Investigating the Behavior of Perturbed Complex Quadratic Maps: Moving the singularity and the critical point*

Matt Arthur †
Bruce Peckham‡

June 28, 2019

Abstract

The study of discrete dynamical systems has substantially grown in popularity over the past 30 years with increases in computing power. The use of computer technology to simulate the behavior of these systems has been instrumental in understanding the inner workings of many different research questions in the field. This research report focuses primarily on explaining dynamical behavior of a certain family of nonanalytic maps of the plane which was constructed by adding a singular perturbation term to the well-studied quadratic complex family: $z_{n+1} = z_n^2 + c$. The dynamical system to be studied in this project is given by

$$z_{n+1} = z_n^2 + c + \frac{.001}{(\overline{z}_n - \overline{b})^2}$$

where z, c, and b are all complex numbers. This report discusses and summarizes the steps that were taken to understand the dynamical behavior of the system given above. One of the main goals of the project was to understand how the values of parameters b and c influence this dynamical behavior. Findings suggest explanations of some of the numerical experiments.

*Note: This report summarizes the results of an undergraduate research project (UROP) performed by the first author under the supervision of the second author from Fall 2014 through Spring 2015 at the Department of Mathematics and Statistics, Univerisity of Minnesota Duluth (UMD), Duluth, MN 55812. The original report was edited and modified in spring 2019 by the second author.

†Email: arth0074@d.umn.edu
‡Email:bpeckham@d.umn.edu.

Contents

1 Introduction

This report summarizes the results of a UROP project that took place Fall 2014 through Spring 2015 at the Univerisity of Minnesota Duluth (UMD). The project was conducted by the first author (MA) through the UMD's Department of Math and Statistics and advised by the second author (BP). The dynamical systems studied in this projected are variants of more heavily studied maps, such as the singularly perturbed maps of the form $z^n + c + \alpha/z^d$ studied by Devaney et al. [3, 4].

Background. Dynamical systems is a broad field of mathematics, and one of the most studied due to its applications to science and nature. Dynamical systems can appear in the form of differential equations or iteration formulas. This study focuses exclusively on the latter. See a textbook such as [2] for an introduction to dynamics.

The particular system studied in this project is defined as follows. Let $f : \Re^2 \mapsto \Re^2$ be defined as

$$f_{(c,b)}(z) = z^2 + c + \frac{\beta}{(\overline{z} - \overline{b})^2} \tag{1}$$

with $z \in \mathcal{C}$, and parameters $c, b \in \mathcal{C}$. The auxiliary parameter β is set to 0.001 for this paper. The system's corresponding iteration formula is then given by: $z_{n+1} = f_{(c,b)}(z_n)$. The initial condition, or seed for this system is denoted z_0. Fixing c and b (and β) defines a particular map in this family, and then selecting z_0 generates a sequence of values for the system. This sequence, $\{z_0, z_1, z_2, ...\}$, is known as an *orbit*. The goal in dynamical systems is to understand the long term behavior of all such orbits for all values of c, b and all initial conditions. The parameter b was introduced to the project with the goal explaining the behavior of orbits as the singularity of the map shifts. The majority of the report, however, considers the special case where $b = 0$, although there is a section on 1D maps which provides more details related to changes in this special parameter. When b is fixed at zero, we often abbreviate $f_{(c,0)}$ as f_c. The goal of understanding the long term behavior of orbits is easier to achieve for some dynamical systems than for others. The system in this report is a perturbation of a map whose dynamics mathematicians for the most part fully understand: $z \mapsto Q_c(z_n) = z^2 + c$. The addition of the perturbation term in this study changes the behavior of orbits in this system in a number of ways. First of all, the pertubation is singular, and so points near $z = b$ map to a large magnitude point, and successive iterates increase without bound (or "go off to infinity", terminology which will be used throughout the report). Perhaps the most significant complication in this case stems from the addition of $\overline{z}$ in the perturbation term. Including this term with $\overline{z}$ changes the function $f_{(c,b)}$ so that it is no longer complex analytic. The full details on this nonanalytic classification are beyond the scope of this report, but essentially $f_{(c,b)}$ is put into another class of functions which run no longer from $\mathcal{C}$ to $\mathcal{C}$, but from $\Re^2$ to $\Re^2$. Many theorems in complex dynamical systems apply only to complex analytic maps, so proving results about this family can require significantly new approaches.

1.1 Methods and Terminology

Research in dynamical systems is typically a combination of numerical experimentation and analysis. Since few mathematicians have studied the dynamics of $f_{(c,b)}$, existing dynamics software had to be modified or built from scratch to accomodate the needs of this project. Over the course of this study, much time and effort was spent attempting to optimize computer programs to carry out certain tasks. This section focuses on the different tools used in this project and the role of computer technololgy in utilizing these tools. We set $b = 0$ in this section and abbreviate $f_{(c,0)}$ by f_c.

Escape Sets One first step that researchers use to begin to understand orbit behaviors for any iteration function is to distinguish orbits which stay bounded from those that are unbounded. For fixed c and b, the set of all z values whose orbits stay bounded is called a Filled Julia Set. To make this a bit more formal, let K_c denote the Filled Julia Set for c. Then

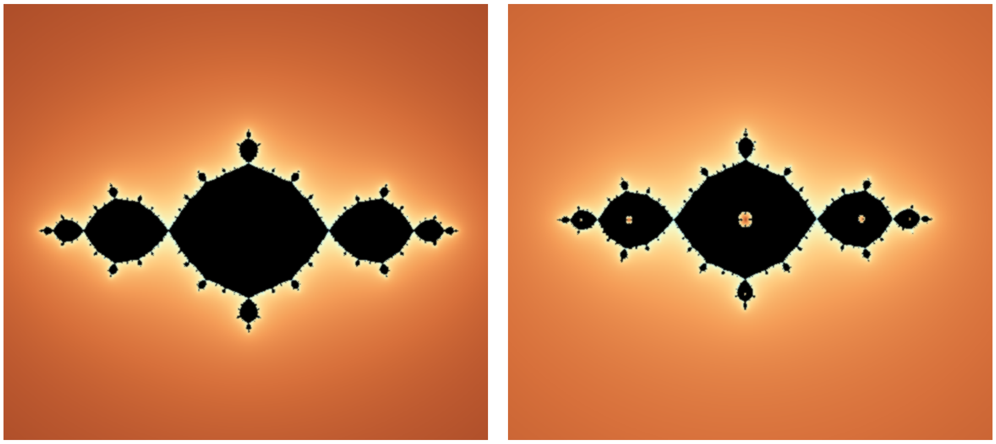
$$K_c = \{z \in \mathcal{C} | \exists k < \infty : \forall n \in N, |f_c^n(z)| \leq k\}.$$

K_c is the set of initial conditions corresponding to bounded orbits; its complement is called the escape set. This terminology makes sense for our family because infinity is attracting. That is, orbits whose entries reach a certain magnitude necessarily have successive entries that escape to infinity. It is a phase-plane escape set because z is the phase variable (also called the dynamic variable, as opposed to a parameter) in this family. Colors are assigned based on how many iterates a given orbit takes to exit a disk of a certain radius. Darker colors indicate faster escape, except that black indicates an orbit that does not escape, at least through the finite number of iterates checked by the escape algorithm. See numerically computed escape picture in Fig. 1. Note the brown region near the origin in Fig. 1b. Following terminology from [3], this escape region is called the *trap door*. Additional brown regions in that figure are preimages of the trap door.

Another kind of escape set which is common for mathematicians to study is the parameter-plane escape set. Recall that K_c is calculated by fixing parameter values and varying initial condition values, z_0. Parameter-plane escape sets are computed by fixing an initial condition value for z_0 and varying parameters. Similar to the definition of K_c, let M_{z_0} denote the parameter plane escape set for a given value of z_0 . Then,

$$M_{z_0} = \{c \in \mathcal{C} | \exists k < \infty : \forall n \in N, |f_c^n(z_0)| \leq k\}.$$

It would also be possible to consider b plane parameter escape sets, although none are considered in this report. The notation M_{z_0} refers exclusively to a c-plane parameter escape set, whose map will be made clear by context. Figures 1a and 1b below show K_{-1} for the quadratic map $Q_{-1}(z) = z^2 - 1$, and function $f_{-1}(z)$, repectively. Note that a different value for z_0 was chosen for the parameter-plane escape sets in Figures 2a and 2b. The reason for this will be addressed shortly, but the reasoning behind this choice of z_0 turns out to be one of the key investigative questions of this report. In 1D maps and complex analytic maps, parameter-plane escape sets are usually computed by choosing



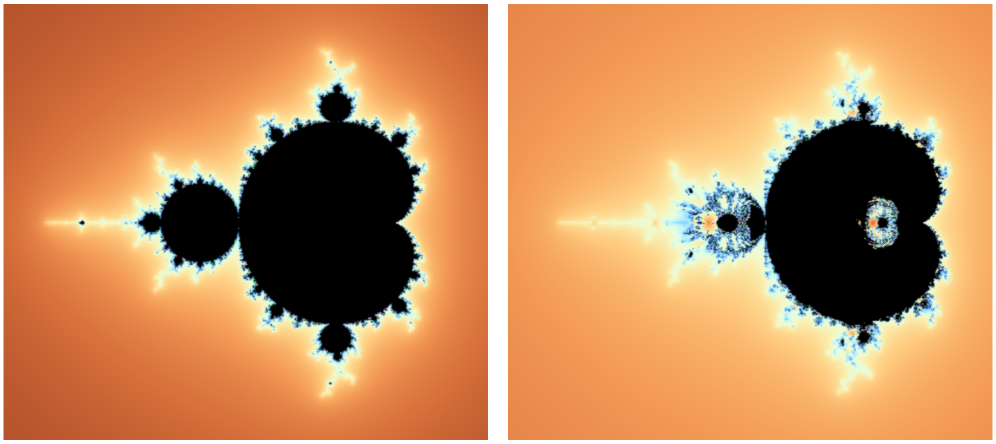
(a) Unperturbed: $z \mapsto z^2 - 1$ (b) Perturbed: $z \mapsto z^2 - 1 + \frac{0.001}{\overline{z}^2}$.

Figure 1: Dynamic plane filled Julia sets.

a value for z_0 which is a critical point (derivative equals zero) of the function in the system. This choice of z_0 is partially due to a theorem in dynamics which states that, for iteration functions satisfying certain conditions, every attracting periodic orbit must attract a critical point. For the map $z \mapsto z^2 + c$, there is only one such critical point, at $z = 0$. Thus, when computing a parameter-plane escape set for this map, choosing the seed $z_0 = 0$ is the clear choice.

The choice is not so clear for the map f_c. For maps of the real plane, critical points are defined as points for which the Jacobian determinant vanishes. It turns out that the critical set f_c is not an isolated set of points, but instead a circle centered at the origin with radius $0.001^{1/4} \approx 0.17783$. This can be verified using "$(z, \overline{z})$" coordinates, or converting to (x, y) coordinates, where $z = x + iy$, and $\overline{z} = x - iy$. This set shall be referred to as the critical circle, and denoted J_0. Its image will be denoted J_1. It is not immediately obvious which point to choose from the critical circle in the calculation of M_{z_0}. Further results on this topic will are presented in Section 2 of the report.

A variety of computer software was used in the computation of escape sets throughout this project. Initially, the mathematical software Sage was used to create escape sets. "To Be Continued...", a dynamical systems software authored by the second author, was also used in the creation of these sets [9]. "To Be Continued..." already had maps of the form of eq. (1) coded. It is also possible to compute critical sets with "To Be Continued...", although *Mathematica* was used more frequently for this task. Towards the end of the project, Fraqtive [7], an escape set explorer application was modified to use the system in this study. This software was optimized well to begin with, and the program can be used to quickly modify K_c based on the position of the mouse pointer in the c plane. This tool ended up being useful as a visual aid and to quickly observe the escape/nonescape sets for any value of c.



(a) Unperturbed Mandelbrot Set: M_0 for $z \mapsto z^2 + c$

(b) Perturbed Mandelbrot Set: $M_{(0.17783,0)}$ for $z \mapsto z^2 + c + \frac{0.001}{\bar{z}^2}$

Figure 2: Parameter plane escape sets.

1.2 Studying Shifts in the Singularity: 1D Maps

Studying shifts in the location of the singularity of the system has the added difficulty that the system's critical sets are no longer necessarily circles. The programming to create parameter-plane escape sets becomes much more difficult. To study variations in the value of b, the system was restricted to the real line. If c and b are restricted to be real, then restricting the system to the real axis is equivalent to studying the dynamics of the real-valued system $x \mapsto x^2 + c + \frac{0.001}{(x-b)^2}$, for $x, c, b \in \Re$. This is not hard to see, as all complex numbers have been removed from the system, and successive iterates of real initial conditions must stay real.

We studied variations in b by fixing a value of c and computing an orbit diagram in the (b, x) plane. See Figure 5. More information and analysis of these concepts is presented later in this report.

Mathematica was used abundantly in the study of the restricted 1D case. It was first used to compute bifurcation curves and orbit diagrams. Symbolic dynamics were considered in studying this problem, and *Mathematica* was used to do a preliminary analysis of the symbolic behavior of orbits in the system, but these results are not presented here.

2 Results

The results presented in this section should provide some insight into the dynamical behavior of this system and give directions for how research for this system might proceed in the future. This section is divided into two parts: Part 1 discusses results about the 2D system using $b = 0$. Part 2 discusses results concerning the one-dimensional system obtained by restricting to the real axis with both c and b also real. Part 2 also discusses future work and further direction of the research into this type of system.

2.1 Part 1: Escape sets

2.1.1 Filled Julia sets

.

Numerical escape experiments suggested that the filled Julia sets for our maps exhibit a symmetry.

Remark 1: For all maps in the family $f_c(z) = z^2 + c + \beta/\overline{z}^2$, K_c is symmetric with respect to the origin.

Proof. = This can be verified because the only dependence on z as z^2 and $\overline{z}^2$. Replacing z with $-z$ in both terms yields the same value. That is, $f_c^2(z) = f_c^2(-z)$. Thus, the orbits of z and $-z$ are identical after the first iterate. So the coloring of the escape experiments for the two points is identical. Since this result is independent of c, we conclude that for all c, K_c is symmetric with respect to the origin. □

The result above is interesting because it means that when analyzing the Filled Julia sets for this system, it is only necessary to analyze dynamical behavior for half of the points on the z plane.

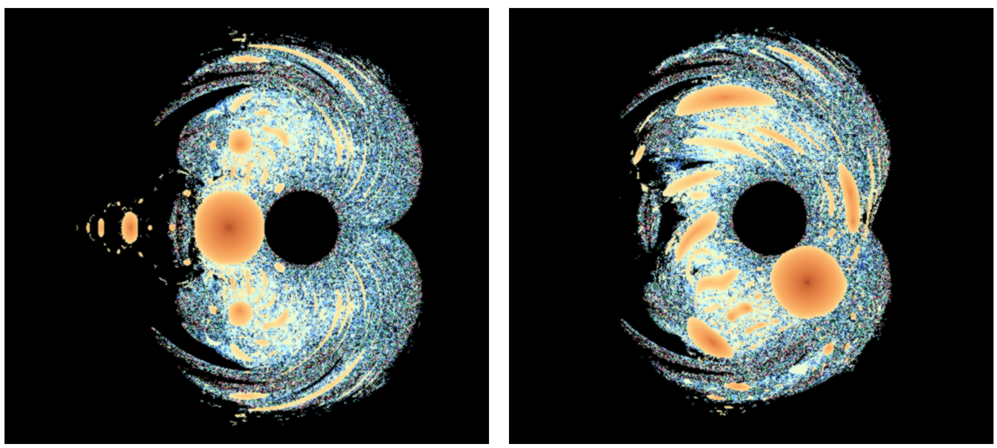
Further, if the value of c is real (and β is a fixed real number), then oribits that start at conjugate points stay at congugate points. That is: $f_c(\overline{z}) = \overline{f_c(z)}$. This means the corresonding sets K_c are all symmetric across the x-axis. See, for example, Fig. 1. For sets with both symmetries, it is only necessary to compute orbits in one quadrant to determine the escape behavior of all orbits.

2.1.2 Parameter space escape sets with varying point on critical circle

. Some of the most interesting results in this section are concerned not with the phase-plane escape sets, but with the parameter plane escapse sets. Section 1.1 refers to the issue of choosing a point on the critical circle J_0 with which to calculate parameter plane escape sets for this system. The results of this study show that M_{z_0} varies somewhat, based on the starting value z_0 on the critical circle. For any c, f_c has a critical circle centered at the origin with radius $|\beta|^{1/4}$. For $\beta = 0.001$, this radius is $0.001^{1/4}$.

Explaining "The Dark Brown Disks in Figures 3a and 3b. Figure 3 shows two zooms of the center of M_{z_0} using, in polar coordinates, $z_0 = 0.001^{1/4}e^{i0}$, and $z_0 = 0.001^{1/4}e^{2\pi/3}$. Note the different location of the circular brown region tangent to the middle black circular region in figure 3a versus 3b. The black disk at the center of these escape sets is known as the Maiers region and has been explained in previous research [6]. For maps corresponding to the Maiers region, the trap door has no preimages, so the escape pictures for corresponding maps have only one "hole": the trap door. The location of the brown disks rotates around the Maiers region (twice) as the critical point z_0 rotates around the critical circle. The dark brown center of this "primary" brown circular region for M_{z_0} is a c value for which z_0 lands on the pole at 0 in one iterate. This dependence can be verified by computation. First, observe that the value of c plays no role in the location of the critical circle J_0, since the constant disappears when differentiating to obtain the Jacobian matrix. However, the first iterate of the critical circle, J_1, does depend on the value for c. To see this relationship, it helps to consider the map in polar coordinates. That is: $z \mapsto f_c(z) = (re^{i\theta})^2 + pe^{i\phi} + \frac{0.001}{(re^{-i\theta})^2}$, where $z = re^{i\theta}$, and $c = pe^{i\phi}$. This expression can be simplified to yield

$$re^{i\theta} \mapsto (r^2 + \frac{0.001}{r^2})e^{2i\theta} + pe^{i\phi}. \tag{2}$$



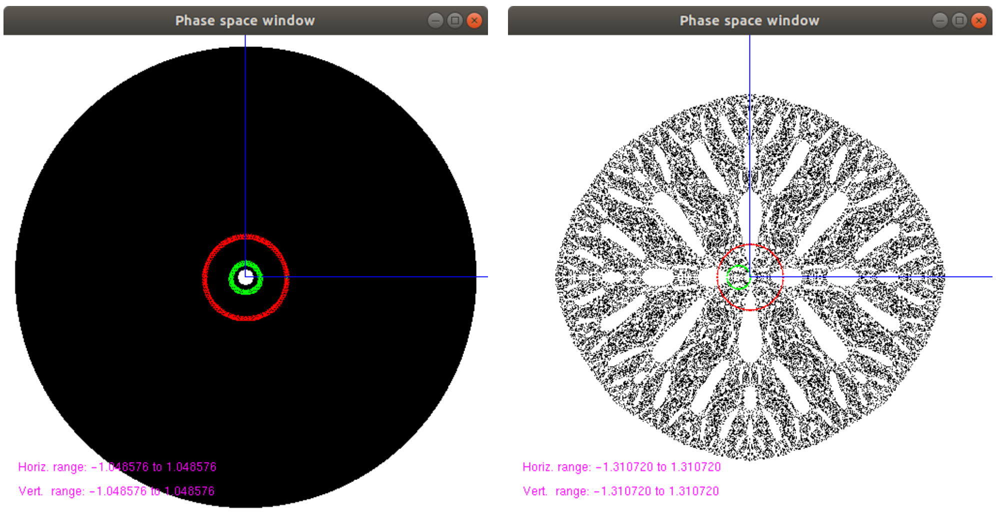
(a) Zoom of Fig. 2b: M_{z_0} for $z_0 = (r, \theta) = (0.001^{(1/4)}, 0)$

(b) Same as Fig. 3a but with z_0 replaced by $(r, \theta) = (0.001^{(1/4)}, \pi/3)$

Figure 3: Zoom of c plane escape figures for $z \mapsto z^2 + c + \frac{0.001}{\bar{z}^2}$. The origin is at the center of the Maiers region, which is the black disk. The location of the brown disk depends on z_0.

The RHS of equation (2) can vanish only if $p = r^2 + \frac{0.001}{r^2}$ and $\phi = 2\theta + \pi$. If, in addition, $re^{i\theta}$ is on the critical circle, then $r = 0.001^{\frac{1}{4}}$, so $p = 2(0.001)^{1/2}$. That is, $c = 2(0.001)^{1/2}e^{2\theta_0+\pi}$. This is illustrated in Fig. 3a, with $\theta_0 = 0$ and $c = -2(0.001)^{1/2} \approx -0.0623$, and in Fig. 3b, with $\theta_0 = \pi/3$ and $c = 2(0.001)^{1/2}e^{2\pi/3+\pi} = 2(0.001)e^{5\pi/3} \approx 0.0623e^{5\pi/3}$. A similar computation can be done for more general families of the form $z^d + c + \beta/\bar{z}^d$.

The center of the disk in Fig. 3a is at a c value of approximately $c = -2(0.001)^{1/2} = -0.0623$. Using this value for c, Fig. 4 shows the critical circle J_0 and its image J_1 for two different values of c, $c = 0$ and $c \approx -0.0623$, the center of the brown disk in Fig. 3a. The critical set J_0 is the same (red) circle for both figures, but while J_1 for $c = 0$ is centered at the origin, J_1 for $c = -0.0623$ is translated so that it (almost exactly) passes through the origin.



(a) $c = 0$: no images inside the green circle, so no orbits land in the trapdoor; the whole black annulus is the bounded set.

(b) $c = -.0623$: the (green) critical circle image J_1 goes through the origin, so part of the trapdoor is outside the green image of the critical circle, so orbits can land in the trapdoor and escape; lots of (white) preimages of the trapdoor.

Figure 4: Filled Julia sets for for $z \mapsto z^2 + c + \frac{0.001}{\overline{z}^2}$. Critical circle in red; its image in green.

Remark 2: If β is real, and z_0 is real, then the corresponding parameter-plane escape set, M_{z_0}, is symmetric with respect to the real axis.

Proof. This follows from the observation that, if $f_c(z) = z^2 + c + \beta/\overline{z}^2$, then $f_{\overline{c}}(\overline{z}) = \overline{z}^2 + \overline{c} + \beta/z^2 = \overline{f_c(z)}$. This means orbits under f_c versus $f_{\overline{c}}$ that start out at conjugate z values have conjugate orbits. Thus, the escape/nonescape behavior of the two orbits is identical. This makes M_{z_0} symmetric across the real axis. □

This result is useful because it means that for the two values of z_0 on the critical circle that lie on the real axis, it is only necessary to explain one half of the parameter plane escape set.

2.2 Part 2: 1D Maps with varying singularity

Recall that at the beginning of this report, the map f was given in full generality as

$$f_{b,c}(z) = z^2 + c + \frac{\beta}{(\overline{z} - \overline{b})^2} \tag{3}$$

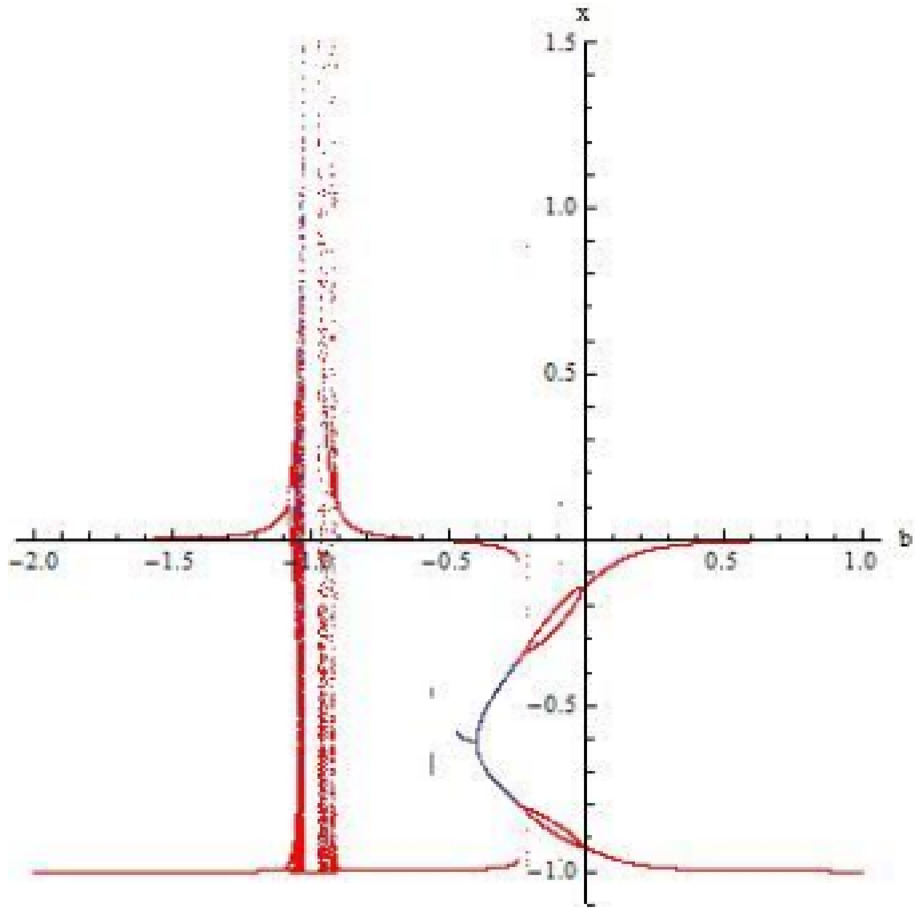
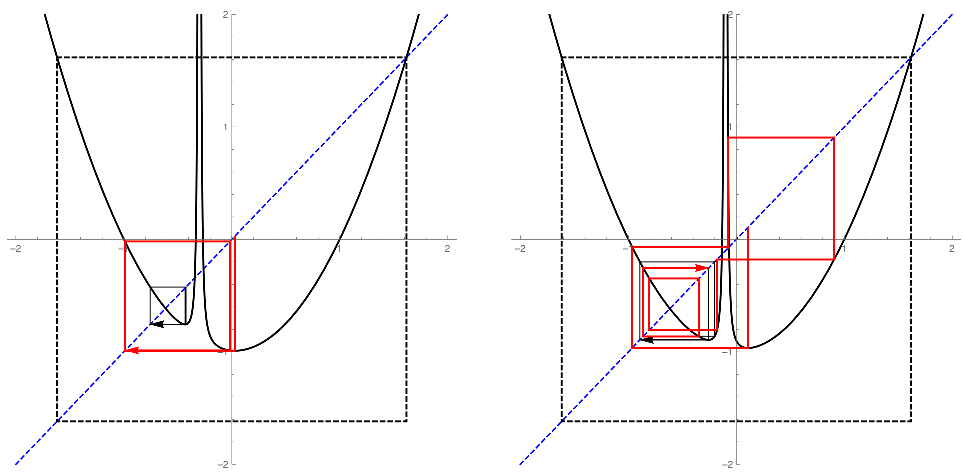


Figure 5: Orbit Diagram x versus b for $c = -1$ and $\beta = 0.001$. The critical orbit of the right-hand (left-hand) ritical point is in red (black).

The majority of this report studies the case where $\beta = 0.001$, and $b = 0$, as this is the case for which the majority of results were able to be proven. However, one of the challenges that will be faced by researchers in this field is the study of the map in full generality – as b is allowed to vary.

Orbit diagrams. One numerical experiment that leads to some understanding of this family is to fix all the parameters except b, the location of the singularity, restrict the phase variable to be real, and compute an orbit diagram in the (b, x) plane. We fixed $c = -1$, and $\beta = 0.001$. Although the orbit diagram allows only one real parameter to vary, it contains information about the phase location of the orbits. So in some ways it includes more information and in other ways less information than the two-parameter escape pictures for M_{z_0}. In particular, it is possible to identify attracting fixed and periodic points as b varies.

The maps in this family all have two critical points. The orbit diagram in Fig. 5 is created by computing both critical orbits at any fixed parameter b, throwing away the first 100 iterates, and plotting the next 100. Sometimes the two critical orbits have the same fate, other times they have different fates. For example, at $b = -.3$, each critical orbit approaches its own attracting period-2 orbit. See Fig. 6a: each period-2 orbit is close to including one of the critical points. At $b = -.1$, each critical orbit approaches the same attracting period-4 orbit. See Fig. 6b. The attracting period-4 orbit is close to including the left-hand critical point. The period-4 orbit can be seen by carefully following the cobweb diagram starting close to that critical point. The right-hand critical point follows a longer path before it gets drawn into the same attracting period-4 orbit. All transients for both critical orbits are shown in Fig. 6b. Compare with Fig. 5 at $b = -0.1$ and $b = -0.3$. The "slice" at $b = -0.3$ shows four points on the orbit diagram: an outside period-2 orbit (red) and an inside period-2 orbit (black). The "slice" at $b = -0.1$ also shows four points on the orbit diagram, but they corresond to a single period-4 orbit.



(a) Fates of both critical orbits for $b = -0.3$; transients included, but barely visible because each critical point is almost on the period-2 orbit to which it is attracted.

(b) The attracting period-4 fate of both critical orbits for $b = -0.1$; transients included. Five iterates of the left-hand critical point are shown in black; nine iterates of the right-hand critical point are shown in red. The orbit from the red arrow maps to the black arrow, and in three further iterates arrives (almost) back at the red arrow.

Figure 6: Fates for both critical orbits for $x^2 - 1 + .001/(x-b)^2$. Dashed black box is for reference.

Note the "interesting" behavior on this diagram occurs close to $b = -1$. The strange behavior around $b = -1$ is partially due to the fact that the critical orbits land "close enough" to the singularity at $x = 0$ to cause their corresponding orbits to escape almost immediately. This explains the thin vertical white strip present in the orbit diagram in Fig. 5 near $b = -1$. Note that the location of the critical points is a function of b only and not c. That is, the orbit diagram would iterate the same set of critical points for any c value, but the fates would, of course, depend on c. Note in Fig. 5 that when $b > 0.4$ or $b < -1.5$, orbits in this system approach an attracting period-2 cycle close to $x = 0$ and $x = -1$. This is because the singularities are far from the period-2 orbit between $x = 0$ and $x = -1$ that exists for $x^2 - 1$. Most of the interesting behavior occurs over the approximate interval $(-1.2, 0)$, at least for the range of b values displayed in Fig. 5. It will be a goal for future researchers to fully explain the behavior of orbits in this more chaotic region.

Two-parameter bifurcation diagrams. Another approach to gain insight into the dynamics for our one-dimensional family in eq. (3) is to numerically compute codimension-one bifurcation curves in the (b, c) parameter plane. The curves we computed used continuation algorithms from "To Be Continued...", a dynamical systems software package [9]. Fig. 7 below shows several fixed-point bifurcation curves in the (b, c) parameter plane. Corresponding cobweb phase diagrams are show in Fig. 8. The curves displayed are only a sample of "all" the bifurcation curves which exist for the two-parameter family. A full discussion of the bifurcations for this family is beyond the scope of this report. In particular, we have computed only fixed-point bifurcations. There is an infinity of other bifucation curves for other periods. All of the missing curves, however, exist between the highest and lowest curves in Fig. 7. That is, no bifurcation curves exist above the the two (green) saddle-node curves (through (a) and (d)), and no bifurcations occur below the the lower of the magenta critical orbit prefixed curve through (j) and the cyan critical orbit prefixed curve through (k). For any c-values "above" the saddle-nodes, all orbits escape to infinity; for any c-values "below" the prefixed curves, the dynamics is (appears to be) conjugate to a full shift on four variables, with the bounded set being a Cantor set created by removing three intervals at each stage of its construction.

Note that when $c = -1$, the two-parameter bifurcation diagram in Fig. 7 is related to the orbit diagram in Fig. 5. For example, the point on the green saddle node curve at $(b, c) \approx (-.472, -1)$ corresponds to the saddle-node point at the left-hand endpoint of the short curve of attracting fixed points at $(b, x) \approx (-0.472, -0.57)$ in Fig. 5.

.

3 Future Work

The family of dynamical systems $f_{(c,b,\beta)}$ studied in this paper (eq. (1)), and its generalizations, are currently a central subject of study for the second author, students, and colleagues. See, for example, the work in [1, 8, 11, 13, 5]. There is still much to explain about these systems. The one-dimensional restriction may be able to be completely explained using symbolic dynamics. See, for example, the symbolic dynamics used in [8] Future work directly related to this study may involve introducing the b parameter into the full system from $\Re^2$ to $\Re^2$. As stated earlier, the critical sets for this more general

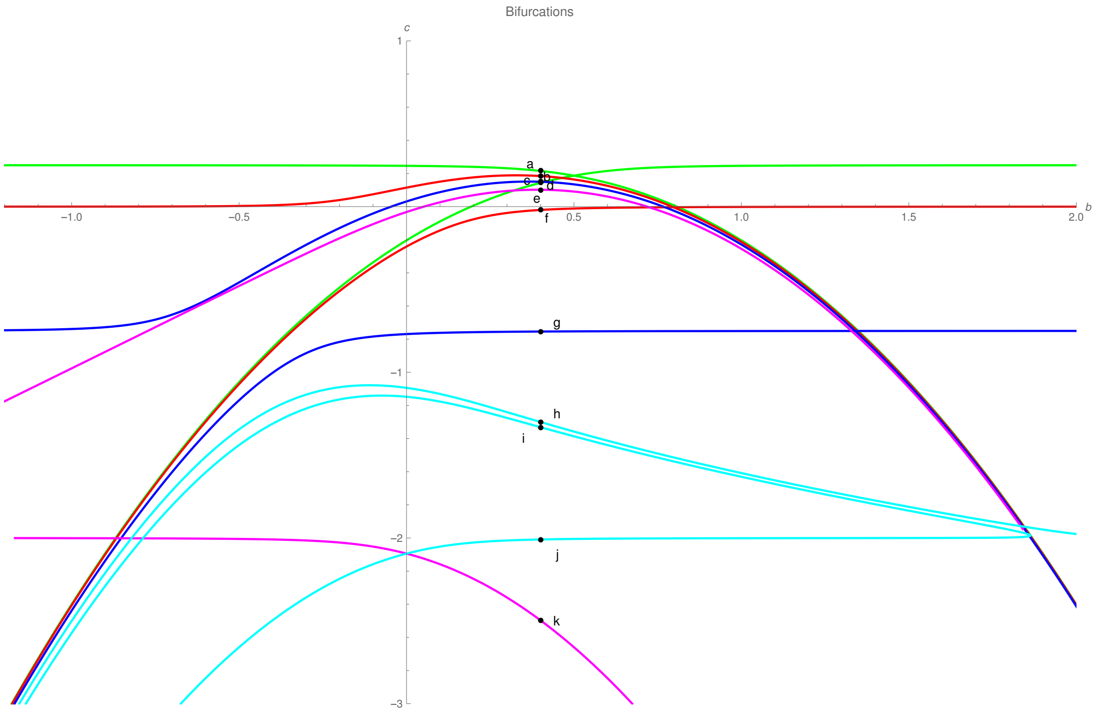
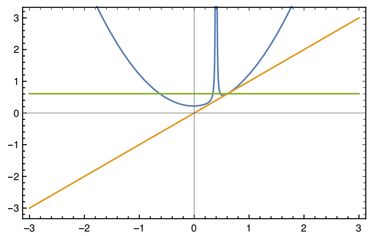


Figure 7: Bifurcation curves in the (b, c) plane for $z \mapsto z^2 + c + \frac{0.001}{(\bar{z}-\bar{b})^2}$. Labelled points along $b = 0.4$ correspond to cobweb diagrams in Fig. 8. Green: saddle-node, Red: superattracting, blue: period-doubling, Magenta: right-hand critical point is fixed after two iterates, Cyan: left-hand critical point is fixed after two iterates.

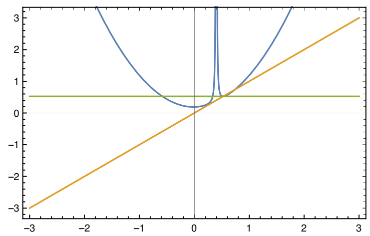
system are curves rather than points. In many examples studied so far, the critical sets are circles. Significant further analysis will be necessary to fully understand the dynamics in the real plane.

4 Conclusions

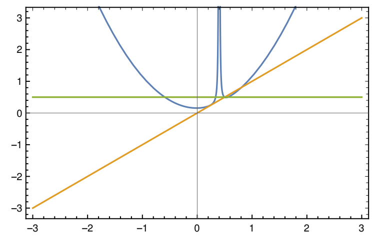
The point of this study was to gain a better understanding of certain rational maps of the plane. This is a massive area of study in discrete dynamical systems. To date, much more is known about complex-analytic rational maps of the complex plane, versus rational maps of the real plane Perturbing a known complex-analytic system by a nonanalytic term begins to generalize the problem away from the complex plane and to the real plane. This study focused on one such perturbed system and a partial analysis of its behavior. The study also sought to compare the perturbed system to the original system to gain insight into the changes brought about by adding in a nonanalytic term. This study used a variety methods from both primary and secondary sources to aid in the research process, and the results from the study have helped shed a small amount of light on the family



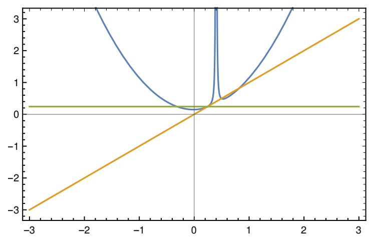
(a) $c = 0.215226$: saddle-node fixed-point



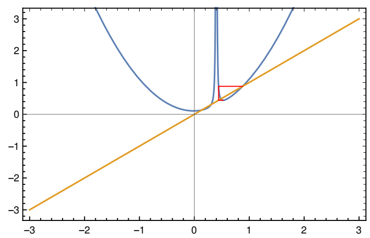
(b) $c = 0.184423$: super-attracting fixed point



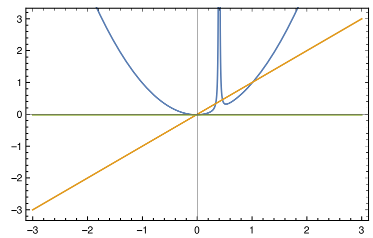
(c) $c = 0.15$: period-doubling



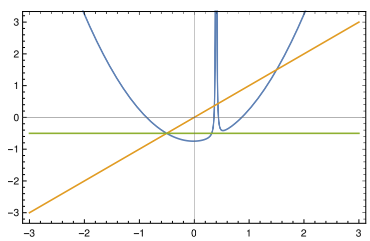
(d) $c = 0.143382$: saddle-node fixed-point



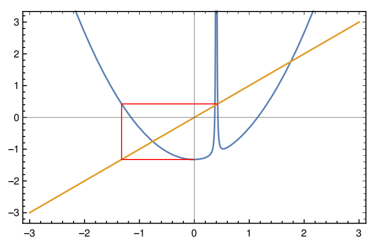
(e) $c = 0.101819$: critical orbit pre-fixed



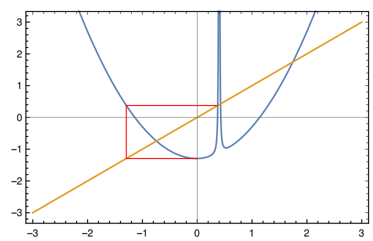
(f) $c = -0.0201147$: superattracting fixed-point



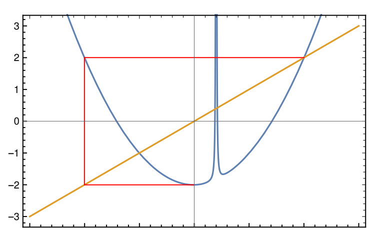
(g) $c = -0.753964$: period-doubling fixed-point



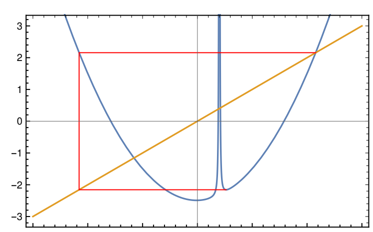
(h) $c = -1.29985$: critical orbit pre-fixed (left of singuarity)



(i) $c = -1.33110$: critical orbit pre-fixed (right of singularity)



(j) $c = -2.0089$: critical orbit pre-fixed



(k) $c = -2.49695$: critical orbit pre-fixed

Figure 8: Cobweb phase graphs at bifurcation c-values for $x^2 + c + 0.001/(x - 0.4)^2$. Green lines emphasize the bifurcation point; red lines mark pre-fixed critical orbits.

of dynamical systems that was studied. There are still far more questions than answers for both this system and perturbed discrete dynamical systems in general. The studies conducted througout this project are a small step in understanding the dynamics of a very broad and complicated class of systems.

5 Acknowledgements

Figures 1, 2, and 3 were computed using a version of the sofware Fraqtive [7], modified by the authors.

References

[1] Bozyk, Brett and Peckham, Bruce B: *Nonholomorphic Singular Continuation: a Case with Radial Symmetry*, International Journal of Bifurcation and Chaos 23: (2013):1-22. Web. http://www.d.umn.edu/ bpeckham/export/Papers/sing-ijbc.pdf

[2] Devaney, Robert L., *A First Course in Chaotic Dynamical Systems: Theory and Experiment*, Reading, MA: Addison-Wesley, (1992). Print

[3] Devaney, Robert L., *Singular perturbations of complex polynomials*, Bull. Amer. Math. Soc. 50 (2013), 391-429.

[4] Devaney, R., Blanchard, P., Cuzzocreo, D., and Fitzgibbon, E. "Accessible Mandelbrot Sets in the Family $z^n + \frac{\lambda}{z^n}$." 31 March 2015.

[5] Liu, Yujiong,"A Connection between Analytic and Nonanalytic Singular Perturbations of the Quadratic Map: A Case Study" Master's project, University of Minnesota Duluth, supervised by B. Peckham, 2017.

[6] Maiers, Jordan. "Singular Perturbations of the Quadratic Map $z \mapsto z^2 + c + \frac{\beta}{\bar{z}^2}$." Submitted in Conjuction with a UROP Project. Advised by Dr. Bruce Peckham. April 2014.

[7] Mecinski, Michal. "Fraqtive". Mandelbrot Family Fractal Generator Software. Version 0.4.8. 2015. Web:

[8] Oman, Evan: *Infinite Levels of Complexity in a Family of One-Dimensional Singular Dynamical Systems*, UMD Master's Project, supervised by B. Peckham (2015): 1-42. Web. (www.d.umn.edu)

[9] Peckham B B, *To Be Continued ...*, dynamical systems software package, under development 1985 - 2019. Web. http://www.d.umn.edu/∼bpeckham/tbc_home.html. Contributions from many students of B. Peckham.

[10] Peckham B B and Kevrekidis I G, *Period-doubling with higher order degeneracies* SIAM J. Math. Analysis, Vol. 22, No. 6, pp. 1552-1574, 1991.

[11] Shang, Haitao. "Dynamics of [perturbations of] Quadratic Families", submitted in conjuction with a 2015 UROP and Honors Project. Advised by B. Peckham. Web (www.d.umn.edu/ bpeckham).

[12] Steven H. Strogatz: *Nonlinear Dynamics and Chaos: with applications to physics, biology, chemistry, and engineering* Perseus Books Publishing, (1994). Print.

[13] Wang, Lesheng. "Dynamics of a Singularly Perturbed Quadratic Family", submitted in conjuction with a 2017 UROP Project. Advised by Bruce Peckham. Revised by B. Peckham 2019. Web (www.d.umn.edu/ bpeckham).