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PERIOD DOUBLING WITH HIGHER-ORDER DEGENERACIES*

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Abstract. A family of local diffeomorphisms of $\mathbf{R}^n$ can undergo a period doubling (flip) bifurcation as an eigenvalue of a fixed point passes through -1. This bifurcation is either supercritical or subcritical, depending on the sign of a coefficient determined by higher-order terms. If this coefficient is zero, the resulting bifurcation is "degenerate." The period doubling bifurcation with a single higher-order degeneracy is treated, as well as the more general degenerate period doubling bifurcation where a fixed point has -1 eigenvalue and any number of higher-order degeneracies. The main procedure is a Lyapunov-Schmidt reduction: period-2 orbits are shown to be in one-to-one correspondence with roots of the reduced "bifurcation function," which has $\mathbf{Z}_2$ symmetry. Illustrative examples of the occurrence of the singly degenerate period doubling in the context of periodically forced planar oscillators are also presented.

Key words. period doubling, bifurcation, bifurcation function, Lyapunov-Schmidt, $\mathbf{Z}_2$ symmetry

AMS(MOS) subject classifications. 39, 15

1. Introduction. This paper describes the local bifurcations that take place when we perturb a diffeomorphism $\mathbf{G}_0$ of $\mathbf{R}^n$ which has a fixed point with a single eigenvalue equal to -1. Since $\mathbf{G}_0$ has a nonhyperbolic fixed point, it is necessary to consider higher-order (nonlinear) terms in order to describe the phase portraits near the fixed point of the map $\mathbf{G}_0$, both by itself and also under perturbation in a family $\mathbf{G}_{\boldsymbol{\mu}}$, $\boldsymbol{\mu} \in \mathbf{R}^k$.

When G_0 is a map of $\mathbf{R}$, any even-order term in its Taylor series expansion can be eliminated by a change of variables. This is a direct result of the normal forms theorem. After eliminating the constant and second-order terms, the linear coefficient will be -1 and the sign of the resulting coefficient of the third-order term will determine whether G_0 will undergo a supercritical or subcritical period doubling (flip) bifurcation [Ar], [GH]. If the third-order coefficient should happen to be zero (a higher-order degeneracy), then the sign of the fifth-order term becomes important. Perturbations of the resulting map ($G_0(x) = -x + cx^5 + o(x^5)$, $c \neq 0$) produce a greater number of topologically distinct phase portraits than do perturbations of the nondegenerate ($G_0(x) = -x + cx^3 + o(x^3)$, $c \neq 0$) map. Two parameters are needed to fully capture all possible phase portraits near the (singly) degenerate map. By the same token, a degenerate bifurcation will generically occur only in families with at least two parameters.

This discussion naturally extends to multiply degenerate period doubling maps: $G_0(x) = -x + cx^{2k+1} + o(x^{2k+1})$, $c \neq 0$ ($k-1$ times degenerate). These codimension-k bifurcations will generically occur only in families with at least k parameters.

In § 2, we consider the model $k-1$ times degenerate period doublings $f_0(x) = -x + \delta x^{2k+1}$ where $\delta = \pm 1$, and the corresponding model k-parameter unfoldings $f_{\varepsilon}(x) = -(\varepsilon_1 + 1)x - \varepsilon_2 x^3 - \cdots - \varepsilon_k x^{2k-1} + \delta x^{2k+1}$. We present the mathematical theory in § 3. We show that the period-2 orbits of the individual maps we study are in one-to-one correspondence with the zeros of a "reduced" bifurcation function. This bifurcation function is obtained by using a standard Lyapunov-Schmidt reduction. Because the

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topological equivalence of the maps we study is determined by the period-2 orbits and their stability, knowledge of the corresponding bifurcation functions is sufficient to provide us with the topological classification of the original maps. When we consider a *family* of maps, the possible behaviors of the bifurcation functions are given by standard singularity theory. We need only interpret the singularity theory results in the bifurcation context of the original family of maps. In particular, we show that each family in the class of period doubling bifurcations that we treat is "equivalent" to one of the model families we describe in § 2.

The singly degenerate period doubling has a special significance in two-parameter families of maps such as those generated by periodically forced oscillators, which possess period-q "resonance horns" whose boundaries typically consist of saddlenode bifurcations for the qth iterate of the map. We and other researchers [KAS], [MSA], [P1], [P2], [P3], [SDCM], [VR] have repeatedly observed such a degenerate period doubling bifurcation on the boundaries of period-2 resonance horns. In § 4 we describe two models of periodically operated chemical reactors (a chemostat with simple predator-prey kinetics, and a continuous stirred tank reactor (CSTR) with a single irreversible exothermic reaction) where this bifurcation occurs.

The bifurcation diagrams we obtained for our degenerate period doublings turned out to be virtually the same as those for a Hopf bifurcation with higher-order degeneracies for a *flow* [GS], [Ta]. Consequently, work on the Hopf bifurcation suggested approaches to the period doubling problem. Our analysis in its final form parallels that of Golubitsky and Schaeffer [GS, Chap. VIII]. In particular, the use of the Lyapunov–Schmidt reduction to obtain a bifurcation function, as well as the unreduced function with which to start, was suggested by their exposition. Using the reduction on a "finite sequence space," however, appears to be a new idea in this paper. (We have since found out that Vanderbauwhede [Va] and Brown and Roberts [BR] have independently started using the Lyapunov–Schmidt reduction on finite sequence spaces in current research as well.) See also the bibliography in [GS] for the original references using the Lyapunov–Schmidt reduction and singularity theory to study the Hopf bifurcation for flows.

The Hopf problem for flows and our problem are analogous because both can be reduced to finding roots of the same $\mathbf{Z}_2$-symmetric bifurcation function. The period doubling problem, interestingly, turns out to be significantly easier to handle than the Hopf bifurcation. Many of the issues that [GS] had to treat simply did not appear in the period doubling analysis. Consequently, we are able to obtain slightly stronger stability information from the bifurcation than was obtained for the Hopf bifurcation in [GS]. We discuss the comparison with the Hopf bifurcation further in § 5.

To place our work in context, we provide Table 1, showing model unfoldings for bifurcations with higher-order degeneracies. The unfoldings in the table are not always exactly as in the corresponding reference, and the references are not intended to be complete. In all cases, $\boldsymbol{\varepsilon} \in \mathbf{R}^k$ is the unfolding parameter of the codimension-k bifurcation; $\delta = \pm 1$.

The most widely known higher-order degeneracy in Table 1 is the saddle-node (for either the flow or map) with a single higher-order degeneracy, commonly called the *cusp* bifurcation. The map and flow cases are exactly analogous. We will encounter saddlenode bifurcations with higher-order degeneracies in this paper for period-2 orbits, because they appear in the unfoldings of period doubling points with more than one higher-order degeneracy. Higher-order degeneracies in the Hopf bifurcation for maps, however, are much more complicated to treat than degeneracies in the Hopf bifurcation for flows. The map case includes not only all the subtleties of the flow

TABLE 1

Flows: Name	Vector field	Unfolding	References
Saddlenode	$x' = \delta x^{k+1}$	$x' = \varepsilon_1 + \varepsilon_2 x + \cdots + \varepsilon_k x^{k-1} + \delta x^{k+1}$	[Ar], [GH]
Hopf	$r' = \delta r^{2k+1}$	$r' = \varepsilon_1 r + \varepsilon_2 r^3 + \cdots + \varepsilon_{2k} r^{2k-1} + \delta r^{2k+1}$	[Ar], [GH],
	$\theta' = \omega + r^2$	$\theta' = \omega + r^2$	[GS], [Ta]
Maps: Name	**Map**	**Unfolding**	**References**
Saddlenode	$x \to x + \delta x^{k+1}$	$x \to \varepsilon_1 + (\varepsilon_2 + 1)x + \cdots + \varepsilon_{k-1} x^{k-1} + \delta x^{k+1}$	[Ar], [GH]
Hopf	$r \to \delta r^{2k+1}$	$r \to \varepsilon_r r + \varepsilon_2 r^3 + \cdots + \varepsilon_{2k} r^{2k-1} + \delta r^{2k+1} + \text{h.o.t.}$	[Ch]
	$\theta \to \theta + \omega + r^2$	$\theta \to \theta + \omega + r^2 + \text{h.o.t.}$	
Period Dblg	$x \to -x + \delta x^{2k+1}$	$x \to -(\varepsilon_1 + 1)x - \cdots - \varepsilon_k x^{2k-1} + \delta x^{2k+1}$	this paper

case, but also some monumental additional problems caused by resonant interaction of periodic orbits, and the existence of invariant sets other than equilibria and closed orbits. Chenciner [Ch] has performed much work on this problem. Note that the higher-order terms must appear, even in the model unfoldings.

We point out that [HW] provides a short description of the period doubling with a single higher-order degeneracy ($k = 2$ in Table 1). That model, but not the theorems in this paper, is relatively well known to bifurcation researchers.

2. The model period doubling families. This section is devoted to describing the bifurcations that take place in the specific families we use as our models. The new results, including the justification for choosing these particular families as models, are given in § 3. The interested reader may skip directly to that section, if desired. We do, however, make some effort in this section to prepare the groundwork for the techniques of § 3. In particular, we use the zeros of several "bifurcation functions" to help us describe the topological classification of our model families. These bifurcation functions will turn out to be special cases of the more general bifurcation functions obtained from the more general maps treated in § 3. (See Corollary 3.13.)

Recall that for maps of $\mathbf{R}$ having a fixed point with a -1 eigenvalue, the normal forms theorem [Ar], [GH] allows us to eliminate any even-order term by a change of variable. Thus the absence of even-order terms from our models should seem reasonable. Keep in mind that, because we are describing local bifurcations, we are only interested in the germs of our functions in phase $\times$ parameter space. The base point of all our model germs is the origin of $\mathbf{R} \times \mathbf{R}^k$.

DEFINITION 2.1. The local (near $(x, \boldsymbol{\varepsilon}) = (0, \mathbf{0})$) family

$$f_{\boldsymbol{\varepsilon};k,\delta}(x) := -(\varepsilon_1 + 1)x - \varepsilon_2 x^3 - \cdots - \varepsilon_k x^{2k-1} + \delta x^{2k+1}, \qquad \delta = \pm 1 \tag{2.2}$$

is called the *model local period doubling bifurcation family with $k-1$ higher-order degeneracies*. The map $f_{\mathbf{0};k,\delta}(x) = -x + \delta x^{2k+1}$ (for x near zero) is called the *model period doubling bifurcation map with $k-1$ higher-order degeneracies.*

Note that the parameter $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k)$ is in $\mathbf{R}^k$, $k \geqq 1$. We will often drop the subscripts k and δ since their values are assumed to be fixed for a given family.

2.1. Individual maps: Stability of periodic orbits. We first describe the behavior of the map $f_{\boldsymbol{\varepsilon}} = f_{\boldsymbol{\varepsilon};k,\delta}$ for fixed values of the parameters. Zero is the unique fixed point near $x = 0$ of $f_{\boldsymbol{\varepsilon}}$ for any $\boldsymbol{\varepsilon}$ near $\mathbf{0}$. The fixed point is attracting for $\varepsilon_1 < 0$ and repelling for $\varepsilon_1 > 0$. Since $f_{\boldsymbol{\varepsilon}}^2$ is an orientation preserving diffeomorphism of $\mathbf{R}$, fixed points are the only form of recurrence it can have. Once these fixed points of f_ε^2 have been located, the topological equivalence class of f_ε^2, and therefore that of the orientation reversing f_ε, is determined by the directions in which iterates of f_ε^2 progress in the intervals of $\mathbf{R}\backslash\{\text{fixed points of } f_\varepsilon^2\}$. (This can be proven by fundamental interval arguments.) For nonzero x, it is apparent that if $f_\varepsilon^2(x)-x=0$, then x is on a period-2 orbit for f_ε; if $f_\varepsilon^2(x)-x>0$, the orbit of x increases under iteration of f_ε^2; if $f_\varepsilon^2(x)-x<0$, the orbit of x decreases under iteration of f_ε^2. Since the behavior of f_ε is completely determined by the roots and sign of the function $f_\varepsilon^2(x)-x$, we call it a "bifurcation function" associated with the family $f_\varepsilon=f_{\varepsilon;k,\delta}$ in (2.2).

Even in the nondegenerate case ($k=1$ in $f_{\varepsilon;k,\delta}$), the second iterate f_ε^2 is somewhat cumbersome to handle. The algebra is greatly reduced by noticing that f_ε is an odd, or $\mathbf{Z}_2$-symmetric, function of $x: f_\varepsilon(-x)=-f_\varepsilon(x)$. The consequence is that x is a period-2 orbit if and only if $f_\varepsilon(x)=-x$. Thus $f_\varepsilon^2(x)-x=0$ is equivalent to $-f_\varepsilon(x)-x=0$. We have chosen to use $-f_\varepsilon(x)-x$ instead of $f_\varepsilon(x)+x$ because when they do not equal zero, $\operatorname{sgn}(f_\varepsilon^2(x)-x)=\operatorname{sgn}(-f_\varepsilon(x)-x)$. Thus the function $-x-f_\varepsilon(x)$ is also a bifurcation function for (2.2).

Furthermore, $-x-f_\varepsilon(x)=xP_\varepsilon(x^2)$, where

$$P_{\varepsilon;k,\delta}(u):=P_\varepsilon(u):=\varepsilon_1+\varepsilon_2u+\cdots+\varepsilon_ku^{k-1}-\delta u^k. \tag{2.3}$$

Since $x=0$ is always a fixed point, the roots of $P_\varepsilon(x^2)$ with $x\neq0$ are precisely the period-2 points. That is, each positive root r^2 of $P_\varepsilon(u)$ corresponds to the period-2 orbit $r\leftrightarrow -r$. For $x\neq 0$ the sign of $-f_\varepsilon(x)-x$ and therefore the sign of $f_\varepsilon^2(x)-x$ is determined by the sign of $P_\varepsilon(x^2)$. So the stability of the fixed point and any period-2 orbit is also determined by the sign of P_ε. Thus, $P_\varepsilon(u)$ becomes our third and simplest bifurcation function.

It may help to keep in mind Fig. 1a, where we graph the three bifurcation functions $f_\varepsilon^2(x)-x$, $-f_\varepsilon(x)-x$, and $P_\varepsilon(x^2)$ for a specific example: $(\boldsymbol{\varepsilon}; k, \delta)=((.000016, .0024, .09), 3, +1)$. Figure 1b shows the phase portrait for f_ε^2 that Fig. 1a determines.

2.2. Bifurcations. We are now ready to analyze the bifurcation sets in phase $\times$ parameter space ($\mathbf{R}\times\mathbf{R}^k$) that exist in the model families $f_{\varepsilon;k,\delta}$ for fixed values of k and δ. These consist of the nonhyperbolic fixed and period-2 points, possibly with higher-order degeneracies.

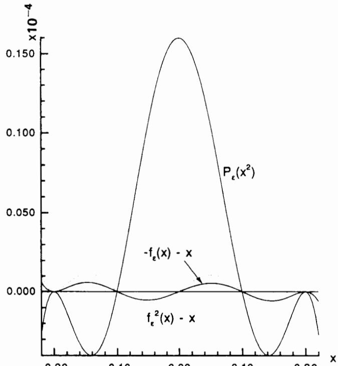

FIG. 1a. *Three bifurcation functions.*

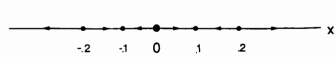

FIG. 1b. *The "flow" of* f_ε^2.

We will treat the fixed-point bifurcations first. Since $f_{\boldsymbol{\varepsilon}}$ is an orientation reversing diffeomorphism of $\mathbf{R}$, the only potential bifurcations for the unique fixed point zero are period doublings. From (2.2), the set in $\mathbf{R}\times\mathbf{R}^k$ of fixed points, which we call D^0, is $\{x=0\}$; the set of period doubling bifurcations is $D^1 := \{x=\varepsilon_1=0\}$; more generally, the set of period doubling bifurcations with at least $i-1$ higher-order degeneracies is apparently (look ahead to Definition 3.1—the model families in (2.2) are already in normal form on the center manifold) the codimension $i+1$ (dimension $k-i$) hyperplane given by

$$(2.4)\qquad D^i := D^i_k = \{(x, \boldsymbol{\varepsilon}) \in \mathbf{R}\times\mathbf{R}^k \colon x=\varepsilon_1=\cdots=\varepsilon_i=0\}, \qquad i=0,\cdots,k.$$

The superscripts have been chosen to indicate the codimension of the corresponding set when projected to the k-dimensional parameter space. The set of simple (nondegenerate) period doubling bifurcation parameters is thus, as usual, a codimension-1 set in the parameter space.

The nonhyperbolic period-2 points are treated by considering $f^2_{\boldsymbol{\varepsilon}}$. Since $f^2_{\boldsymbol{\varepsilon}}$ is an orientation preserving diffeomorphism of $\mathbf{R}$, the only potential bifurcations for the period-2 orbits are saddlenodes, possibly with higher-order degeneracies. By definition [Ar], [GH], a map $g:\mathbf{R}\to\mathbf{R}$ has a saddlenode with $i-1$ higher-order degeneracies at y_0 if $g(y)-y$ has a zero of multiplicity $i+1$ at $y=y_0$. So the period-2 points in our models have saddlenode bifurcations with $i-1$ higher-order degeneracies at x_0 if, for a fixed value of $\boldsymbol{\varepsilon}$, $f^2_{\boldsymbol{\varepsilon}}(x)-x$ has a zero of multiplicity $i+1$ at $x=x_0\neq 0$. But $f^2_{\boldsymbol{\varepsilon}}(x)-x$ having a zero of multiplicity $i+1$ at $x=x_0\neq 0$ is equivalent to $P_{\boldsymbol{\varepsilon}}(x^2)$ having a zero of multiplicity $i+1$ at $x=x_0\neq 0$. If we define $S^0_{k,\delta}$ as the set of period-2 points and $S^j_{k,\delta}$ as the set of period-2 saddlenode points with at least $j-1$ higher-order degeneracies for $1\leqq j\leqq k-1$, then these sets are

$$(2.5)\qquad S^j := S^j_{k,\delta} = \{(x, \boldsymbol{\varepsilon}) \in \mathbf{R}\times\mathbf{R}^k \colon P^{(i)}_{\boldsymbol{\varepsilon}}(x^2)=0 \text{ for } 0\leqq i\leqq j,\ x\neq 0\}.$$

2.3. The low codimension period doublings. We can now use (2.4), (2.5), and the sign of $P_{\boldsymbol{\varepsilon}}$ to determine the bifurcation diagrams and phase portraits for the codimension-k bifurcations with $k=1,2,3$.

$k=1$. When $k=1$ then $\boldsymbol{\varepsilon}=\varepsilon_1$ and (2.2) becomes the *simple* (nondegenerate) *period doubling bifurcation*: $f_{\varepsilon;1,\delta}(x)=-(\varepsilon_1+1)x+\delta x^3$. $P_{\boldsymbol{\varepsilon}}(u)=\varepsilon_1-\delta u$ and $P'_{\boldsymbol{\varepsilon}}(u)=-\delta\neq 0$. From (2.5) we see that period-2 points exist whenever $\delta\varepsilon>0$ and are located on the parabola $x=\pm\sqrt{\delta\varepsilon}$. The period-2 orbits are stable for $\delta=+1$ and unstable for $\delta=-1$. Since $P'_{\boldsymbol{\varepsilon}}(u)\neq 0$ all period-2 points are hyperbolic. A bifurcation diagram with three representative phase portraits for $\delta=+1$ is shown in Fig. 2. This is the *supercritical* case. The arrows on these phase portraits indicate the direction of travel of second iterates of $f_{\boldsymbol{\varepsilon}}$. The same figure can be used for $\delta=-1$, the *subcritical* case, by reversing the direction of the ε-axis and the direction of the arrows on the phase portraits. Changing the arrow directions means that the stability of the fixed point and any period-2 orbits for $\delta=-1$ will be the reverse of the stability for $\delta=+1$.

$k=2$. In this case, which really motivated the whole paper, (2.2) represents the *singly degenerate period doubling bifurcation* $f_{\varepsilon;2,\delta}(x)=-(\varepsilon_1+1)x-\varepsilon_2x^3+\delta x^5$. Since the coefficient ε_2 of the x^3 term, which determines the criticality of the simple period doubling bifurcation, is allowed to change from positive to negative, we will have both supercritical and subcritical period doublings. All the fixed-point bifurcations have already been identified in (2.4). For the period-2 bifurcations, we use the bifurcation function $P_{\varepsilon;2,\delta}(u)=P_{\boldsymbol{\varepsilon}}(u)=\varepsilon_1+\varepsilon_2u-\delta u^2$, so $P'_{\boldsymbol{\varepsilon}}(u)=\varepsilon_2-2\delta u$ and $P''_{\boldsymbol{\varepsilon}}(u)=-2\delta\neq 0$.

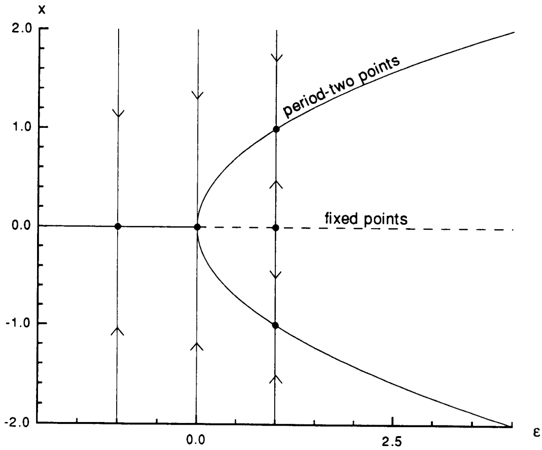

FIG. 2. *Supercritical simple period doubling.*

By (2.5), the period-2 points in $\mathbf{R}\times\mathbf{R}^k$ are $S^0=\{\varepsilon_1=-\varepsilon_2x^2+\delta x^4,\ \varepsilon_2\neq 2\delta x^2\}$ and they project to $\pi_\varepsilon(S^0)=\{\delta\varepsilon_1>0\}\cup\{\delta\varepsilon_2>0$ and $\varepsilon_2^2\geqq -4\delta\varepsilon_1\}$ in the $\boldsymbol{\varepsilon}$-parameter plane $\mathbf{R}^2$. The nonhyperbolic period-2 points are all (nondegenerate) saddlenode bifurcations. They are given by $S^1=\{\varepsilon_1=-\delta x^4,\ \varepsilon_2=2\delta x^2,\ x\neq 0\}$ and project to $\pi_\varepsilon(S^1)=\{\varepsilon_1=(-\delta/4)\varepsilon_2^2,\ \delta\varepsilon_2>0\}$. The formulas for the projections to the $\boldsymbol{\varepsilon}$ parameter plane are obtained by eliminating x from the expressions for S^0 and S^1.

Figure 3 shows sketches of the above sets for $\delta=+1$ in phase $\times$ parameter space. The projections to parameter space are drawn on the fixed-point plane $\{x=0\}$. The surface $S^0_{2,+1}$ of period-2 points, the plane D^0_2 of fixed points, the period doubling line D^1_2, the saddlenode curve $S^1_{2,+1}$, and its projection $\pi_\varepsilon(S^1_{2,+1})$ to the $\boldsymbol{\varepsilon}$ parameter plane, drawn in the $\{x=0\}$ plane, are all indicated in the figure. Note that all the bifurcation points occur on the "folds" of the period-2 surface $S^0_{2,+1}$.

Various two-dimensional bifurcation diagrams (pieces of Fig. 3) are shown in Fig. 4: 4a gives the projection of the bifurcation sets S^1 (saddlenodes) and D^1 (period doublings) to the parameter space; the other three are representative one-parameter cuts of Fig. 3: 4b and 4c each have a fixed value for ε_2, while a small circular path

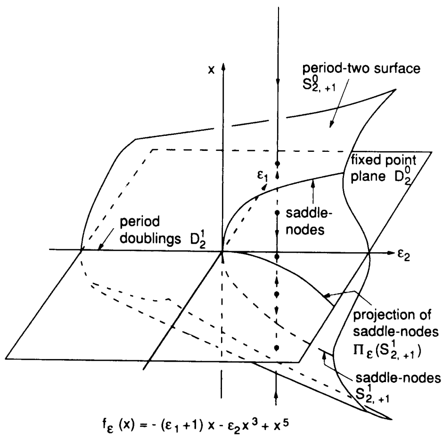

FIG. 3. *Singly degenerate period doubling.*

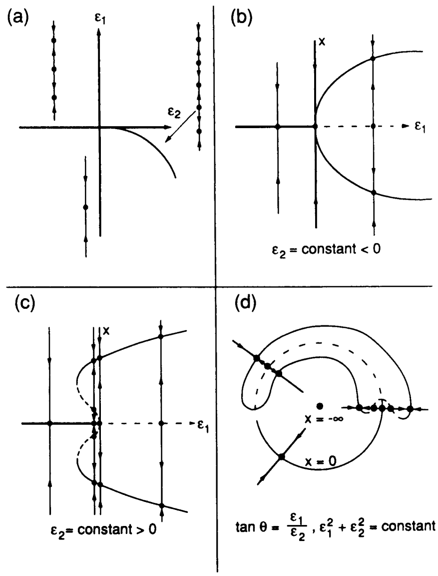

FIG. 4. *Aspects of singly degenerate period doubling.*

around the origin in the ε-plane yields 4d. Arrows all indicate the "flow" of the second iterate of $f_{\varepsilon;2,+1}$. (Compare Figs. 4b and 4c with Fig. 3.1 in [GS, p. 260]; compare Fig. 4d with Fig. 136 in [Ar, p. 283].)

As in the simple period doubling case, Fig. 3 and all Fig. 4 diagrams could be converted from the $\delta=+1$ case to the $\delta=-1$ case by reversing the directions of the ε_1 axis, the ε_2 axis, and the "flow" lines. The stability of the fixed point and all period-2 orbits is opposite for the two cases.

$k \geqq 3$. The program for computing the bifurcation submanifolds can obviously be continued for the model period doublings of any codimension. Because the computations are more lengthy but not much more enlightening, we merely list the results, with special attention to the $(k, \delta)=(3,+1)$ case.

The fixed-point bifurcation sets satisfy $D_k^0 \supset D_k^1 \supset \cdots \supset D_k^{k-1} \supset D_k^k$ where $D_k^{j-1} \backslash D_k^j$ is the codimension-j manifold in $\mathbf{R} \times \mathbf{R}^k$ of period doubling points with exactly $j-2$ higher-order degeneracies. Similarly, the period-2 bifurcation sets satisfy $S_{k,\delta}^0 \supset S_{k,\delta}^1 \supset \cdots \supset S_{k,\delta}^{k-2} \supset S_{k,\delta}^{k-1}$ where $S_{k,\delta}^{j-1} \backslash S_{k,\delta}^j$ is the codimension-j manifold in $\mathbf{R} \times \mathbf{R}^k$ of period-2 saddlenodes with exactly $j-2$ higher-order degeneracies. The set $S_{k,\delta}^{k-1}$ and its projection to parameter space have the explicit parametric representations

$$S_{k,\delta}^{k-1}=\left\{(x,[\varepsilon_1,\cdots,\varepsilon_k])=\left(x,-\delta\left[(-1)^k\binom{k}{k}x^{2k},\cdots,\right.\right.\right.$$
$$\left.\left.\left.-\binom{k}{3}x^6,\binom{k}{2}x^4,-\binom{k}{1}x^2\right]\right), x\neq 0\right\}, \tag{2.6}$$

$$\pi_\varepsilon(S_{k,\delta}^{k-1})=\left\{(\varepsilon_1,\cdots,\varepsilon_k)=\left((-1)^{k+1}\frac{\delta\binom{k}{k}}{k^k}\varepsilon_k^{2k-2},\cdots,\right.\right.$$
$$\left.\left.\frac{\delta\binom{k}{3}}{k^3}\varepsilon_k^4,\frac{-\delta\binom{k}{2}}{k^2}\varepsilon_k^2,\varepsilon_k\right),\delta\varepsilon_k>0\right\}. \tag{2.7}$$

When $k=3$ we obtain

$$S^0_{3,\delta}=\{(x,\varepsilon_1,\varepsilon_2,\varepsilon_3)=(x,-\varepsilon_2x^2-\varepsilon_3x^4+\delta x^6,\varepsilon_2,\varepsilon_3),\ x\neq 0\},$$
$$S^1_{3,\delta}=\{(x,\varepsilon_1,\varepsilon_2,\varepsilon_3)=(x,\varepsilon_3x^4-2\delta x^6,-2\varepsilon_3x^2+3\delta x^4,\varepsilon_3),\ x\neq 0\},$$
$$S^2_{3,\delta}=\{(x,\varepsilon_1,\varepsilon_2,\varepsilon_3)=(x,\delta x^6,-3\delta x^4,3\delta x^2),\ x\neq 0\}$$

(cf. (2.6)). $\pi_\varepsilon(S^0_{3,+1})$, the set of all parameter values with period-2 orbits, is described below.

$$\pi_\varepsilon(S^1_{3,+1})=\Big\{(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(\varepsilon_3[W_-(\varepsilon_2,\varepsilon_3)]^2+2[W_-(\varepsilon_2,\varepsilon_3)]^3,\varepsilon_2,\varepsilon_3),\quad \varepsilon_3<0,\ 0<\varepsilon_2\leqq\frac{\varepsilon_3^2}{3}\Big\}$$
$$\cup\Big\{(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(\varepsilon_3[W_+(\varepsilon_2,\varepsilon_3)]^2+2[W_+(\varepsilon_2,\varepsilon_3)]^3,\varepsilon_2,\varepsilon_3),\quad \varepsilon_3\geqq 0,\ \varepsilon_2<0 \text{ or } \varepsilon_3<0,\ \varepsilon_2\leqq\frac{\varepsilon_3^2}{3}\Big\},$$

where

$$W_\pm(\varepsilon_2,\varepsilon_3):=\frac{-\varepsilon_3\pm\sqrt{\varepsilon_3^3-3\varepsilon_2}}{3};$$
$$\pi_\varepsilon(S^2_{3,+1})=\Big\{(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\Big(\frac{\varepsilon_3^4}{27},\frac{-\varepsilon_3^2}{3},\varepsilon_3\Big),\ \varepsilon_3>0\Big\}$$

(cf. (2.7)).

Because the full phase × parameter space is now four-dimensional, the best pictures we can draw are either "slices" or projections of the four-dimensional space. Figure 5 shows the slice corresponding to $\varepsilon_3\equiv$ constant <0. Note the appearance of the cusp point on the curve of saddlenodes, so named for its location on the projection of the saddlenode curve to the parameter plane. Such a point appears only for $k\geqq 3$. The slice corresponding to $\varepsilon_3\equiv$ constant >0 we do not show, because it is qualitatively the same as Fig. 3.

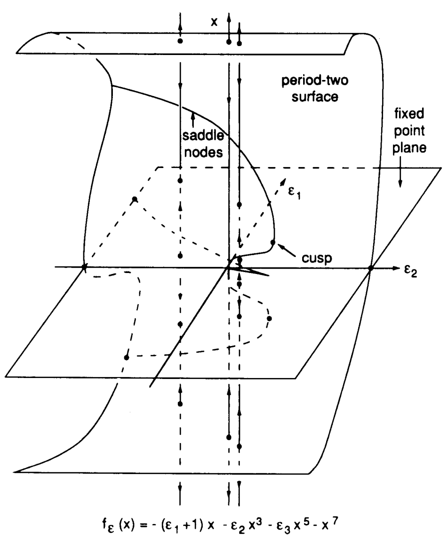

FIG. 5. *Doubly degenerate period doubling.*

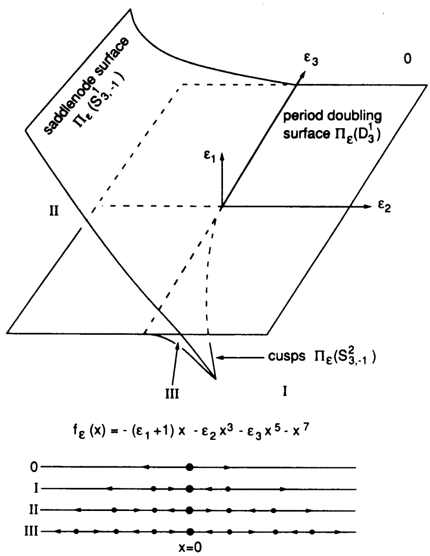

FIG. 6. *Doubly degenerate period doubling: parameter space.*

Figure 6 shows the saddlenode surfaces $\pi_\varepsilon(S^1_{3,+1})$, the cusp curve $\pi_\varepsilon(S^2_{3,+1})$ (where the two saddlenode surfaces, one defined with W_+ and the other with W_-, meet), and the period doubling plane $\pi_\varepsilon(D^1_3)$ in the three-dimensional parameter space. The set $\pi_\varepsilon(S^0_{3,+1})$ is bounded "above" in Fig. 6 by the higher of the saddlenode surfaces (inclusive) and the period doubling plane (not inclusive). Compare our Fig. 6 with Fig. 6 in [Ta]. Note that the plane $\{x=0, \varepsilon_3=\text{const}<0\}$ appears in both Fig. 5, as the fixed-point plane, and in Fig. 6, as the leading edge of the graph.

3. General period doubling families. In § 2 we analyzed the local topological behavior of the special families of diffeomorphisms of $\mathbf{R}: f_{\varepsilon;k,\delta}(x) = -(\varepsilon_1+1)x - \varepsilon_2 x^3 - \cdots - \varepsilon_k x^{2k-1} + \delta x^{2k+1}$. We now treat the more general case of a local family of diffeomorphisms of $\mathbf{R}^n$.

DEFINITION 3.1. Fix $k \geqq 1$. Let $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu}) = \mathbf{G}_{\boldsymbol{\mu}}(\mathbf{x})$ be a representative of the germ of a C^{2k+1} function satisfying

(1) $\mathbf{G}: U \to \mathbf{R}^n$, U is a neighborhood of $(\mathbf{x}_0, \boldsymbol{\mu}_0)$ in $\mathbf{R}^n \times \mathbf{R}^m$.

(2) $\mathbf{G}(\mathbf{x}_0, \boldsymbol{\mu}_0) = \mathbf{x}_0$.

(3) $D_{\mathbf{x}}\mathbf{G}(\mathbf{x}_0, \boldsymbol{\mu}_0)$ has a single eigenvalue of -1 and no other eigenvalues on the unit circle.

(4) On its one-dimensional center manifold, the map $\mathbf{G}_{\boldsymbol{\mu}_0}$ can be transformed by a C^{2k+1} change of coordinates to a C^{2k+1} map of the form $y \to -y + cy^{2k+1} + o(y^{2k+1})$, $c \neq 0$.

Then $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ is a *local period doubling bifurcation family with* $k-1$ *higher-order degeneracies*, and $\mathbf{G}_{\boldsymbol{\mu}_0}$ is *a local period doubling bifurcation map with* $k-1$ *higher-order degeneracies.*

The main goal of this section is to establish Theorem 3.15, where we show that on its center manifold, every k-parameter period doubling bifurcation family with $k-1$ higher-order degeneracies is, at least generically, the "same" as one of the model families $f_{\varepsilon;k,\delta}$, where $\delta = \operatorname{sign}(c)$. The main technical tools for Theorem 3.15 are the existence of a "$\mathbf{Z}_2$-symmetric bifurcation function" related to the original period doubling family (Theorem 3.3) and the universal unfolding theorem from $\mathbf{Z}_2$-singularity theory (Lemma 3.21). We are then able to compare $\mathbf{G}_{\boldsymbol{\mu}}$ to the appropriate model family via their respective bifurcation functions.

The Lyapunov–Schmidt reduction. Let $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ be a period doubling family with any number of higher-order degeneracies. For simplicity, we will assume $(\mathbf{x}_0, \boldsymbol{\mu}_0) = (\mathbf{0}, \mathbf{0})$. As with our special functions $f_{\varepsilon;k,\delta}$ in (2.2), the implicit function theorem guarantees that $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ has a unique fixed point near $\mathbf{x} = \mathbf{0}$ for each $\boldsymbol{\mu}$ near zero. Having only one phase variable (along with the m parameters) on the center manifold implies that the only other local recurrence can be in the form of period-2 points [CMY].

The period-2 points (including the fixed point) of $\mathbf{G}$ are characterized by the roots of the function $\boldsymbol{\Phi}: \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n \times \mathbf{R}^n$ defined by

$$\boldsymbol{\Phi}(\mathbf{x}, \mathbf{y}, \boldsymbol{\mu}) := \boldsymbol{\Phi}_{\boldsymbol{\mu}}(\mathbf{x}, \mathbf{y}) := (\mathbf{y} - \mathbf{G}(\mathbf{x}, \boldsymbol{\mu}), \mathbf{x} - \mathbf{G}(\mathbf{y}, \boldsymbol{\mu})). \tag{3.2}$$

The reason this function turns out to be more useful than $\mathbf{G}_{\boldsymbol{\mu}}^2(\mathbf{x}) - \mathbf{x}$ is twofold: $\boldsymbol{\Phi}$ deals only with first iterates of $\mathbf{G}_{\boldsymbol{\mu}}$, and it has an obvious symmetry that will be quite useful. Specifically, $\boldsymbol{\Phi}_{\boldsymbol{\mu}}\mathfrak{R} = \mathfrak{R}\boldsymbol{\Phi}_{\boldsymbol{\mu}}$, where $\mathfrak{R}$ is the reflection that interchanges the variables $\mathbf{x}$ and $\mathbf{y}$ in both the domain and range of $\boldsymbol{\Phi}$. That is, $\boldsymbol{\Phi}_{\boldsymbol{\mu}}\mathfrak{R}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\Phi}_{\boldsymbol{\mu}}(\mathbf{y}, \mathbf{x}) = (\mathbf{x} - \mathbf{G}(\mathbf{y}, \boldsymbol{\mu}), \mathbf{y} - \mathbf{G}(\mathbf{x}, \boldsymbol{\mu})) = \mathfrak{R}(\mathbf{y} - \mathbf{G}(\mathbf{x}, \boldsymbol{\mu}), \mathbf{x} - \mathbf{G}(\mathbf{y}, \boldsymbol{\mu})) = \mathfrak{R}\boldsymbol{\Phi}_{\boldsymbol{\mu}}(\mathbf{x}, \mathbf{y})$.

We now perform the Lyapunov–Schmidt reduction [GS, § I.3] on $\boldsymbol{\Phi}$ to get the following theorem. Although the theorem is stated for C^ρ functions, we will be interested mainly in the case $\rho = \infty$.

THEOREM 3.3. *Let $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ be a C^ρ, $2k+1 \leqq \rho \leqq \infty$, local period doubling bifurcation family with $k-1$ higher-order degeneracies as in Definition* 3.1, *with* $(\mathbf{x}_0, \boldsymbol{\mu}_0) = (\mathbf{0}, \mathbf{0})$. *Define $\boldsymbol{\Phi}(\mathbf{x}, \mathbf{y}, \boldsymbol{\mu})$ by* (3.2). *Then there exists a C^ρ* bifurcation function $b: \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}$ *of the form $b(s, \boldsymbol{\mu}) = sB(u, \boldsymbol{\mu})$, $u := s^2$, such that solutions of $\boldsymbol{\Phi}(\mathbf{x}, \mathbf{y}, \boldsymbol{\mu}) = \mathbf{0}$ for $(\mathbf{x}, \mathbf{y}, \boldsymbol{\mu})$ near $(\mathbf{0}, \mathbf{0}, \mathbf{0})$ are in one to one correspondence with solutions of $b(s, \boldsymbol{\mu}) = 0$ for $(s, \boldsymbol{\mu})$ near* $(0, \mathbf{0})$.

Proof. The Lyapunov–Schmidt reduction to prove Theorem 3.3 is standard [GS, § 1.3], but we include most of the computations since we will be interested in the specific bifurcation function we get via the reduction, as well as some of the intermediate functions defined in the proof.

Case 1: $x \in \mathbf{R}$. In standard coordinates, the linearization of $\boldsymbol{\Phi}_\mathbf{0}$ at $(0, 0)$ is $L := D_{x,y}\boldsymbol{\Phi}_0(0, 0) = \left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$. Thus the kernel of L, $\ker L = \langle(1, -1)\rangle$, and range $L = \langle(1, 1)\rangle$. Note that $\mathbf{R}^2 = \ker L \oplus \operatorname{range} L$ so that $E(x, y) := ((x+y)/\sqrt{2}, (x+y)/\sqrt{2})$ is the projection onto range L, and $(I - E)(x, y) := ((x-y)/\sqrt{2}, -(x-y)/\sqrt{2})$ is the projection onto $\ker L$. The equation $\boldsymbol{\Phi}(\mathbf{x}, \mathbf{y}, \boldsymbol{\mu}) = \mathbf{0}$, which we wish to solve, is equivalent to the two equations (with the $\sqrt{2}$ factor introduced for convenience):

$$\sqrt{2}\, E\boldsymbol{\Phi}(x, y, \boldsymbol{\mu}) = (0, 0), \tag{3.4a}$$

$$\sqrt{2}(I - E)\boldsymbol{\Phi}(x, y, \boldsymbol{\mu}) = (0, 0). \tag{3.4b}$$

These two equations are more conveniently expressed in coordinates with respect to the splitting $\mathbf{R}^2 = \ker L \oplus \operatorname{range} L$. Formally, this can be defined by the change of coordinates from (x, y) with respect to the standard basis on $\mathbf{R}^2$ to (s, r) with respect to the new basis which we choose as $\{(1, -1), (1, 1)\}$. The coordinates are related by $x(1, 0) + y(0, 1) = s(1, -1) + r(1, 1)$, or $x = s + r$ and $y = r - s$. Since the s component of the new coordinate version of (3.4a) is automatically satisfied by definition of E, as is the r component of the new coordinate version of (3.4b), the two vector equations in (3.4) are equivalent to the two scalar equations

$$Q(s, r, \boldsymbol{\mu}) := \tfrac{1}{2}\{2r - G(s + r, \boldsymbol{\mu}) - G(-s + r, \boldsymbol{\mu})\} = 0, \tag{3.5a}$$

$$\tfrac{1}{2}\{-2s - G(s + r, \boldsymbol{\mu}) + G(-s + r, \boldsymbol{\mu})\} = 0. \tag{3.5b}$$

Equation (3.5a) is the r component of (3.4a); equation (3.5b) is the s component of (3.4b).

Since $Q(0,0,\mathbf{0})=0$ and $(\partial Q/\partial r)(0,0,\mathbf{0})=2\neq 0$, then the implicit function theorem implies that there exists a unique C^p function $R(s,\boldsymbol{\mu})$ satisfying $R(0,\mathbf{0})=0$ and $Q(s,R(s,\boldsymbol{\mu}),\boldsymbol{\mu})=0$ for $(s,\boldsymbol{\mu})$ near $(0,\mathbf{0})$. Plugging this new function $R(s,\boldsymbol{\mu})$ into the left-hand side of (3.5b), we get our reduced bifurcation function $b(s,\boldsymbol{\mu})$:

$$(3.6)\qquad \begin{aligned} b(s,\boldsymbol{\mu}) &:= \tfrac{1}{2}\{-2s - G(s+R(s,\boldsymbol{\mu}),\boldsymbol{\mu}) + G(-s+R(s,\boldsymbol{\mu}),\boldsymbol{\mu})\} \\ &= -G(s+R(s,\boldsymbol{\mu}),\boldsymbol{\mu}) + (-s+R(s,\boldsymbol{\mu})). \end{aligned}$$

The latter form is obtained by substituting $Q(s,R(s,\boldsymbol{\mu}),\boldsymbol{\mu})=0$ from (3.5a) into the first line of (3.6).

It can be verified directly that $R(-s,\boldsymbol{\mu})=R(s,\boldsymbol{\mu})$, and therefore that $b(-s,\boldsymbol{\mu})=-b(s,\boldsymbol{\mu})$, but this is really a consequence of the equivariance of the original function $\boldsymbol{\Phi}$ with respect to the reflection $\mathfrak{R}$. This is because $b(s,\boldsymbol{\mu})$ is really the coordinate representation of a map from $\ker \boldsymbol{\Phi}\times \mathbf{R}^k$ to $\ker\boldsymbol{\Phi}$, and $\mathfrak{R}$ acts on $\ker\boldsymbol{\Phi}$ by $\mathfrak{R}(s(1,-1))=\mathfrak{R}(s,-s)=(-s,s)=-s(1,-1)$.

That $b(s,\boldsymbol{\mu})$ has the form $sB(s^2,\boldsymbol{\mu})$ is immediate from the odd symmetry of $b(s,\boldsymbol{\mu})$. The one-to-one correspondence between solutions of $b(s,\boldsymbol{\mu})=0$ and $\boldsymbol{\Phi}(x,y,\boldsymbol{\mu})=0$ is

$$(3.7)\qquad (s,\boldsymbol{\mu})\leftrightarrow(s+R(s,\boldsymbol{\mu}),\,-s+R(s,\boldsymbol{\mu}),\,\boldsymbol{\mu}).$$

Note that if $s\neq 0$, solutions s and $-s$ correspond to the same period-2 orbit, but these are distinct solutions for $\boldsymbol{\Phi}:\boldsymbol{\Phi}(x,y,\boldsymbol{\mu})=0$ and $\boldsymbol{\Phi}(y,x,\boldsymbol{\mu})=0$. This completes the proof for $x\in\mathbf{R}$.

Case 2: $\mathbf{x}\in\mathbf{R}^n$, $n>1$, and the coordinates $\mathbf{x}=(x_1,\cdots,x_n)$ have been chosen with respect to the basis $\{\mathbf{e}_i\}_{i=1}^n$ so that matrix of $D_{\mathbf{x}}\mathbf{G}(\mathbf{0},\mathbf{0})$ has the block form $\hat{B}=\left(\begin{smallmatrix}-1 & 0\\ 0 & B\end{smallmatrix}\right)$, where B is an $(n-1)\times(n-1)$ matrix. The $2n\times 2n$ matrix of the linearization of $L=D_{\mathbf{x,y}}\boldsymbol{\Phi}_0(\mathbf{0},\mathbf{0})$ with respect to the induced basis $\{\mathbf{f}_1,\cdots,\mathbf{f}_{2n}\}=\{(\mathbf{e}_1,\mathbf{0}),\cdots,(\mathbf{e}_n,\mathbf{0}),(\mathbf{0},\mathbf{e}_1),\cdots,(\mathbf{0},\mathbf{e}_n)\}$ becomes $L=\left(\begin{smallmatrix}-\hat{B} & I\\ I & -\hat{B}\end{smallmatrix}\right)$. The first and $(n+1)$st rows of L are identical, but using the fact that no other eigenvalues are on the unit circle, it can be shown that the remaining rows are independent. (This would be easier to see if $\{\mathbf{e}_i\}$ were a basis putting $\hat{B}$ into Jordan canonical form.) So we still have the dimension of $\ker L=1$. In fact, $\ker L=\langle\mathbf{f}_1-\mathbf{f}_{n+1}\rangle$ and range $L=\langle\mathbf{f}_1+\mathbf{f}_{n+1},\mathbf{f}_2,\cdots,\mathbf{f}_n,\mathbf{f}_{n+2},\cdots,\mathbf{f}_{2n}\rangle$. We also still have $\mathbf{R}^{2n}=\ker L\oplus\text{range } L$. The coordinates with respect to this splitting are s on $\ker L$, and $(r,x_2,\cdots,x_n,y_2,\cdots,y_n)$ on range L, where $x_1=s+r$ and $y_1=r-s$, and $\mathbf{x}=(x_1,\cdots,x_n)$ and $\mathbf{y}=(y_1,\cdots,y_n)$ are in coordinates with respect to $\{e_i\}_{i=1}^n$. Solving $\sqrt{2}\mathbf{E}\boldsymbol{\Phi}(\mathbf{x},\mathbf{y},\boldsymbol{\mu})=0$ in the new coordinates is equivalent to $\mathbf{Q}=\mathbf{0}$, where

$$(3.8)\qquad \begin{aligned} &\mathbf{Q}(s,r,x_2,\cdots,x_n,y_2,\cdots,y_n,\boldsymbol{\mu}) \\ &\quad:=(\tfrac{1}{2}(2r-G_1(s+r,x_2,\cdots,x_n,\boldsymbol{\mu})-G_1(-s+r,y_2,\cdots,y_n,\boldsymbol{\mu})), \\ &\qquad y_2-G_2(s+r,x_2,\cdots,x_n,\boldsymbol{\mu}),\cdots,y_n-G_n(s+r,x_2,\cdots,x_n,\boldsymbol{\mu}), \\ &\qquad x_2-G_2(-s+r,y_2,\cdots,y_n,\boldsymbol{\mu}),\cdots,x_n-G_n(-s+r,y_2,\cdots,y_n,\boldsymbol{\mu}))=\mathbf{0}. \end{aligned}$$

This equation can be solved uniquely by the implicit function theorem for C^p functions $r,x_2,\cdots,x_n,y_2,\cdots,y_n$, all in terms of s and $\boldsymbol{\mu}$ in a neighborhood of $(s,\boldsymbol{\mu})=(0,\mathbf{0})$. We shall call these solutions $R(s,\boldsymbol{\mu})$, $X_j(s,\boldsymbol{\mu})$, and $Y_j(s,\boldsymbol{\mu})$. That is,

$$(3.9)\qquad \mathbf{Q}(s,\mathbf{W}(s,\boldsymbol{\mu}),\boldsymbol{\mu})=\mathbf{0},$$

where $\mathbf{W}(s,\boldsymbol{\mu}):=(R(s,\boldsymbol{\mu}),X_2(s,\boldsymbol{\mu}),\cdots,\mathrm{X}_n(\mathrm{s},\boldsymbol{\mu}),Y_2(s,\boldsymbol{\mu}),\cdots,Y_n(s,\boldsymbol{\mu}))$. Differentiation of (3.9) with respect to s and using the block form of $D_{\mathbf{x}}\mathbf{G}(\mathbf{0},\mathbf{0})$ yields

$$(3.10)\qquad \frac{\partial\mathbf{W}}{\partial s}=\mathbf{0}.$$

As for any Lyapunov-Schmidt reduction involving a symmetry, the symmetry $\mathfrak{R}$ is inherited as $\mathbf{W}\mathfrak{R}(s, \boldsymbol{\mu}) = \mathfrak{R}\mathbf{W}(s, \boldsymbol{\mu})$, interpreted as

$$\begin{aligned}(R(-s, \boldsymbol{\mu}), X_2(-s, \boldsymbol{\mu}), \cdots, X_n(-s, \boldsymbol{\mu}), Y_2(-s, \boldsymbol{\mu}), \cdots, Y_n(-s, \boldsymbol{\mu}))\\ = (R(s, \boldsymbol{\mu}), Y_2(s, \boldsymbol{\mu}), \cdots, Y_n(s, \boldsymbol{\mu}), X_2(s, \boldsymbol{\mu}), \cdots, X_n(s, \boldsymbol{\mu})).\end{aligned}$$

Thus $Y_j(s, \boldsymbol{\mu}) = X_j(-s, \boldsymbol{\mu}), j = 2, \cdots, n$, and $R(s, \boldsymbol{\mu}) = R(-s, \boldsymbol{\mu})$. The bifurcation function, analogous to (3.6), is

$$\begin{aligned}b(s, \boldsymbol{\mu}) = \tfrac{1}{2}\{-2s - G_1(s + R(s, \boldsymbol{\mu}), X_2(s, \boldsymbol{\mu}), \cdots, X_n(s, \boldsymbol{\mu}), \boldsymbol{\mu})\\ + G_1(-s + R(s, \boldsymbol{\mu}), X_2(-s, \boldsymbol{\mu}), \cdots, X_n(-s, \boldsymbol{\mu}), \boldsymbol{\mu})\},\end{aligned} \tag{3.11}$$

where $\mathbf{G} = (G_1, \cdots, G_n)$. It is clear from the first line, since $R(s, \boldsymbol{\mu}) = R(-s, \boldsymbol{\mu})$, that we still have our $\mathbf{Z}_2$-symmetric bifurcation function: $b(-s, \boldsymbol{\mu}) = -b(s, \boldsymbol{\mu})$. So $b(s, \boldsymbol{\mu})$ is still of the form $sB(s^2, \boldsymbol{\mu})$. The one-to-one correspondence between roots of $b(s, \boldsymbol{\mu})$ and $\boldsymbol{\Phi}(\mathbf{x}, \mathbf{y}, \boldsymbol{\mu})$, analogous to (3.7), is given by $(s, \boldsymbol{\mu}) \leftrightarrow (\mathbf{X}(s, \boldsymbol{\mu}), \mathbf{Y}(s, \boldsymbol{\mu}), \boldsymbol{\mu})$, where

$$\begin{aligned}\mathbf{X}(s, \boldsymbol{\mu}) &:= (s + R(s, \boldsymbol{\mu}), X_2(s, \boldsymbol{\mu}), \cdots, X_n(s, \boldsymbol{\mu})),\\ \mathbf{Y}(s, \boldsymbol{\mu}) &:= \mathbf{X}(-s, \boldsymbol{\mu}) = (-s + R(s, \boldsymbol{\mu}), X_2(-s, \boldsymbol{\mu}), \cdots, X_n(-s, \boldsymbol{\mu})).\end{aligned} \tag{3.12}$$

Thus the theorem is true for $\mathbf{x} \in \mathbf{R}^n$, with the assumed coordinate system.

Case 3: $\mathbf{x} \in \mathbf{R}^n$, $n > 1$. Change this general case into the special coordinate form of Case 2 by a linear change of variable. Then follow the procedure outlined in that case. □

We now prove two corollaries that give some insight into the mechanics of the Lyapunov-Schmidt reduction of Theorem 3.3.

COROLLARY 3.13. *For our model families* $f_{\varepsilon;k,\delta}(x) = -(\varepsilon_1 + 1)x - \varepsilon_2 x^3 - \cdots - \varepsilon_k x^{2k-1} + \delta x^{2k+1}$, *the bifurcation function* $b_f(s, \boldsymbol{\varepsilon}) = sB_f(s^2, \boldsymbol{\varepsilon}) = -f_{\varepsilon}(s) - s$. *Also,* $B_f(u, \boldsymbol{\varepsilon}) = P_{\boldsymbol{\varepsilon};k,\delta}(u)$, *where* $P_{\boldsymbol{\varepsilon};k,\delta}(u)$ *is as defined in* (2.3).

Proof. Because our model families $f_{\boldsymbol{\varepsilon};k,\delta}$ are odd, it is apparent from (3.5a) by letting $G(x, \boldsymbol{\varepsilon}) = f_{\boldsymbol{\varepsilon};k,\delta}(x)$ for any fixed values of k and δ that $Q(s, 0, \boldsymbol{\varepsilon}) = 0$, so $R(s, \boldsymbol{\varepsilon}) = 0$ must be the unique solution to $Q(s, R(s, \boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}) = 0$. Thus, from (3.6), $b(s, \boldsymbol{\varepsilon}) = sB_f(s^2, \boldsymbol{\varepsilon})$ becomes $-s - f_{\boldsymbol{\varepsilon}}(s)$. But $= -s - f_{\boldsymbol{\varepsilon}}(s) = \varepsilon_1 s + \varepsilon_2 s^3 + \cdots + \varepsilon_k s^{2k-1} - \delta s^{2k+1} = sP_{\boldsymbol{\varepsilon};k,\delta}(s^2)$. So $B_f(u, \boldsymbol{\varepsilon}) = P_{\boldsymbol{\varepsilon};k,\delta}(u)$. □

So the seemingly ad hoc method we used in § 2 to analyze our model families turns out to be merely a special case of the more general Lyapunov-Schmidt reduction.

COROLLARY 3.14. *Let* $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ *be a local period doubling family with* $k-1$ *higher-order degeneracies at* $(\mathbf{x}, \boldsymbol{\mu}) = (\mathbf{0}, \mathbf{0}) \in \mathbf{R}^n \times \mathbf{R}^m$. *If* $\{(\mathbf{x}, \boldsymbol{\mu}): x_2 = \cdots = x_n = 0\}$ *is the center manifold* (*instead of just the center eigenspace*) *of* $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$, *then*

(A) *the functions* $X_j(s, \boldsymbol{\mu})$ *and* $Y_j(s, \boldsymbol{\mu})$, *defined in* (3.12), *are zero for* $j = 2, 3, \cdots, n$,

(B) *the bifurcation function of* $[\mathbf{G}]$ = *the bifurcation function of* [$\mathbf{G}$ *restricted to its* $1+m$ *dimensional center manifold*].

Proof. (A). We can show there exists a solution to (3.8) with $x_j = y_j = 0$, $j = 2, \cdots, n$. By uniqueness of solutions, the functions $X_j(s, \boldsymbol{\mu})$ and $Y_j(s, \boldsymbol{\mu})$ *must* be zero for $j \geqq 2$.

(B) This follows from (A) by directly computing the two bifurcation functions using (3.8) and (3.11). □

Corollaries 3.13 and 3.14 suggest that using the Lyapunov-Schmidt reduction to obtain the bifurcation function $b_{\mathbf{G}}(s, \boldsymbol{\mu})$ should be compared to the more topological alternative of obtaining a bifurcation function $-\tilde{f}_{\boldsymbol{\mu}}(s) - s$ from $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ by the following steps:

(1) Restrict $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ to its $1+m$ dimensional center mainfold: $f(x_1, \boldsymbol{\mu}) := f_{\boldsymbol{\mu}}(x_1) := G_1((x_1, \mathbf{H}(x_1, \boldsymbol{\mu})), \boldsymbol{\mu})$, where the center manifold is the graph of $\mathbf{H}: \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^{n-1}$.

(2) Put the resulting function into its normal form $\tilde{f}(s, \boldsymbol{\mu}) := \tilde{f}_{\boldsymbol{\mu}}(s) := h_{\boldsymbol{\mu}} \circ f_{\boldsymbol{\mu}} \circ h_{\boldsymbol{\mu}}^{-1}(s)$, where $h(x_1, \boldsymbol{\mu}) := h_{\boldsymbol{\mu}}(x_1)$ are the coordinate changes to put $f_{\boldsymbol{\mu}}$ into its normal form $\tilde{f}_{\boldsymbol{\mu}}$.

(3) Use the resulting odd symmetry to replace the bifurcation function $\tilde{f}^2_{\boldsymbol{\mu}}(s) - s$ with the simpler function $-\tilde{f}_{\boldsymbol{\mu}}(s) - s$.

Besides being a single step, the Lyapunov–Schmidt reduction has another major advantage over the center manifold/normal forms technique. Although the normal forms theorem guarantees a polynomial change of coordinates to put $f_{\boldsymbol{\mu}}(x_1)$ into its normal form up to any finite order, the existence of a coordinate change to eliminate *all* even-order terms in x_1 is not guaranteed. Thus step (2) above may not even be possible. On the other hand, if we put the function $f_{\boldsymbol{\mu}}(x_1)$ into its normal form only up to some finite order, step (3) would not be possible because the resulting function would be odd only up to that finite order. Note also that the original function $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ being C^∞ does not imply that its center manifold realization is C^∞. The Lyapunov–Schmidt bifurcation function $b_{\mathbf{G}}(s, \boldsymbol{\mu})$, however, is C^∞.

Universality of the model families. We now use Theorem 3.3 and some standard results from singularity theory to show that the model unfoldings we considered in Chapter 2 are "universal unfoldings." More specifically, we prove that, when restricted to a center manifold, any map in a local period doubling family is topologically equivalent to one of the model family maps. If certain nondegeneracy conditions are satisfied, the whole family of center manifold maps will be "equivalent" to one of the model families. Our notion of equivalence is embodied in the statement of the theorem.

We use the following notation. Let $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ be any C^∞ period doubling family with $k-1$ higher-order degeneracies at $(\mathbf{0}, \mathbf{0})$. Let $b_{\mathbf{G}}(s, \boldsymbol{\mu}) = sB_{\mathbf{G}}(s^2, \boldsymbol{\mu})$ be a bifurcation function obtained from $\mathbf{G}$ as in Theorem 3.3. Assume x_1 is a coordinate along the eigenspace corresponding to the -1 eigenvalue for the fixed point $\mathbf{x} = \mathbf{0}$ for $\boldsymbol{\mu} = \mathbf{0}$. Let $g_{\boldsymbol{\mu}}(x_1) := g(x_1, \boldsymbol{\mu})$ be the realization of $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ on its $1+m$ dimensional center manifold. By Definition 3.1, the center manifold map in normal form up to order $2k+1$ for $\boldsymbol{\mu} = \mathbf{0}$ is $y \to -y + cy^{2k+1} + o(y^{2k+1})$. Let $f_{\boldsymbol{\varepsilon}}(z) := f_{\boldsymbol{\varepsilon};k,\delta}(z)$ be the model family $-(\varepsilon_1+1)z - \varepsilon_2 z^3 - \cdots - \varepsilon_k z^{2k-1} + \delta z^{2k+1}$, where $\delta = -\operatorname{sgn}(c)$. Recall the definitions of the bifurcation sets D^i_k and $S^i_{k,\delta}$ in (2.4) and (2.5) for the model families $f_{\boldsymbol{\varepsilon};k,\delta}$. We now analogously define the bifurcation sets for $\mathbf{G}$.

$D^0_g = \{(x_1, \boldsymbol{\mu}) \in \mathbf{R} \times \mathbf{R}^m : x_1$ is a fixed point for $g_{\boldsymbol{\mu}}\}$,
$D^i_g = \{(x_1, \boldsymbol{\mu}) \in \mathbf{R} \times \mathbf{R}^m : x_1$ is a fixed point for $g_{\boldsymbol{\mu}}$ with eigenvalue -1 and at least $i-1$ higher-order degeneracies$\}$ for $i \geqq 1$,
$S^0_g = \{(x_1, \boldsymbol{\mu}) \in \mathbf{R} \times \mathbf{R}^m : x_1$ is a period-2 point for $g_{\boldsymbol{\mu}}\}$,
$S^i_g = \{(x_1, \boldsymbol{\mu}) \in \mathbf{R} \times \mathbf{R}^m : x_1$ is a period-2 point for $g_{\boldsymbol{\mu}}$ with eigenvalue 1 and at least $i-1$ higher-order degeneracies$\}$ for $i \geqq 1$.

THEOREM 3.15. *Let $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ be a C^∞ period doubling family with $k-1$ higher-order degeneracies. Define its center manifold representation $g_{\boldsymbol{\mu}}(x_1)$ and the model family $f_{\boldsymbol{\varepsilon}}(z)$ as in the above paragraph. Assume the "eigenvalue crossing condition:" $\nabla_{\boldsymbol{\mu}}\lambda(\boldsymbol{\mu}) \neq \mathbf{0}$, where $\lambda(\boldsymbol{\mu})$ is the eigenvalue of the unique fixed point of $g_{\boldsymbol{\mu}}$. Then*

(a) *There exists a neighborhood N of $(0, \mathbf{0})$ in $\mathbf{R} \times \mathbf{R}^m$ and a C^∞ function $\boldsymbol{\Psi}: N \to \{\mathbf{R} \times \mathbf{R}^k\}: (x_1, \boldsymbol{\mu}) \to (z, \boldsymbol{\varepsilon})$ of the form $\boldsymbol{\Psi}(x_1, \boldsymbol{\mu}) = (Z_{\boldsymbol{\mu}}(x_1), \boldsymbol{\psi}(\boldsymbol{\mu}))$ with the following properties*:

(1) $\boldsymbol{\Psi}: (0, \mathbf{0}) \to (0, \mathbf{0})$.

(2) *For each fixed parameter value $\boldsymbol{\mu}$, $g_{\boldsymbol{\mu}}(x_1)$ restricted to the neighborhood N and $f_{\boldsymbol{\psi}(\boldsymbol{\mu})}(z)$ restricted to $\boldsymbol{\Psi}(N)$ are topologically conjugate to each other.*

(3) $\boldsymbol{\Psi}$ *maps fixed points, period-2 points, and bifurcation manifolds of g to fixed points, period-2 points, and corresponding bifurcation manifolds of f, respectively.* (*That is,* $\boldsymbol{\Psi}: D_g^i \to D_k^i$ *for* $i = 0, \cdots, k$, *and* $S_g^i \to S_{k,\delta}^i$ *for* $i = 0, \cdots, k-1$.)

(b) *Let k and δ be fixed. Any family that can replace* $f_{\varepsilon:k,\delta}$ *in Theorem* 3.15(a) *must have at least k parameters.* (*This justifies calling the period doubling bifurcation with* $k-1$ *higher-order degeneracies a codimension-k bifurcation.*)

(c) *If* $\boldsymbol{\mu} \in \mathbf{R}^k$ *and*

$$\left\{ \nabla_{\boldsymbol{\mu}} \left(\frac{\partial^i B_G(u, \boldsymbol{\mu})}{\partial u^i} \right) \Bigg|_{(0,\mathbf{0})}, \qquad i = 0, \cdots, k-1 \right\}$$

is independent, then $\boldsymbol{\psi}$ *and* $\boldsymbol{\Psi}$ *are* C^∞ *diffeomorphisms.*

Before beginning the proof of this theorem, we make the following comments:

(1) Recall that in the proof of Theorem 3.3, $b_G(s, \boldsymbol{\mu})$ and therefore $B_G(s, \boldsymbol{\mu})$ were defined using the implicit function theorem. Although this means the bifurcation functions and their derivatives are not usually computable, their values at $(s, \boldsymbol{\mu}) = (0, \mathbf{0})$ are computable. (See, for example, Lemma 3.16.) Consequently, the nondegeneracy conditions in part (c) of the theorem *are* computable.

(2) The nondegeneracy conditions in part (c) will generically be true. Thus, for a generic k-parameter family of maps, $\boldsymbol{\Psi}$ will be a diffeomorphism. Since the C^∞ diffeomorphism $\boldsymbol{\Psi}$ preserves the bifurcation sets, and the bifurcation sets for the models are analytic, this is what guarantees that the bifurcation manifolds will all be C^∞ and that the pictures obtained from applications (see § 4) all "look like" the bifurcation pictures obtained from the model families in § 2. In particular, the orders of tangency of corresponding bifurcation manifolds will be the same as in the model families. In the codimension-2 case, with only one higher-order degeneracy, the projection to the parameter space of the bifurcation manifolds will always (generically) show a curve of saddlenodes for the second iterate of the map being tangent to a period doubling curve where it terminates. (Look ahead to Figs. 7–9 in comparison to the model family bifurcation diagrams in Figs. 3 and 4.)

(3) Note that the center eigenspace coordinate x_1 can be replaced by any phase space coordinate not perpendicular to x_1 by a one-dimensional linear change of coordinates independent of the parameter. Consequently, any generic phase variable coordinate can be used in place of a center eigenspace coordinate x_1 in drawing the bifurcation sets. This is exactly what was done to obtain Figs. 7–9.

(4) This is a technical comment comparing our notion of "equivalence" implied by the existence in the theorem of the function $\boldsymbol{\Psi}$ to the oft-used notion of "topological conjugacy." Recall that $g(x_1, \boldsymbol{\mu})$ and $f(z, \boldsymbol{\varepsilon})$ are (locally) topologically conjugate families if there exists a local homeomorphism $\boldsymbol{\Phi}(x_1, \boldsymbol{\mu}) = (h_{\boldsymbol{\mu}}(x_1), \boldsymbol{\phi}(\boldsymbol{\mu}))$ such that $g_{\boldsymbol{\mu}} = h_{\boldsymbol{\mu}}^{-1} \circ f_{\boldsymbol{\psi}(\boldsymbol{\mu})} \circ h_{\boldsymbol{\mu}}$. If the individual topological conjugacies $h_{\boldsymbol{\mu}}(x_1)$ do not necessarily vary continuously with respect to the parameter $\boldsymbol{\mu}$, then the families are said to be "mildly topologically conjugate" [NPT]. Because Theorem 3.15 guarantees that $g_{\boldsymbol{\mu}}(x_1)$ and $f_{\boldsymbol{\psi}(\boldsymbol{\mu})}(z)$ will be topologically conjugate to each other for each fixed value of $\boldsymbol{\mu}$, our equivalence implies the two families $g(x_1, \boldsymbol{\mu})$ and $f(z, \boldsymbol{\varepsilon})$ are at least mildly topologically conjugate (by letting $\boldsymbol{\phi} = \boldsymbol{\psi}$) as long as the parameter space map $\boldsymbol{\psi}(\boldsymbol{\mu})$ is a homeomorphism.

We point out that although the conjugacies $h_{\boldsymbol{\mu}}(x_1)$ and the functions $Z_{\boldsymbol{\mu}}(x_1)$ of the theorem are not the same, they are related. Specifically, they will agree on all the bifurcation sets D_g^i and S_g^i. This includes the fixed and period-2 sets. Thus, when restricted to the bifurcation sets, $h_{\boldsymbol{\mu}}(x_1)$ will not only vary continuously with respect to the parameter $\boldsymbol{\mu}$, but will also be C^∞.

Consequently, when the parameter space map $\psi(\mu)$ is a diffeomorphism, the existence of the function Ψ of Theorem 3.15 is a stronger property than mild topological conjugacy but not comparable to topological conjugacy. Topological conjugacies have the stronger property that the individual conjugacies $h_\mu(x_1)$ should vary continuously with the parameter; our equivalence has the stronger property that the function Ψ is a (C^∞) diffeomorphism, and consequently that the individual conjugacies $h_\mu(x_1)$ *restricted to the bifurcation surfaces* are also diffeomorphisms.

The rest of this section is devoted to the proof of Theorem 3.15. We begin with the following lemmas.

LEMMA 3.16. *If* $x \in \mathbf{R}$ *and* $c \neq 0$, *then* $G(x, \mathbf{0}) = -x + cx^{2k+1} + o(x^{2k+1})$ *implies* $b_G(s, \mathbf{0}) = -cs^{2k+1} + o(x^{2k+1})$.

Proof. We differentiate the definition of $b_G(s, \mu)$ in (3.6), using the derivatives of $R(s, \mu)$ at $(0, \mathbf{0})$, which we obtain from (3.5a) by repeated implicit differentiation. Since R is even in s, we immediately know that $(\partial^j R/\partial s^j)(0, \mathbf{0}) = 0$ for odd j. We also know from the proof of Theorem 3.3 that $R(0, \mathbf{0}) = 0$. It is relatively straightforward to show that the implicit differentiation yields

$$\frac{\partial^2 R}{\partial s^2}(0, \mathbf{0}) = \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(0, \mathbf{0}),$$

$$\frac{\partial^4 R}{\partial s^4}(0, \mathbf{0}) = \frac{1}{2}\frac{\partial^4 G}{\partial x^4}(0, \mathbf{0}) + \frac{3}{4}\frac{\partial^2 G}{\partial x^2}(0, \mathbf{0})\left\{2\frac{\partial^3 G}{\partial x^3}(0, \mathbf{0}) + \frac{1}{2}\left[\frac{\partial^2 G}{\partial x^2}(0, \mathbf{0})\right]^2\right\}.$$

In general,

$$\frac{\partial^k R}{\partial s^k}(0, \mathbf{0}) = \frac{1}{2}\frac{\partial^k G}{\partial x^k}(0, \mathbf{0}) + \cdots,$$

where the omitted terms all have factors of $(\partial^j G/\partial x^j)(0, \mathbf{0})$ with $2 \leqq j \leqq k-1$.

Using these derivatives, and the fact that $b_G(s, \mu)$ is odd in r (so that all even derivatives of b_G with respect to r vanish), we obtain

$$\frac{\partial b_G}{\partial r}(0, \mathbf{0}) = 0,$$

$$\frac{\partial^3 b_G}{\partial r^3}(0, \mathbf{0}) = -\frac{\partial^3 G}{\partial x^3}(0, \mathbf{0}) - \frac{3}{2}\left\{\frac{\partial^2 G}{\partial x^2}(0, \mathbf{0})\right\}^2, \tag{3.17}$$

$$\begin{aligned}\frac{\partial^5 b_G}{\partial r^5}(0, \mathbf{0}) = &-\frac{\partial^5 G}{\partial x^5}(0, \mathbf{0}) - 5\frac{\partial^4 G}{\partial x^4}(0, \mathbf{0})\frac{\partial^2 G}{\partial x^2}(0, \mathbf{0}) - \frac{15}{4}\frac{\partial^3 G}{\partial x^3}(0, \mathbf{0})\left(\frac{\partial^2 G}{\partial x^2}(0, \mathbf{0})\right)^2 \\ &-5\left[\frac{1}{2}\frac{\partial^4 G}{\partial x^4}(0, \mathbf{0}) + \frac{3}{4}\frac{\partial^2 G}{\partial x^2}(0, \mathbf{0})\left\{2\frac{\partial^3 G}{\partial x^3}(0, \mathbf{0}) + \frac{1}{2}\left(\frac{\partial^2 G}{\partial x^2}(0, \mathbf{0})\right)^2\right\}\right].\end{aligned} \tag{3.18}$$

The expressions for the seventh-order derivative are not pretty. In general, however, we have the relation

$$\frac{\partial^k b_G}{\partial r^k}(0, \mathbf{0}) = -\frac{\partial^k G}{\partial x^k}(0, \mathbf{0}) + \cdots,$$

where the omitted terms all have factors of $(\partial^j G/\partial x^j)(0, \mathbf{0})$ with $2 \leqq j \leqq k-1$.

The lemma follows immediately. □

Note. The sign of (3.17) determines the criticality of the nondegenerate period doubling bifurcation. If it is negative, the bifurcation is supercritical; if it is positive, the bifurcation is subcritical; if it equals zero, there is at least one higher-order degeneracy. If both (3.17) and (3.18) are zero, there are at least two higher-order degeneracies.

LEMMA 3.19. *Let the C^∞ period doubling family $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$, its center manifold realization $g_{\boldsymbol{\mu}}(x_1)$, and the bifurcation function $b_{\mathbf{G}}(s, \boldsymbol{\mu})$ be as in the paragraph preceding Theorem 3.15. Then there exists a neighborhood N of $(0, \mathbf{0})$ in $\mathbf{R} \times \mathbf{R}^m$ such that for $(s, \boldsymbol{\mu}) \in N$, $g_{\boldsymbol{\mu}}^2(x_1) - x_1$ has the same sign as $b_{\mathbf{G}}(s, \boldsymbol{\mu})$, where $(s, \boldsymbol{\mu})$ and $(x_1, \boldsymbol{\mu})$ are related by the C^∞ diffeomorphism $(x_1, \boldsymbol{\mu}) = (s + R(s, \boldsymbol{\mu}), \boldsymbol{\mu})$ (as in (3.12)).*

Furthermore, for each fixed $\boldsymbol{\mu}$, the multiplicity of the corresponding zeros of $g_{\boldsymbol{\mu}}^2(x_1) - x_1$ and $b_{\mathbf{G}}(s, \boldsymbol{\mu})$ is the same.

Proof. Theorem 3.3 guarantees that roots of $G_{\boldsymbol{\mu}}^2(\mathbf{x}) - \mathbf{x}$ are in one-to-one correspondence with roots of $b_{\mathbf{G}}(s, \boldsymbol{\mu})$. Since roots of $G_{\boldsymbol{\mu}}^2(\mathbf{x}) - \mathbf{x}$ must be on the center manifold of $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$, the roots of $g_{\boldsymbol{\mu}}^2(x_1) - x_1$ must also be in one-to-one correspondence with roots of $\mathbf{G}_{\boldsymbol{\mu}}^2(\mathbf{x}) - \mathbf{x}$, and therefore with roots of $b_{\mathbf{G}}(s, \boldsymbol{\mu})$. The correspondence is indicated by (3.12) in the proof of Theorem 3.3:

$$(3.20) \quad s \leftrightarrow \mathbf{x} = \mathbf{X}(s, \boldsymbol{\mu}) = (s + R(s, \boldsymbol{\mu}), X_2(s, \boldsymbol{\mu}), \cdots, X_n(s, \boldsymbol{\mu})) \leftrightarrow x_1 = s + R(s, \boldsymbol{\mu}).$$

For each fixed $\boldsymbol{\mu}$, the multiplicities of corresponding roots of $g_{\boldsymbol{\mu}}^2(x_1) - x_1$ and $b_{\mathbf{G}}(s, \boldsymbol{\mu})$ must be the same, because if they are not, then a perturbation of $\mathbf{G}$ could be made so that their roots would not correspond. (It can be shown that an arbitrarily C^∞ small perturbation of $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ can be chosen to perturb $g_{\boldsymbol{\mu}}^2(x_1) - x_1$ or $b_{\mathbf{G}}(s, \boldsymbol{\mu})$ from a zero of multiplicity ρ to a function with ρ distinct real roots.)

We have left only to show that the signs of the two functions are equal. Since for fixed $\boldsymbol{\mu}$ we already have the zeros and their multiplicities corresponding for $g_{\boldsymbol{\mu}}^2(x_1) - x_1$ and $b_{\mathbf{G}}(s, \boldsymbol{\mu})$, and since these two functions are perturbations of $g_{\mathbf{0}}^2(x_1) - x_1$ and $b_{\mathbf{G}}(s, \mathbf{0})$, respectively, the signs will be the same for $g_{\boldsymbol{\mu}}^2(x_1) - x_1$ and $b_{\mathbf{G}}(s, \boldsymbol{\mu})$ if and only if the signs of the leading coefficients of $g_{\mathbf{0}}^2(x_1) - x_1$ and $b_{\mathbf{G}}(s, \mathbf{0})$ are the same.

According to Definition 3.1, if $x \in \mathbf{R}$ then in normal form up to order $2k+1$, $G(x, \mathbf{0}) = -x + cx^{2k+1} + o(x^{2k+1})$, $c \neq 0$. This makes $G_{\mathbf{0}}^2(x) - x = g_{\mathbf{0}}^2(x) - x = -2cx^{2k+1} + o(x^{2k+1})$. Lemma 3.16 implies $b_G(s, \mathbf{0}) = -cs^{2k+1} + o(s^{2k+1})$. If $\boldsymbol{\mu} = \mathbf{0}$, then $s = 0$ corresponds to $x = x_1 = 0 + R(0, \mathbf{0}) = 0$, so the signs of the leading coefficients of $g_{\mathbf{0}}^2(x) - x$ and $b_G(s, \mathbf{0})$ correspond. If $x \in \mathbf{R}$ but $G(x, \mathbf{0}) = g(x, \mathbf{0})$ is not in normal form up to order $2k+1$, a near identity polynomial change of coordinates $x = h(y)$ can put $g_{\boldsymbol{\mu}}(x)$ into this normal form. That is, $\tilde{g}_{\mathbf{0}}(y) := h^{-1}(g_{\mathbf{0}}(h(y)))$ is in normal form up to order $2k+1$. By perturbation arguments as in the second paragraph of this proof, the multiplicity of the zeros of $\tilde{g}_{\mathbf{0}}^2(y) - y$, $g_{\mathbf{0}}^2(x) - x$, $b_g(s, \mathbf{0})$, and $b_{\tilde{g}}(\tilde{s}, \mathbf{0})$ must all be the same. The same logic works along a whole path of coordinate changes from h_t, $t \in [0, 1]$, from the $h_0 :=$ identity to $h_1 := h$. Therefore, by continuity, the sign of the leading coefficient of $\tilde{g}_{\mathbf{0}}^2(y) - y$ and $g_{\mathbf{0}}^2(x) - x$ must be the same, as must be the sign of the leading coefficient of $b_g(s, \mathbf{0})$ and $b_{\tilde{g}}(\tilde{s}, \mathbf{0})$. Since the sign of the leading coefficients of $\tilde{g}_{\mathbf{0}}^2(y) - y$ and $b_{\tilde{g}}(\tilde{s}, \mathbf{0})$ are equal by the previous paragraph, this forces the sign of the leading coefficients of $g_{\mathbf{0}}^2(x) - x$ and $b_g(s, \mathbf{0})$ to be the same.

If $x \in \mathbf{R}^n$ with $n > 1$, then the realization of $\mathbf{G}$ on its center manifold can also be obtained by a near identity change of coordinates. So by a continuity argument similar to that in the paragraph above, the leading coefficient of $g_{\mathbf{0}}^2(x_1) - x_1$ will have the same sign as the leading coefficient of $b_{\mathbf{G}}(s, \mathbf{0})$. □

One consequence of Lemma 3.19 is that the period doubling map with $k-1$ higher-order degeneracies can be alternatively characterized by

$$\left. \frac{\partial^i B_{\mathbf{G}}(u, \boldsymbol{\mu})}{\partial u^i} \right|_{(0,\mathbf{0})} = 0 \quad \text{for } i = 0, \cdots, k-1,$$

but

$$\left.\frac{\partial^k B_G(u, \boldsymbol{\mu})}{\partial u^k}\right|_{(0,\mathbf{0})} \neq 0.$$

Another consequence is that the sign of b_G or B_G can be used to determine stability of the fixed and period-2 orbits of $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ and $g(x_1, \boldsymbol{\mu})$. It is usually more practical, however, to do this by eigenvalue computations, especially because, as mentioned after the statement of Lemma 3.19, the bifurcation functions are defined via the implicit function theorem.

Technical note. Lemma 3.19 and Theorem 3.15 are both stated under the assumption that the coordinate x_1 is already a coordinate on the center eigenspace for $\boldsymbol{\mu} = \mathbf{0}$. When $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ does not originally come in this form, there is some leeway in choosing x_1. Its choice, however, involves a change of coordinates from the given form of $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$. If the change of coordinates is orientation preserving, a path to the identity argument as in the last two paragraphs of the proof of Lemma 3.19 can be used to show that the leading coefficient of $g_{\mathbf{0}}^2(x_1) - x_1$ will have the same sign as the leading coefficient of $b_G(s, \mathbf{0})$. The case of an orientation reversing change of coordinates is converted to the orientation preserving case by noting that the change of variables $x_1 \to -x_1$ leaves $b_g(s, \boldsymbol{\mu})$ the same and leaves the leading coefficient of $g_{\mathbf{0}}^2(x_1) - x_1$ the same.

This note shows that even though the bifurcation function constructed in the proof of Theorem 3.3 is not necessarily unique (there is a choice of coordinates made in reducing Case 3 to Case 2), the zeros, including multiplicities, and signs at corresponding nonzero points of any two bifurcation functions arising from the same original function must all be equal.

We now recall the universal unfolding theorem for $\mathbf{Z}_2$-symmetric bifurcation functions.

LEMMA 3.21. *Define the k-parameter family of $\mathbf{Z}_2$-symmetric bifurcation functions $U(S, \boldsymbol{\varepsilon}) := \varepsilon_0 S + \varepsilon_1 S^3 + \cdots + \varepsilon_k S^{2k-1} + \delta S^{2k+1}$, $\delta = \pm 1$. Let $V(s, \boldsymbol{\mu})$ be any family of $\mathbf{Z}_2$-symmetric bifurcation functions satisfying $V(s, \mathbf{0}) = cs^{2k+1} + \cdots$, with* sgn $(c) = \delta$, *and*

$$\left.\nabla_{\boldsymbol{\mu}}\left(\frac{\partial^i V(s, \boldsymbol{\mu})}{\partial s^i}\right)\right|_{(0,\mathbf{0})} \neq \mathbf{0}.$$

Then in a neighborhood of $(0, \mathbf{0})$, *there exist* C^∞ *functions* M, Σ, *and* $\boldsymbol{\phi}$ *such that*

$$V(s, \boldsymbol{\mu}) = M(s, \boldsymbol{\mu})\, U(\Sigma(s, \boldsymbol{\mu}), \boldsymbol{\phi}(\boldsymbol{\mu})) \tag{3.22}$$

with

$$M(s, \boldsymbol{\mu}) > 0,\ (\partial\Sigma/\partial s)(s, \mathbf{0}) > 0,\ \Sigma(s, \mathbf{0}) = 0,$$

$$\boldsymbol{\phi}(\mathbf{0}) = \mathbf{0},\ M(-s, \boldsymbol{\mu}) = M(s, \boldsymbol{\mu}),\ \Sigma(-s, \boldsymbol{\mu}) = -\Sigma(s, \boldsymbol{\mu}).$$

Furthermore, there is no family having the properties of $U(S, \boldsymbol{\varepsilon})$ with fewer than k parameters.

Proof. Combine Proposition 2.14 [GS, p. 256] and Proposition 3.4 [GS, p. 259]. □

Proof of Theorem 3.15. (a) Recall from the paragraph preceding the statement of Theorem 3.15 that $g(x_1, \boldsymbol{\mu})$ is the center manifold realization of $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ and $f(z, \boldsymbol{\varepsilon})$ is the appropriate model family. We will define the function $\boldsymbol{\Psi}$ so that the sign of $g_{\boldsymbol{\mu}}^2(x_1) - x_1$ will be the same as the sign of $f_{\boldsymbol{\varepsilon}}^2(z) - z$ for $(z, \boldsymbol{\varepsilon}) = \boldsymbol{\Psi}(x_1, \boldsymbol{\mu})$. As previously noted in § 2.1, this will guarantee that $g_{\boldsymbol{\mu}}$ and $f_{\boldsymbol{\varepsilon}}$ will be topologically conjugate to each other for fixed values of the parameters (and appropriately restricted neighborhoods).

Let $b_{\mathbf{G}}(s, \boldsymbol{\mu})$ and $b_f(S, \boldsymbol{\varepsilon})$ be the bifurcation functions determined from $\mathbf{G}(\mathbf{x}, \boldsymbol{\mu})$ and $f(z, \boldsymbol{\varepsilon})$, respectively, as in the proof of Theorem 3.3. Let $R^{\mathbf{G}}(s, \boldsymbol{\mu})$ and $R^f(S, \boldsymbol{\varepsilon})$ be the respective functions defined following (3.9), with the superscripts added to distinguish the R's arising from the different functions $\mathbf{G}$ and f.

By Lemma 3.19, $g^2_{\boldsymbol{\mu}}(x_1) - x_1$ has the same sign as $b_{\mathbf{G}}(s, \boldsymbol{\mu})$, where $(s, \boldsymbol{\mu})$ and $(x_1, \boldsymbol{\mu})$ are related by the diffeomorphism $(x_1, \boldsymbol{\mu}) = (s + R^{\mathbf{G}}(s, \boldsymbol{\mu}), \boldsymbol{\mu})$. Also by Lemma 3.19, $f^2_{\boldsymbol{\varepsilon}}(z) - z$ has the same sign as $b_f(S, \boldsymbol{\varepsilon})$, where $(S, \boldsymbol{\varepsilon})$ and $(z, \boldsymbol{\varepsilon})$ are related by the diffeomorphism $(z, \boldsymbol{\varepsilon}) = (S + R^f(S, \boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}) = (S, \boldsymbol{\varepsilon})$. This last equality follows from the proof of Corollary 3.13, where we showed that $R^f(S, \boldsymbol{\varepsilon}) = 0$.

Also, by Corollary 3.13, $b_f(S, \boldsymbol{\varepsilon}) = \varepsilon_0 S + \varepsilon_1 S^3 + \cdots + \varepsilon_k S^{2k-1} + \delta S^{2k+1}$, which equals $U(S, \boldsymbol{\varepsilon})$ as defined in Lemma 3.21. Lemma 3.21 can therefore be used to show that there exist functions Σ and $\boldsymbol{\phi}$ such that $b_{\mathbf{G}}(s, \boldsymbol{\mu})$ and $b_f(S, \boldsymbol{\varepsilon})$ have the same sign for $(S, \boldsymbol{\varepsilon}) = (\Sigma(s, \boldsymbol{\mu}), \boldsymbol{\phi}(\boldsymbol{\mu}))$. Note that this C^∞ map will be a diffeomorphism if $\boldsymbol{\phi}(\boldsymbol{\mu})$ is a diffeomorphism.

Combining the results of the two paragraphs above, we see that the signs of $g^2_{\boldsymbol{\mu}}(x_1) - x_1$, $b_{\mathbf{G}}(s, \boldsymbol{\mu})$, $b_f(S, \boldsymbol{\varepsilon})$ and $f^2_{\boldsymbol{\varepsilon}}(z) - z$ are all the same for $x_1 = s + R^{\mathbf{G}}(s, \boldsymbol{\mu})$, $(S, \boldsymbol{\varepsilon}) = (\Sigma(s, \boldsymbol{\mu}), \boldsymbol{\phi}(\boldsymbol{\mu}))$, and $S = z$. These relationships define the map $\boldsymbol{\Psi}(x_1, \boldsymbol{\mu})$ by the composition

$$(x_1, \boldsymbol{\mu}) \to (s, \boldsymbol{\mu}) \to (S, \boldsymbol{\varepsilon}) \to (z, \boldsymbol{\varepsilon}). \tag{3.23}$$

Each map in the composition is C^∞ in a neighborhood of $(0, \mathbf{0})$ and each fixes $(0, \mathbf{0})$. Therefore the same is true of $\boldsymbol{\Psi}$. This establishes (a)(1) and (a)(2) of Theorem 3.15. Part (a)(3) is true because each map in (3.23) preserves not only the zeros but also their multiplicities. (This is true for the first and third maps by Lemma 3.19, and for the middle map by (3.22).)

(b) If there existed a family that could replace $f_{\boldsymbol{\varepsilon}}$ in Theorem 3.15(a), then its bifurcation function would be a "universal unfolding" in the space of $\mathbf{Z}_2$ bifurcation functions with fewer than k parameters. This would contradict the last sentence of the universal unfolding theorem for $\mathbf{Z}_2$-symmetric bifurcation functions, Lemma 3.21.

(c) The condition that

$$\left\{ \nabla_{\boldsymbol{\mu}} \left(\frac{\partial^i B_{\mathbf{G}}(u, \boldsymbol{\mu})}{\partial u^i} \right) \Bigg|_{(0,\mathbf{0})}, \qquad i = 0, \cdots, k-1 \right\}$$

be independent is equivalent to the Jacobian determinant $|\partial \varepsilon_i / \partial \mu_j|_{\mu = \mathbf{0}} \neq 0$ *and therefore is equivalent to the map* $\boldsymbol{\varepsilon} = \boldsymbol{\psi}(\boldsymbol{\mu})$ *being a local diffeomorphism. In this case* $\boldsymbol{\Psi}$ *is also a local diffeomorphism.* □

4. Applications. Theorem 3.15 states that any period doubling diffeomorphism with $k-1$ higher-order degeneracies is equivalent, both in terms of its topological behavior under iteration (restricted to its center manifold) and in terms of its bifurcation sets, to one of our model families of § 2. In order to support these theoretical results, we used a version of the continuation routine AUTO [DK] that we adapted for use with maps to investigate two examples where we knew a period doubling with a higher-order degeneracy to exist. Both are two-parameter families of maps generated by flows of periodically forced planar oscillators. The stroboscopic map and its derivatives were calculated using ODESSA [LK]. Because our applications involved only two parameters, we would not expect to see a period doubling with more than the single higher-order degeneracy. The bifurcation diagrams we produced from these applications should be compared to Figs. 3 and 4 for our model period doubling map with a single higher-order degeneracy.

4.1. Resonance horns in forced oscillators. Consider a system of two autonomous coupled nonlinear ODE's

$$d\mathbf{x}/dt=\mathbf{f}(\mathbf{x},p),\qquad \mathbf{f}:\mathbf{R}^2\times\mathbf{R}\to\mathbf{R}^2,$$

where $p\in\mathbf{R}$ is a parameter. Assume that for $p=p_0$ the system above has an asymptotically attracting closed orbit with frequency ω_0. Consider the two-parameter family of forced oscillators

$$d\mathbf{x}/dt=\mathbf{f}(\mathbf{x},p_0+\alpha g(\omega t)),$$

where α and ω are the parameters (α is the amplitude of the forcing and g has period $T=1/\omega$). A more convenient second parameter is the ratio ω/ω_0 of the forcing to the natural frequency. Taking the time T return map of this flow (sometimes referred to as the stroboscopic map) gives us a two-parameter family of invertible, orientation preserving maps of the plane. The asymptotic attractivity of the limit cycle of the unforced oscillator guarantees the existence of a normally hyperbolic attracting invariant circle for small forcing amplitude α. According to standard circle map theory [Ar], [Ha], we expect resonance horns (also called entrainment regions of Arnol'd tongues) entering the first quadrant of the ω/ω_0-α parameter plane for every rational value of ω/ω_0. The boundaries of the "q/p resonance horn" emanating from $\omega/\omega_0=q/p$ are saddlenode bifurcation points for the qth iterate of the map. Inside this q/p resonance horn, the corresponding map has at least one (typically two: a stable and unstable pair) period-q orbit. In particular, we are interested in the situation where $q=2$, when the boundaries of the $2/p$ horns are saddlenode bifurcations for the second iterate of the map. In continuing these saddlenode curves towards higher values of α, we have repeatedly found them to terminate at a degenerate period doubling where they collide with a period doubling curve. (This was a much easier and less expensive ways of locating the degenerate period doubling points than the method suggested by Definition 3.1 or comment 1 following the statement of Theorem 3.15. To compute the normal form of a map on its center manifold and/or $(\partial^i B_G(u,\boldsymbol{\mu})/\partial u^i)|_{(0,0)}$, we would need higher derivatives of the stroboscopic map generated by numerically integrating the forced oscillator flows.)

Figures 7 and 8 show various features of the period doubling with a single higher-order degeneracy in the context of a 2/3 resonance horn for our first system of periodically forced ODEs:

$$\frac{dx_1}{d\tau}=-(p_0+\alpha\cos(\omega\tau))x_1+\frac{ax_2}{b+x_2}x_1,$$

$$\frac{dx_2}{d\tau}=-(p_0+\alpha\cos(\omega\tau))x_2+\frac{z_f-x_1-x_2}{1+z_f-x_1-x_2}x_2-\frac{ax_2}{b+x_2}x_1.$$

These ODEs model a predator-prey system (protozoa preying on bacteria in a chemostat). Here x_1 is the dimensionless concentration of protozoa, x_2 is the dimensionless concentration of bacteria, and z_f is the dimensionless feed concentration of a substrate on which the bacteria grow with Monod-type kinetics [PK]. The parameter we vary periodically is the flow rate of the chemostat. The autonomous system for $a=0.4$, $b=2.8125$, $z_f=12.4$, and $p_0=0.2$ has a single attracting limit cycle of period $T=18.999$ units of dimensionless time τ.

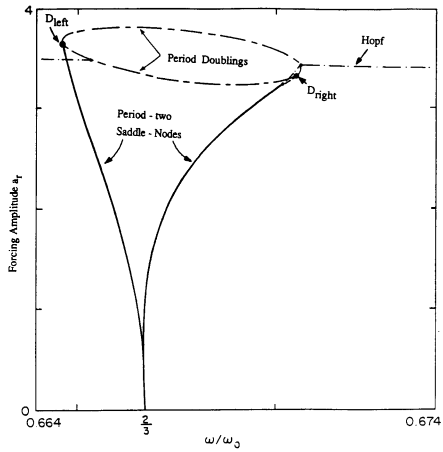

FIG. 7. *Predator-prey parameter plane.*

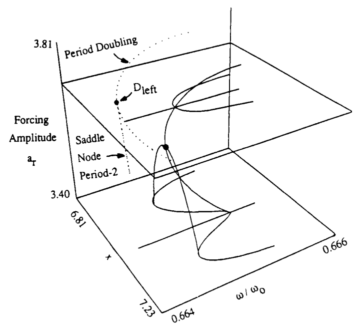

FIG. 8. *Predator-prey: singly degenerate period doubling.*

Figure 7 shows the boundaries of the 2/3 resonance horn for this model ($a_r = \alpha/0.00265$). As we follow both sides of the horn boundary towards higher values of α we encounter degenerate period doubling points D_{left} and D_{right}. Figure 8 is a three-dimensional representation of the full four-dimensional phase $\times$ parameter space of the solution surface and the codimension-1 bifurcation curves in the neighborhood of D_{left}. Compare this diagram to Fig. 3.

Another example where we also observed this phenomenon is the Continuous Stirred Tank Reactor (CSTR) in which a simple exothermic reaction $A \rightarrow B$ takes place. This classical chemical reaction engineering system can be modeled by the following set of dimensionless ODEs:

$$\frac{dx_1}{d\tau} = -x_1 + Da(1 - x_1)\exp(x_2),$$

$$\frac{dx_2}{d\tau} = -x_2 + B\,\mathrm{Da}\,(1 - x_2)\exp(x_2) + \beta(T_c - x_2),$$

where x_1 is a dimensionless concentration of reactant A, x_2 is a dimensionless temperature, and Da (the Damkoehler number), B (the dimensionless heat of reaction),

$T_c = T_{c,0} + \alpha \cos(\omega\tau)$ (the coolant temperature), and β (the dimensionless heat transfer coefficient between the reactor and the coolant fluid) are parameters. For $B = 22$, $\mathrm{Da} = 0.085$, $\beta = 3$, and $T_{c,0} = 0$ the autonomous system ($\alpha = 0$) has an attracting limit cycle of period $T_0 = 1.094996$ surrounding an unstable steady state. In a previous publication [KAS] degenerate period doublings were observed on both $2/p$ horns studied (the 2/1 and the 2/3 horns). Figure 9 is a three-dimensional representation of the full four-dimensional phase $\times$ parameter space of the solution surface and the codimension-1 bifurcation curves in the neighborhood of the equivalent of the D_{right} point of Fig. 7 for the 2/1 resonance horn of the periodically forced CSTR ($a_r = 0.063036$). Compare Fig. 9 also to Fig. 3.

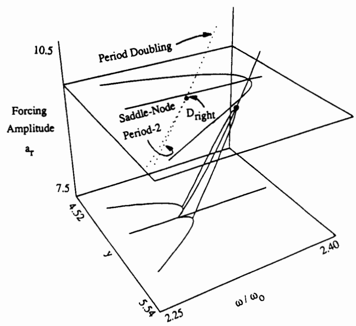

FIG. 9. *Forced* CSTR: *singly degenerate period doubling.*

Recent studies by McKarnin, Schmidt, and Aris [MSA] (a periodically forced surface reaction model), Schreiber et al. [SDCM] (a periodically forced Brusselator), as well as by Vance and Ross [VR] (a periodically forced CSTR) have also repeatedly revealed degenerate period doublings on the boundaries of $2/p$ resonance horns. This bifurcation appears therefore to be ubiquitous in models of periodically forced oscillators arising in various disciplines.

4.2. High-amplitude closing of the resonance horns. In our example (Fig. 7), as well as in the numerous studies of periodically forced oscillators we referred to above, the phenomenon of high-amplitude "closing" of the $2/p$, and generally of the q/p resonance horns was observed. It has been shown that this "closing" phenomenon implies the existence of certain codimension-2 bifurcations for the maps [AMKA], [P1], [P2], [P3]. In most horns, the boundary consists of codimension-1 saddlenode bifurcation curves for the qth iterate of the map along with certain codimension-2 points on these curves. For a $2/p$-horn, however, this boundary typically changes from a saddlenode curve for the 2nd iterate of the map to a period doubling curve in order for the horn to close. The point at which they change is the codimension-2 degenerate period doubling point.

See the references above for details and [Ga] for a related analytical study.

5. Discussion.

5.1. The Hopf bifurcation with higher-order degeneracies. As we mentioned in the introduction, certain higher-order degeneracies in the Hopf bifurcation for flows generate bifurcation diagrams almost identical to those for the period doubling bifurcation with higher-order degeneracies. This is not surprising if we look at the model flows of Table 1:

$$r' = \varepsilon_1 r + \varepsilon_2 r^3 + \cdots + \varepsilon_{2k-1} r^{2k-1} + \delta r^{2k+1},$$

$$\theta' = \omega + r^2.$$

Circular limit cycles exist whenever r satisfies $r(\varepsilon_1 + \varepsilon_2 r^2 + \cdots + \varepsilon_{2k-1} r^{2k-2} + \delta r^{2k}) = 0$. That is, the roots of this function determine the topological phase portraits of the corresponding flows. But this function is precisely $rP_{\varepsilon;k,\delta}(r^2)$, the bifurcation function we defined in (2.3) and used for our model period doublings in § 2. In both cases, the root at $r=0$ corresponds to a "center" fixed point; other roots correspond to limit cycles for the Hopf flow and period-2 orbits for the period doubling map. Roots of higher multiplicity determine higher codimension bifurcation manifolds in both cases.

To prove that the general Hopf bifurcations are all like the above models, Golubitsky and Schaeffer ([GS] and references therein) define a function, analogous to $\mathbf{\Phi}$ in § 3, whose roots determine the limit cycles for a given flow. Among several factors complicating the Hopf analysis are the facts that $\mathbf{\Phi}$ is defined on an infinite-dimensional function space and that its kernel is two-dimensional. After performing a Lyapunov–Schmidt reduction on this function, however, they obtain the same "reduced" bifurcation function as we obtained in Theorem 3.3. That is, both problems can be reduced to finding roots of the *same* bifurcation function.

We illustrate a more geometric connection between the Hopf bifurcation for flows and the period doubling bifurcation for some fixed parameter value in Fig. 10. The flow in $\mathbf{R}^2$ induces a map in $\mathbf{R}^1$ by taking a return map of the flow along a line (not a ray) through the origin. (Let the origin be a fixed point of the map.) Limit cycles of the flow correspond to period-2 orbits of the induced map.

5.2. Other "finite sequence spaces." We characterized period-2 points of $\mathbf{G}(\mathbf{x})$ in this paper as roots of the function $\mathbf{\Phi}(\mathbf{x},\mathbf{y}) = (\mathbf{y} - \mathbf{G}(\mathbf{x}), \mathbf{x} - \mathbf{G}(\mathbf{y}))$ and then used the Lyapunov–Schmidt procedure to reduce $\mathbf{\Phi} = \mathbf{0}$ to a simpler system. Brown and Roberts [BR] and Vanderbauwhede [Va] have recently used Lyapunov–Schmidt reduction for functions on similar "finite sequence spaces" whose roots characterize periodic points of periods other than 2. In general, a period-k orbit $\{\mathbf{x}^1, \cdots, \mathbf{x}^k\}$ of $\mathbf{G}: \mathbf{R}^n \to \mathbf{R}^n$ is characterized as a root of the function $\mathbf{\Phi}: (\mathbf{R}^n)^k \to (\mathbf{R}^n)^k$ defined by $\mathbf{\Phi}(\mathbf{x}^1, \cdots, \mathbf{x}^k) = (\mathbf{x}^2 - \mathbf{G}(\mathbf{x}^1), \mathbf{x}^3 - \mathbf{G}(\mathbf{x}^2), \cdots, \mathbf{x}^1 - G(\mathbf{x}^k))$. The Lyapunov–Schmidt reduction starts from this function.

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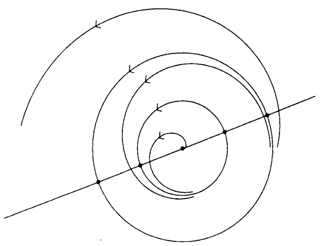

FIG. 10. *Period doubling and Hopf bifurcations.*

REFERENCES

[Ar] V. I. ARNOL'D, *Geometrical methods in the theory of ordinary differential equations*, Grundlehren, 250, Springer-Verlag, New York, 1983.

[AMKA] D. G. ARONSON, R. P. MCGEHEE, I. G. KEVREKIDIS, AND R. ARIS, *Entrainment regions for periodically forced oscillators*, Phys. Rev. A, 33 (1986), pp. 2190–2192.

[BR] A. G. BROWN AND R. M. ROBERTS, *Subharmonic bifurcations of equivariant maps*, in preparation.

[Ch] A. CHENCINER, *Bifurcations de points fixes elliptiques* II, Invent. Math., 80 (1985), pp. 81–106.

[DK] E. J. DOEDEL AND J. P. KERNEVEZ, AUTO: *Software for continuation and bifurcation problems in ordinary differential equations* (*including the* AUTO 86 *User Manual*), Report, Applied Mathematics, California Institute of Technology, Pasadena, CA, 1986.

[CMY] S. N. CHOW, J. MALLET-PARET, AND J. YORKE, *A Periodic Orbit Invariant Which Is a Bifurcation Invariant*, Lecture Notes in Math. 1007, Springer-Verlag, New York, 1983, pp. 100–131.

[Ga] J. M. GAMBAUDO, *Perturbation of a Hopf bifurcation by an external time-periodic forcing*, J. Differential Equations, 57 (1985), pp. 172–199.

[GS] M. GOLUBITSKY AND D. SCHAEFFER, *Singularities and Groups in Bifurcation Theory*, Vol. I, Appl. Math. Sci., 51, Springer-Verlag, New York, 1985.

[GH] J. GUCKENHEIMER AND P. HOLMES, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Appl. Math. Sci., 42, Springer-Verlag, New York, 1983.

[Ha] G. R. HALL, *Resonance zones in two-parameter families of circle homeomorphisms*, SIAM J. Math. Anal., 15 (1984), pp. 1075–1081.

[HW] P. HOLMES AND D. WHITLEY, *Bifurcations of one- and two-dimensional maps*, Philos. Trans. Roy. Soc. London Ser. A, 311 (1984), pp. 43–102.

[LK] J. R. LEIS AND M. A. KRAMER, ODESSA—*An ordinary differential equation solver with explicit simultaneous sensitivity analysis*, ACM Trans. Math. Software, 14 (1985), pp. 61–67.

[KAS] I. G. KEVREKIDIS, R. ARIS, AND L. D. SCHMIDT, *The stirred tank forced*, Chem. Engrg. Sci., 41 (1986), pp 1549–1560.

[MSA] M. A. MCKARNIN, L. D. SCHMIDT, AND R. ARIS, *Forced oscillations of a self-oscillating bimolecular surface reaction model*, Proc. Roy. Soc. London Ser. A, 417 (1988), pp. 363–388.

[NPT] S. NEWHOUSE, J. PALIS, AND F. TAKENS, *Bifurcations and stability of families of diffeomorphisms*, Publ. Math. IHES, 1982.

[PK] S. PAVLOU AND I. G. KEVREKIDIS, *Microbial predation in a periodically operated chemostat*, Math. Biosci., in press.

[P1] B. B. PECKHAM, *Global properties of resonance surfaces and resonance horns*, Ph.D. thesis, University of Minnesota, Minneapolis, MN, 1988.

[P2] ———, *The necessity of the Hopf bifurcation in periodically forced oscillators*, Nonlinearity, 3 (1990), pp. 261–280.

[P3] ———, *Typical bifurcation diagrams for periodically forced oscillators*, in preparation.

[SDCM] I. SCHREIBER, M. DOLNIK, P. CHEE, AND M. MAREK, *Resonance behavior in two-parameter families of periodically forced oscillators*, Phys. Lett. A, 128 (1988), pp. 66–70.

[Ta] F. TAKENS, *Unfoldings of certain singularities of vectorfields: Generalized Hopf bifurcations*, J. Differential Equations, 14 (1973), pp. 476–493.

[VR] W. VANCE AND J. ROSS, *A detailed study of a forced chemical oscillator: Arnol'd tongues and bifurcation sets*, J. Chem. Phys., 91 (1989), pp. 7654–7670.

[Va] A. VANDERBAUWHEDE, *Branching of periodic solutions in time reversible systems*, *Geometry and Analysis in Nonlinear Dynamics*, Res. Notes in Math., Longman Pitman, Boston, MA, to appear.