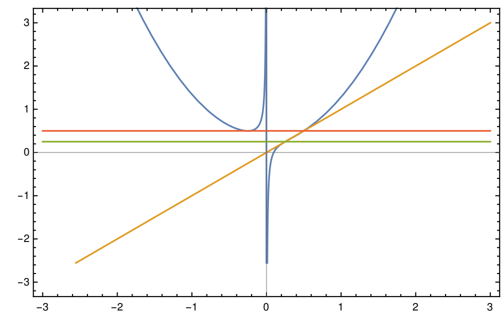
Figure 11: The (β, c) parameter plane: enlargement of Fig. **??**, with eight codimension-two points labelled. Greeen: saddle-node, Blue: period-doubling, Red: superattracting fixed point (critical point fixed), Red dot-dashed: critical value fixed; Red dashed: critical orbit fixed after two iterates; Magenta: critial orbit pre-pole. See text for more explanaton.

3.6 Dynamics

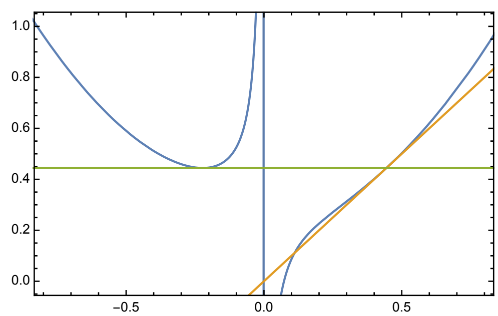
So far in this paper we have focussed on bifurcation curves related to fixed points in our family, but, other than displaying some orbit diagrams, we have not directly addressed the behavior of the critical orbit, and we have addressed even less the fates of other orbits. In this section we will provide some observations about the dynamics of the critical orbit,



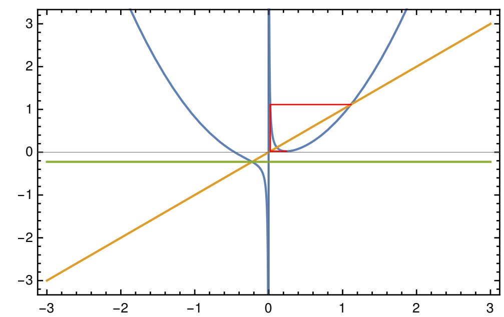
(q) $f_{\beta,c}(x) = x^2 + \frac{(-\frac{1}{27})}{x} + (\frac{1}{3})$: cusp point (cubic tangency of graph at fixed point)



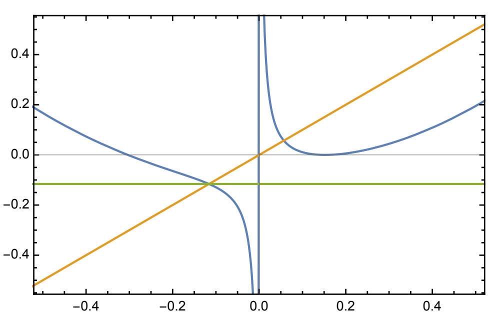
(r) $f_{\beta,c}(x) = x^2 + \frac{(-\frac{1}{32})}{x} + (\frac{5}{16})$: critical orbit pre-fixed (RED) and separate saddle-node (GREEN)



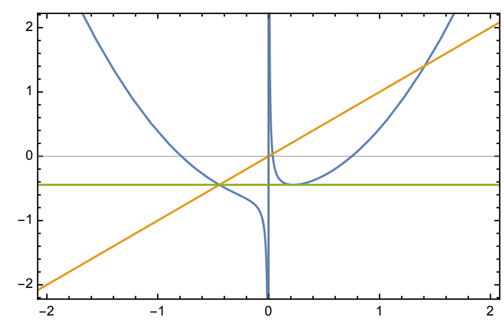
(s) $f_{\beta,c}(x) = x^2 + \frac{(-\frac{16}{729})}{x} + (\frac{8}{27})$: critical orbit pre-fixed at saddle-node



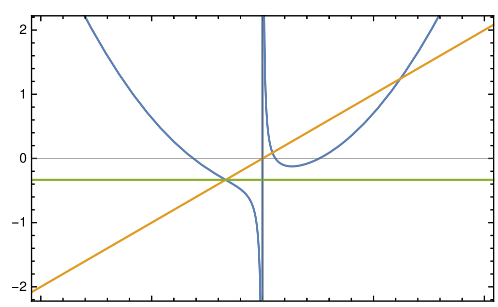
(t) $f_{\beta,c}(x) = x^2 + \frac{(0.027796)}{x} + (-0.151435)$: period-doubling (GREEN) and separate critical orbit fixed after two iterates (RED)



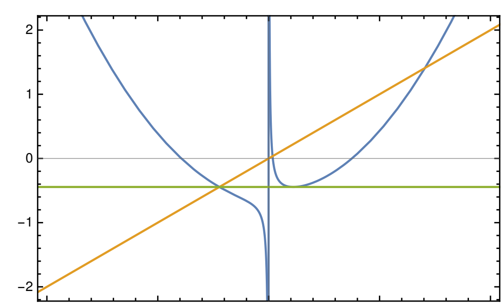
(u) $f_{\beta,c}(x) = x^2 + \frac{(1/32)}{x} + (-3/16)$: critical oribit pre-pole (x axis) and separate period-doubling (GREEN)



(v) $f_{\beta,c}(x) = x^2 + \frac{(16/729)}{x} + (-16/27)$: critical orbit fixed after one iterate at period-doubling point



(w) $f_{\beta,c}(x) = x^2 + \frac{(\frac{1}{27})}{x} + (-\frac{1}{3})$: period-doubling at inflection point



(x) $f_{\beta,c}(x) = x^2 + \frac{(16/729)}{x} + (-16/27)$: degenerate period-doubling. Critical value just happens to be the period-doubling pt.

Figure 12: Graphs at some codimension-two parameter points for $x^2 + c + \beta/x$.

and the consequent fates for all orbits for certain groups of parameter values. These descriptions are not fully proven, nor are they complete. They serve more as a starting point for future investigation.

For example, consider parameter values in Fig. **??** with $\beta > 0$ and (β, c) above the saddle-node bifurcation curve (which runs through point (a)). A representative is shown in Fig.r̃efFig.label.a0 below. By just doing graphical iteration, it is clear that the critical orbit escapes. Further, any orbit starting at x_0 with $x_0 > 0$ will go off to infinity. Other than the one fixed point, any orbit starting with $x_0 < 0$ that does not land on $x = 0$ will eventually have some iterate $x_n > 0$, and therefore will iterate to infinity. If we extend the map (as is customary with rational maps) to the circle which is the one-point compactification of the real line by adding the "point at infinity", then 0 will map to infinity, so that all the orbits that land on 0 will also iterate to infinity. Thus, all orbits except the fixed point escape to infinity.

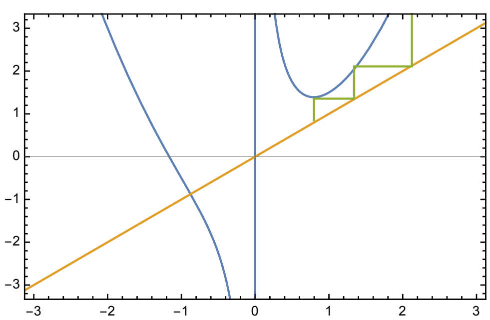


Figure 13: Bifurcation for $\beta > 0$ and (β, c) above the saddle node curve

Similarly, consider parameter values in Fig. **??** with $\beta < 0$ and (β, c) above all the bifurcation curves shown. That is, consider β values above the top "critical orbit prefixed" bifurcation curve (which runs through point (i)), and also above the part of the saddle-node bifurcation curve which runs through point (a) and extends just barely into the left-half plane. For all these parameter values, the critical orbit escapes, and all orbits other than the unique fixed point, now with a positive value, escape.

At the other extreme are "sufficiently negative" c parameter values. For c values below the two "critical orbit prefixed bifurcation curves through the points labelled h and p in Fig. **??**, the critical orbit escapes, and it appears that all other orbits except a Cantor set escape. Further, the dynamics on that Cantor set appears to be conjugate to a full shift on three symbols, since the graph of the iteration function maps three times "across the interval" determined by the rightmost fixed point, and its leftmost preimage. Compare with Figs. **??** and **??** by decreasing c and therefore dropping the graph slightly.

For β values between these upper and lower extremes the description of the dynamics is more complicated, especially in cases where positive values can map to negative values. See, for example, Fig. **??**, where the interval determined by the small (red) square is not invariant, but contains and invariant Cantor set. The interval determined by the large (green) square is not invariant either, but contains additional points whose orbits stay bounded. Full dynamical descriptions for all parameter values is not trivial. It will be left for a future project.

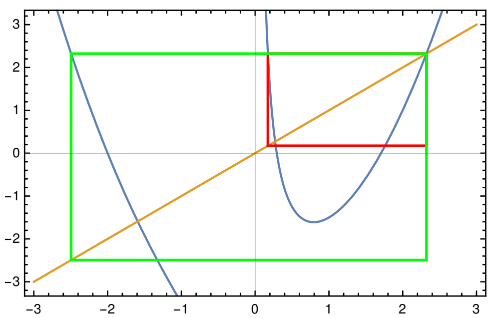


Figure 14: Example graph where neither the interval of the small (RED) square nor the interval of the large (GREEN) square is invariant, but both intervals contain invariant sets that remain inside the respective intervals.

4 Singular perturbations with $d = 2$: $x^2 + c + \beta/x^2$

Now that we have developed many tools and scripts to study our main family, $x^2+c+\beta/x$, we use them to show analogous results for the similar family with the power of the singular perturbation changed from $d = 1$ to $d = 2$.

4.1 Orbit diagram

Define $f_{\beta,c}$ by

$$f_{\beta,c}(x) \equiv x^2 + c + \frac{\beta}{x^2}$$

There are two critical points for this family, but they both have the same critical value, so we can compute "the" orbit diagram by following the positive critical orbit. The result is Fig. **??**.

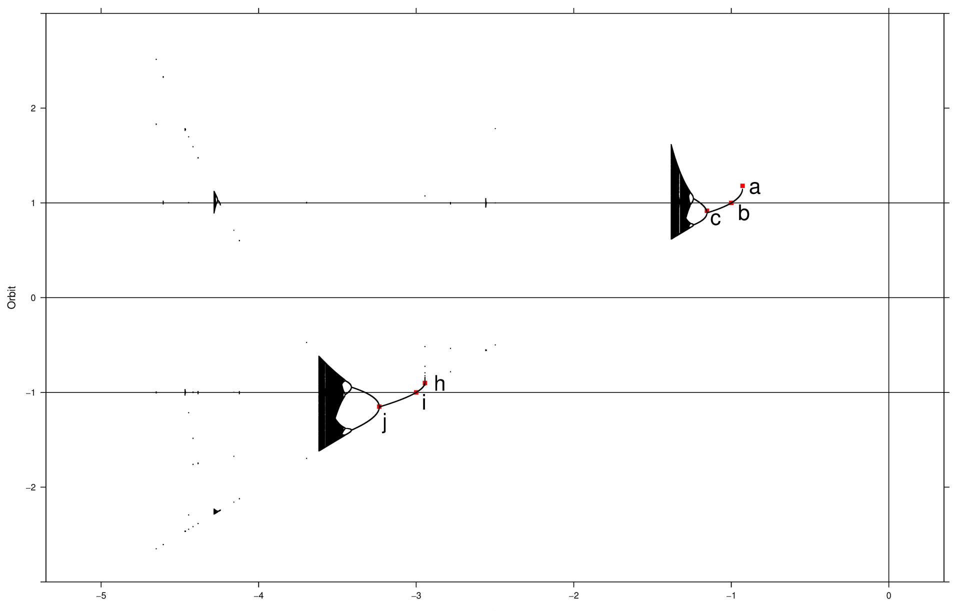


Figure 15: An Orbit Diagram for c with $\beta = 1$. Labels correspond to bifurcation curves in Fig. **??** and iteration graphs in Fig. **??**.

Since this family has two unimodal branches, one with $x > 0$ and one with $x < 0$, the orbit diagram has two main "full period-doubling route to chaos" parameter intervals, extending from a saddle-node bifurcation (labelled (a) (respectively (d))) through the left-hand edge of each of the two primary features, determined by the critical orbit landing on a repelling fixed point in two iterates. But there are many other smaller features as well, many corresponding to higher periodic orbits that include both positive x values and negative x values. See the master's project by Oman [**?**] for a detailed explanation of a similar one-parameter cut, but with $\beta = .001$ instead of $\beta = 1.0$.

4.2 Parameter plane

Figure **??** is the beginning of a (β, c) parameter plane bifurcation diagram for $f_{\beta,c}(x) = x^2+c+\frac{\beta}{x^2}$. Only fixed-point bifurcation curves are shown, but they suggest features of the whole two-parameter family. Primarily, the bifurcation structure in the $\beta > 0$ half-plane is similar (but not exactly the same), to the bifurcations for $d = 1$, shown in Fig. **??**. Much more work would need to be done to fully understand these maps.

The $\beta < 0$ half-plane, on the other hand, has almost no bifurcation curves. The green saddle-node curve (through point (a))barely crosses the c axis to arrive at a cusp that has $\beta < 0$ and $c > 0$, and the blue period-doubling curve barely crosses into the $\beta < 0$ half-plane for c approximately in $[-0.76, 0]$, although this is not clear from Fig.**??**. It is clear in the enlargement in Fig. **??**.

4.3 Bifurcations for $\beta = 1$ with c varying

Figure **??** shows a sequence of graphs illustrating the iteration function at the 13 bifurcation points labelled in Figs. **??** and **??**.

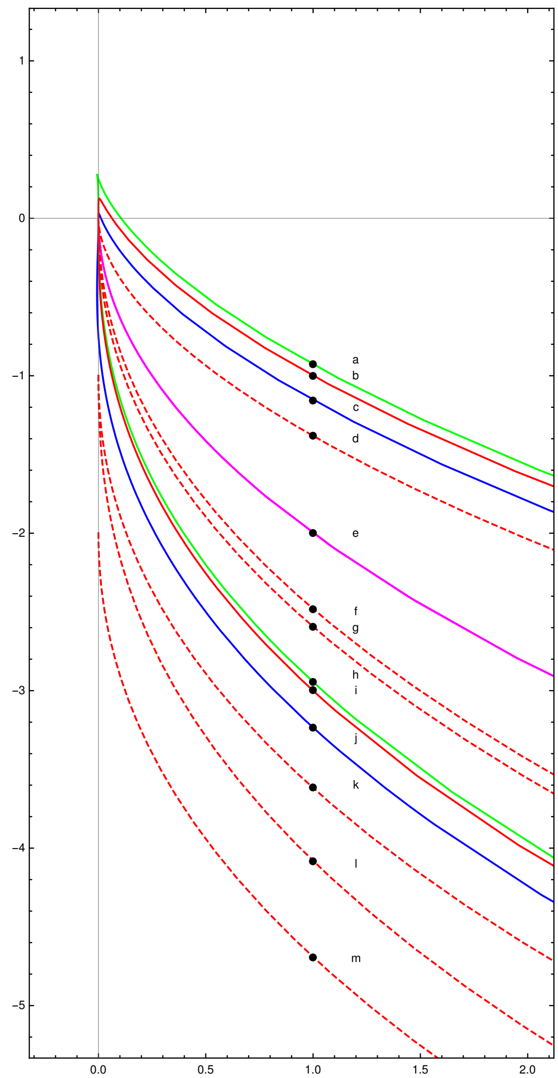


Figure 16: The (β, c) parameter plane. Greeen: saddle-node, Blue: period-doubling, Red: superattracting fixed point (critical point fixed), Red dot-dashed: critical value fixed; Red dashed: critical orbit fixed after two iterates; Magenta: critial orbit pre-pole. Labelled points are all along $\beta = 1$; corresponding graphs are in Figs. **??**a-m.

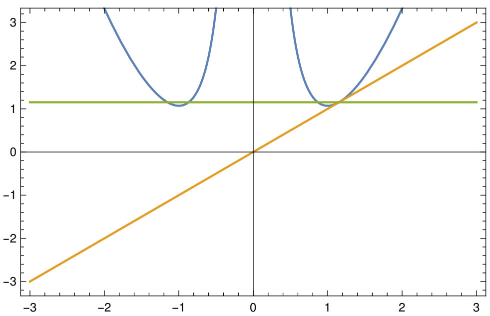
4.4 Codimension-two bifurcation points

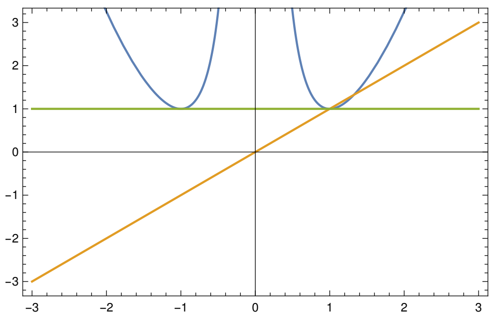
As we have done for the $d = 1$ parameter plane, we have identified several codimension-two bifurcation points. Four are labelled in Fig. **??**. All of them have been described above in subsection **??**.

4.5 Dynamics

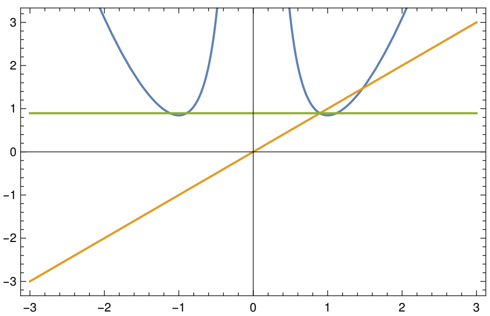
We provide an incomplete set of observations about the dynamics for our family.

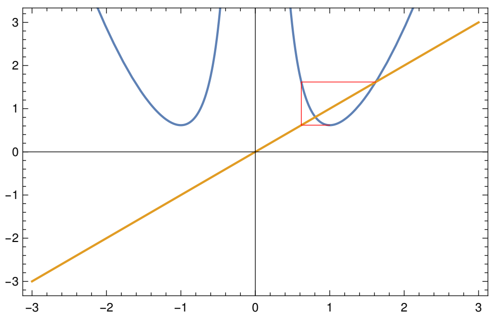
When $\beta < 0$, the graph has two monotonic branches. As long as parameters are also to the left of the saddle-node and period-doubling curves in Fig. **??**, all orbits outside the interval determined by the right-most fixed point and its leftmost preimage will escape.



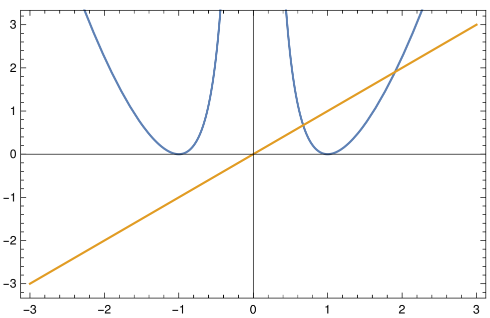


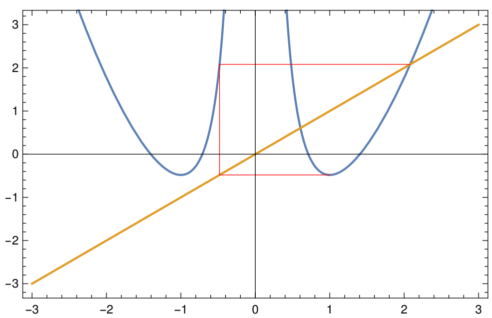
(a) Saddle-node; $c = -0.928623$. (b) Super-attracting fixed point; $c = -1$.



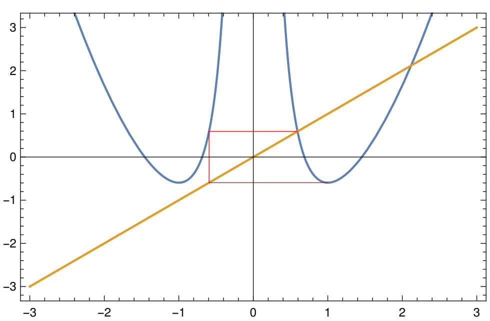


(c) Period-doubling; $c = -1.15447$. (d) Critical orbit prefixed; $c = -1.38197$.



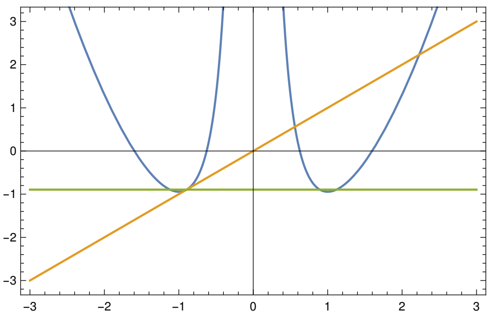


(e) Critical orbit prepole; $c = -2.0$. (f) Critical orbit prefixed; $c = -2.48053$.

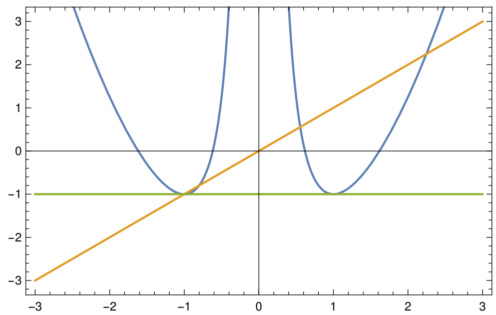


(g) Critical orbit prefixed; $c = -2.5939$.

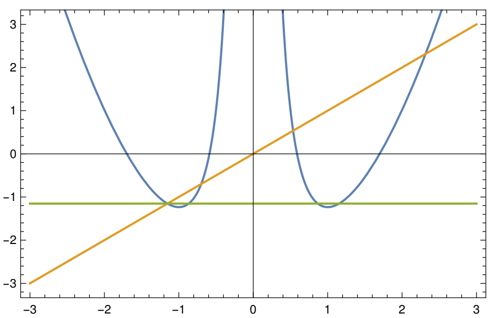
Figure 17: Graphs at some bifurcation values of c for $x^2 + c + \beta/x^2$ with $\beta = 1$. See corresponding parameter values in Fig. **??**. The c values are listed by each subfigure. Green lines emphasize the location of the bifurcation feature. Red lines show the orbit of the critical point(s).



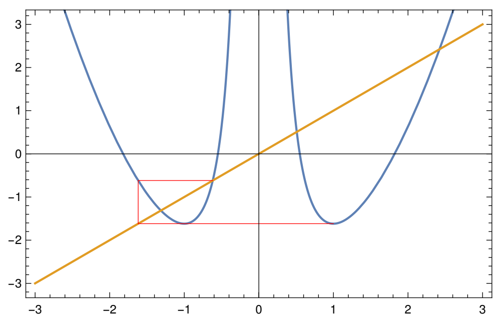
(h) Saddle-node; $c = -2.94443$.



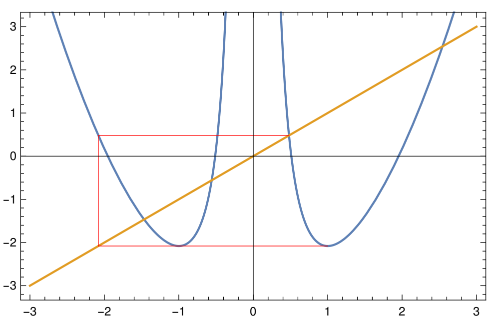
(i) Super-attracting fixed point; $c = -3.0$.



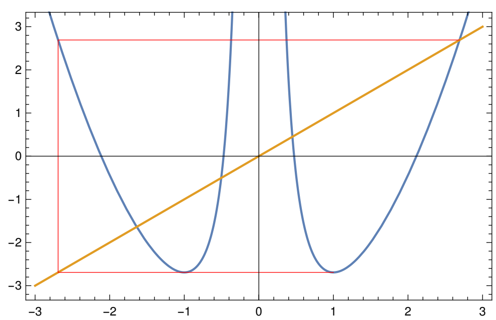
(j) Period-doubling; $c = -3.23418$.



(k) Critical orbit prefixed; $c = -3.61803$.



(l) Critical orbit prefixed; $c = -4.08102$.



(m) Critical orbit prefixed; $c = -4.69174$.

Figure 17: Continued.

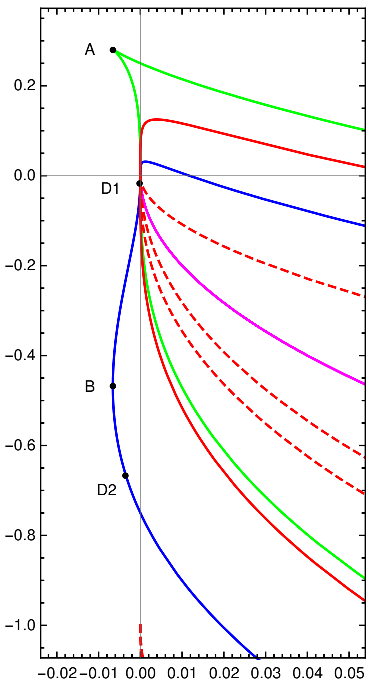


Figure 18: The (c, b) parameter plane. Enlargement of Fig. **??** Green: saddle-node; Red: superattracting fixed point; Blue: period-doubling; Magenta: critical orbit a prepole; Red dashed: critical orbit prefixed in 2 iterates. A: cusp (saddle-node with higher order degeneracies); B: period-doubling at inflection point; D1 and D2: period doublings with higher order degeneracy.

It appears that the points inside that interval also escape, except for an invariant Cantor set. The dynamics appears to be conjugate to a shift on two symbols as the maps all appear to behave similarly to a "V-map", whose dynamics is well-known.

When $\beta > 0$ the dynamics is roughly similar to the description for $d = 1$, but there are some differences. For c above the saddle-node curve in Fig. **??**, all orbits escape. Compare with Fig. **??** by shifting the graph up slightly. For all c below the lowest dashed red line (for critical orbits fixed after two iterates), there is no invariant interval, but the graphs stretch four times across the interval determined by the right-most fixed point and its leftmost preimage. Compare with Fig. **??** by shifting the graph down slightly. The top horiontal red line in that figure is the "interval." Therefore we expect that only a Cantor set of points stays bounded, and the dynamics on this Cantor set is conjugate to the full shift on four symbols. This is for certain the dynamics when c is sufficiently negative that the parts of the graph with both domain and range in this interval have slopes with magintude greater than one.

5 2-D Singular Perturbation: $f_{\beta,c}(z) \equiv z^2 + c + \frac{\beta}{\bar{z}}$

In this section we mention the relationship of our work to a much more difficult problem that we are also interested in understanding: determining the dynamics of $f_{\beta,c}(z) \equiv z^2 + c + \frac{\beta}{\bar{z}}$, where z is a complex dynamic variable, and β and c are complex parameters. Our family from Sec. **??** lives on the real axis of this family whenever c and β are real parameters. Our family also lives on the real axis of $f_{\beta,c}(z) \equiv z^2 + c + \frac{\beta}{z}$, which is in the category of rational complex dynamics. See [**?**] for a survey of results on singular perturbations in rational dynamics. Since our example has a $\bar{z}$ in its formula, it is not a complex analytic rational map, and therefore must be instead considered a rational map of the real plane.

Figure **??** displays the results of an escape experiment for one such family. Black indicates initial conditions whose orbits are bounded up through the first 100 iterations. The picture is partially explained by looking at the real map resticted to the real axis. The graph of this real maps is displayed in Fig. **??**. It has a superattracting period-4 orbit on the real line. The two largest black blobs in Fig. **??** each have two of the points in this period-4 orbit. The other disconected black regions appear to all be preimages of these two largest regions. Clearly there is much more to the dynamics of even this one map of the plane.

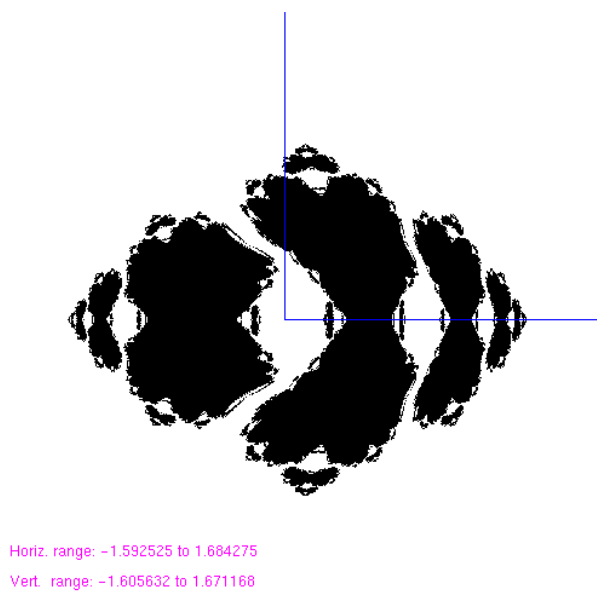


Figure 19: An escape picture for $f(z) = z^2 + \beta/z + c$ where c=-.25 and $\beta = -.2$

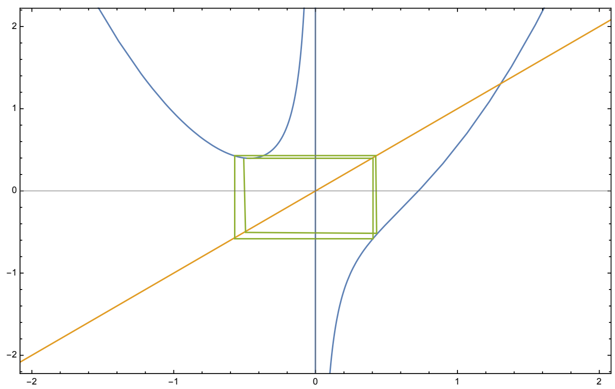


Figure 20: A period-4 critical orbit for $f_{\beta,c}(x) = x^2 + (-0.25) + \frac{-0.2}{x}$

6 Conclusion

This paper has been primarily a study of bifurcations in the families $f_{\beta,c}(x) = x^2 + c + \frac{\beta}{x^d}$ for $d = 1$ and $d = 2$. We conjecture that the bifurcation behavior for any positive d will resemble these two cases, with d odd resembling $d = 1$ and d even resembling $d = 2$ because of the similar shapes of the corresponding graphs. These families are a small subset of the set of all rational maps of the real line, but they have the advantage that they can be compared to the well-known family $x^2 + c$.

Our longer term goal is to use understand the dynamics of these real rational maps as a starting point for understanding non-analytic perturbations of the well-known complex quadratic famlily $z^2 + c$ of the form $z^n + c + \frac{\beta}{\bar{z}^d}$. See some work on special cases of these families in work by former students supervised by B. Peckham: Brett Bozyk 2013 [**?**], Jordan Maiers 2014 [**?**], Evan Oman 2015 [**?**], Matt Arthur 2015 [**?**], and Yujiong Liu 2017 [**?**].

Much is still waiting to be discovered.

7 Acknowledgements

The initial research for this paper was largely performed by the first author, L. Wang, under the supervision of the second author, B. Peckham. The work by L. Wang was partially funded in 2016-2017 by the University of Minnesota as an Undergraduate Research Opportunities Project (UROP). The second author edited and added results in 2019.

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