CHAOS, ATTRACTORS AND THE LORENZ CONJECTURE: NONINVERTIBLE TRANSITIVE MAPS OF INVARIANT SETS ARE SENSITIVE

A THESIS
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

ADVISOR: BRUCE PECKHAM

July 2010

Acknowledgements

I would like to acknowledge the uncountable support of my advisor, Bruce Peckham, who picked a great topic to work on. I would like to acknowledge Guihua Fei and John Pastor, the two other professors who make up my thesis committee, especially for having the patience to read through my pithy outline and offer some great ideas before the writing process began. I would like to acknowledge Edward Lorenz for asking a really great question, a question that allowed a student to earn a master's thesis 21 years later. I would like to acknowledge all the earlier proofs that I wrote for this thesis, although they did not make the final cut, I learned a lot about the importance of all the pesky details when writing proofs. Finally, I would like to acknowledge Hongbo Lin, she helped in more ways than I care to write.

Abstract

In 1989, Edward Lorenz published a paper entitled, “Computational chaos- a prelude to computational instability” [L]. His paper looked at Euler approximations to differential equations. If the time increment of the approximating function was increased, he found that computational chaos set in. Since the numerics suggested transitivity and noninvertibility, he conjectured that transitive, noninvertible maps of an attractor were chaotic. To set the stage for investigating this conjecture, this thesis looked to examine the relationships between some of the standard definitions of chaos and attractor used throughout the literature. In addition to offering a proof of the Lorenz conjecture, a review of a number of related results was conducted. A side product of the work done was a partial result that tried to address whether topological transitivity carries sensitivity at a point to sensitivity on a set.

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Ch 0. Background

In 1989, Edward Lorenz published a paper entitled, “Computational chaos- a prelude to computational instability” [L]. His paper was concerned with the emergence of chaos when difference equations with large time steps are used to approximate the solutions to ordinary differential equations. In particular he was looking at bad Euler approximations. The forward differencing scheme $\mathbf{X}_{n+1} = \mathbf{X}_n + \tau\mathbf{F}(\mathbf{X}_n)$ would be used to approximate solutions to the system of ODEs $d\mathbf{X}/dt = \mathbf{F}(\mathbf{X})$. If τ was chosen to be so large as to give rise to sensitive dependence on initial conditions, for an ordinary differential equation without sensitivity, then the system was said to be computationally chaotic. Similarly, if τ was chosen large enough such that the approximation approached infinity, whereas the corresponding solution to the ODE approached an equilibrium point, then this was said to be computational instability. It was pointed out that not all Euler approximations give rise to computational chaos or computational instability for large τ. For example the Euler method applied to $dx/dt = -x(1+x^2)$ shows chaotic behavior for large enough τ but never reaches computational instability. He then pointed out that even one of the simplest nonlinear flows, $dx/dt = x - x^2$, can elicit computational chaos via bad Euler approximations. Since $dx/dt = x - x^2$ can be solved analytically to $x = e^t/(e^t + \mathrm{c})$ it can be found that particular solutions approach either the stable fixed point $x = 1$ or $x = -\infty$; however, Euler’s method gives $x_{n+1} = (1+\tau)x_n - \tau x_n^2$ where for values of τ between 2 and 3 the map varies chaotically . Then for $\tau > 3$ computational instability sets in.

The main system Lorenz studied is given in the caption of Figure 1. As can be seen in panel A, this system of contains an attracting equilibrium point. As Lorenz simulated “solutions” to this system with increasingly large time steps, he noticed a series of images that showed the formation of what appeared to a smooth attracting invariant circle, which he called an attractor. As τ was increased the attractor progressed from a smooth invariant circle (Fig. 1B), to a smooth circle with bumps (Fig. 1C), to a set with cusps (Fig. 1D/E), to set with a strong numerical indication of chaos (Fig. 1F). Note: Panel A was created using *Mathematica* and its *NDSolve* and *StreamPlot* commands; whereas Panels B-F were modified from [L]. Since Figure 1F provides a strong indication of chaos, a question that was asked by Lorenz was where did the transition to chaos first appear? Lorenz posited that this transition happened when the cusps turned to loops. This led Lorenz to conjecture that a sufficient condition for a system to be chaotic was that a map be not one-to-one on an attractor. For the rest of the paper this will be called the Lorenz Conjecture.

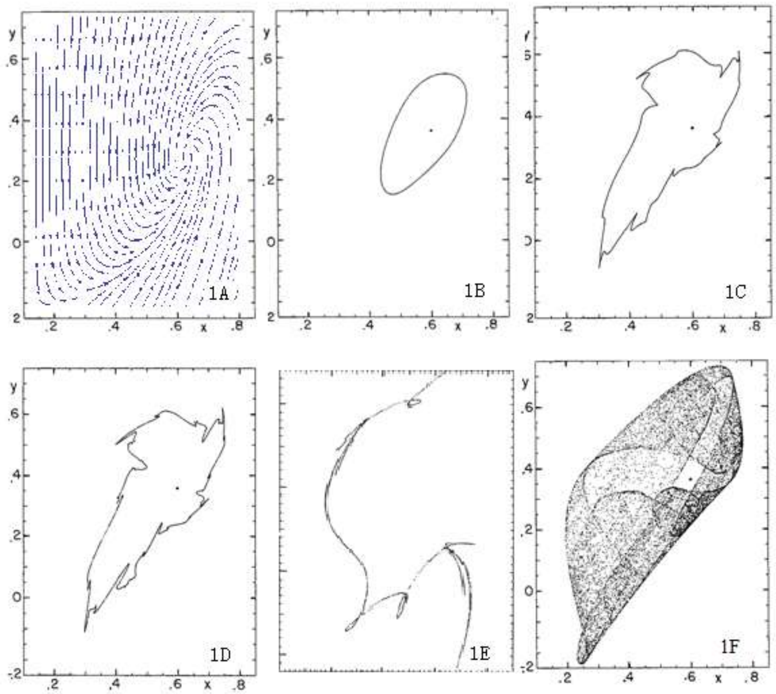

Figure 1. A. The phase portrait of dx/dt =0.36x-xy, dy/dt =x^2-y with x(0)=y(0)=1.
B. The attractor of Xn+1 = (1+.36t)Xn-tXnYn, Yn+1 =(1-t)Yn+tXn^2 with t =1.50
C. Same as B, but t =1.775
D. Same as C, but t =1.785
E. Magnification of 1D
F. Same as D, but t =1.92

Although the motivation for this thesis ultimately came from Lorenz's paper, specifically his conjecture, this thesis has two purposes. The first is to prove the conjecture that was made in Lorenz's paper about a transitive, noninvertible function acting on an attractor being chaotic is true. This led directly into the second purpose of the thesis. Since Lorenz's claim used terms such as attractor and chaos, terms for which there are numerous definitions of in the literature, it was deemed worthwhile to present some of the more widely accepted and used definitions of *chaos* and *attractor* and present them in a hierarchical fashion. In other words, show the implications between the various definitions from the literature.

It was found that the main differences with respect to varying definitions of attractor stem from whether there is a subattractor and what, if anything, does it attract. The definitions of chaos that were considered were the classic Devaney definition (topological transitivity, dense periodic points and sensitive dependence on initial conditions), Li-Yorke chaos (an uncountable scrambled set) and what was named in this paper as Lorenz chaos (topological transitivity and sensitive dependence on initial conditions).

Several results related to the Lorenz conjecture were found in the literature. The most directly related came from [Si], [BV], [AAB] and [GW]. Both [Si] and [BV] showed that for the special case where f is a map of an interval, transitivity, or equivalently a dense orbit, implies chaos, thus validating the Lorenz Conjecture without the need for noninvertibility. Silverman also went on to prove that a dense orbit and noninvertibility imply chaos in the special case when f is a map of a circle. The Lorenz conjecture was shown to be true in a general metric space, but in the special case that the system is minimal, by Akin, Auslander and Berg [AAB].

The rest of this paper will proceed by presenting definitions from the literature of basic terms that will be utilized in defining chaos and attractor. The more enlightened reader can feel free to skim over this section or skip it entirely. Next the definitions of attractor and chaos will be presented, followed by explanations of which definition of attractor (chaos) implies another definition of attractor (chaos). The next section will go into more detail about how much of Lorenz's Conjecture has been answered in the literature. The final section will contain this author's humble proof of the Lorenz conjecture. Also included are some results in an attempt to determine whether transitivity plus sensitivity at a point implies sensitivity over the whole set.

Before proceeding, it should be noted that in [K], the Lorenz conjecture was stated to be true where he cited [AAB] and [GW]. As mentioned above, in [AAB] it was shown that the conjecture was true only when the system was minimal. Furthermore, there is no explicit reference to transitivity and noninvertibility implying chaos in [GW]; however, the Lorenz Conjecture follows from Lemma 1.2 of [GW], provided one adds a few details.

Ch 1. Definitions from the literature

This section presents definitions from the literature, with references, of basic notions that will be used later in the thesis, especially when deriving the definitions of attractor and chaos. Unless stated otherwise assume $f:(X,d)\to(X,d)$, where f is continuous and X is a metric space with metric d. When the metric is understood $f:(X,d)\to(X,d)$ will be denoted $f:X\to X$ or (X,f) for brevity. When working with a subset $A\subset X$, the self map restricted to A will be denoted $f|_A$ or (A,f).

Asymptotic: A point x is asymptotic to y if $\lim d(f^n(x), f^n(y)) = 0$. [HY]

Attracted: The orbit $\{f^n(x_1)\}$ or the point x_1 is attracted to the forward limit set, $\omega(x_0)$, if $\omega(x_1)$ is contained in $\omega(x_0)$. [ASY]

Computational Chaos: Chaotic behavior in a numerical approximation that owes its existence to the use of an excessively large time increment. [L]

Computational Instability: A rapid and unbounded amplification of the variables in a numerical approximation that owes its existence to the use of an excessively large time increment. [L]

Dense Orbit (DO): f has a dense orbit if there is an $x_0 \in X$ whose orbit $\{f^n(x_0)\}_{n=0}^{\infty} = \{x_n\}_{n=0}^{\infty}$ is dense in X. [Si]

Dense periodic points (DPP): $f : X \to X$ has DPP if the closure of the set of periodic points of f is X. [D]

Distal: A point x is distal to y if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) > 0$. [HY]

First/Second Category: A set E in a topological space X is nowhere dense if its closure $\bar{E}$ contains no non-empty open set, or in other words, $\mathrm{int}(\bar{E}) = \varnothing$. A countable union of nowhere dense sets is said to be of first category. Every other subset is said to be of second category. [G]

Forward/Omega Limit Set: The forward/omega limit set of the orbit $\{f^n(x_0)\}$ is the set $\omega(x_0) = \{\, x :$ for all N and $\varepsilon \;\; \exists n > N \;\ni\; |f^n(x_0) - x| < \varepsilon \,\}$. [ASY]

G$_\delta$ Set: In a topological space a G$_\delta$ set is a countable intersection of open sets. [Wikipedia]

Indecomposable: a closed *f*-invariant set is indecomposable if it is not the union of two disjoint closed invariant subsets. [M]

Invariant set: $A \subseteq X$ is an invariant set for A if $x \in A$ then $f^n(x) \in A$ for all n. [St]

Lift: $G : \mathbb{R} \to \mathbb{R}$ is a lift of $g : S \to S$ if G is continuous, the function $G(x) - x$ is periodic with period 1, and $G(x) \bmod 1 = g(x \bmod 1)$. [Si]

Likely Limit Set: The likely limit set $\Lambda = \Lambda(f)$ is the smallest closed subset of X with the property that $\omega(x) \subset \Lambda$ for every point $x \in X$ outside of a set of measure zero. [M]

Li-Yorke Pair: $\{x, y\} \in X$ is called a Li-Yorke pair if

$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$. [K]

Minimal: The system (X, f) is said to be minimal if every point is a transitive point (i.e., a point with a dense orbit). [AAB]

Not Sensitive (~S): A system is ~S if for every $\varepsilon > 0$ there exist an $x \in X$ and a neighborhood U of x such that for every $y \in U$ and $n \in \mathbb{N}$, $d(f^n(x), f^n(y)) \le \varepsilon$. [GW]

Proximal: A point x is proximal to y if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$. [HY]

Realm of Attraction: For the attractor A, the realm of attraction, $\rho(A)$, consists of all points $x \in X$ for which $\omega(x) \subset A$. [M]

Scattering/ 2-Scattering: Given a cover $\mathcal{C}$ of a dynamical system (X, f), usually open or closed, its complexity function $C(\mathcal{C}, n)$ is the minimal number of a sub-cover of refinement $\mathcal{C}_0^n = V_{i=0}^n f^{-i}\mathcal{C}$. A dynamical system is scattering if any cover by non-dense open sets has unbounded complexity, and 2-scattering if the same is true for 2-set covers only. Scattering implies 2-scattering. [HY]

Scrambled set: a set $S \subseteq X$ is called a scrambled set if any pair of different points $\{x, y\} \subseteq S$ is a Li-Yorke pair. [K]

Sensitive dependence on initial conditions (SIC) at a point: $f : X \to X$ is said to be sensitive at $x \in X$ if there exists a $\beta > 0$ such that for every neighborhood N of x there exists a $y \in N, n \geq 0$ such that $\left| f^n(x) - f^n(y) \right| > \beta$.

Sensitive dependence on initial conditions(SIC) on a set: $f : X \to X$ has sensitivity on X if there exists a uniform $\beta > 0$ such that for every $x \in X$ and for every neighborhood N of x there exists a $y \in N, n \geq 0$ such that $\left| f^n(x) - f^n(y) \right| > \beta$. [D]

Separable: A metric space is separable if it contains a countable dense subset. [P]

Topological Transitivity(TT): $f : X \to X$ is TT if for any pair of open sets $U, V \in X$ there exists a $k > 0$ such that $f^k(U) \cap V \neq \varnothing$. [D]

Transitive point: A point $x \in X$ is said to be a transitive point of f, or $x \in \mathrm{Trans}_f(X)$, if x has a dense orbit. If $x \notin \mathrm{Trans}_f(X)$ then x is called an intransitive point, or $x \in \mathrm{Intr}_f(X)$.

Trapping region: A compact region $N \subset X$ is a trapping region for f if $f(N) \subset \mathrm{int}(N)$. [Ro]

Wandering point: An element x of X is a wandering point if there is a neighborhood U of x and an integer N such that, for all $n \geq N$, $f^n(U) \cap U = \varnothing$. If x is not wandering, we call it a nonwandering point $\Omega(f) = \{x \in X : x \text{ is nonwandering}\}$. [Web]

Ch 2. Attractor

Being the Lorenz conjecture stated there was an attractor as one of its assumptions, it was determined to be worthwhile to present some of the more common definitions from the literature and to make comparisons when possible.

Ch 2.1. Definitions

Attractor Definition 1: A subset $A \subset X$ is called an *attractor* if there is a trapping region N such that $A = \bigcap_{n \geq 0} f^n(N)$. [D]

Attractor Definition 2: A subset A of $\Omega(f)$ is an *attractor* provided it is indecomposable and has a neighborhood U such that

$f(U) \subset U$ and $\bigcap_{n>0} f^n(U) = A$, where $\Omega(f)$ is the non-wandering set of *f*. [M]

Attractor Definition 3: A subset $A \subset X$ is called an *attractor* provided there is a trapping region N such that $A = \bigcap_{n>0} f^n(N)$ and $f|_A$ is transitive. [Ro]

Attractor Definition 4: A closed subset $A \subset X$ will be called an *attractor* if the realm of attraction has strictly positive measure and there is no strictly smaller closed set $A' \subsetneq A$ such that $\rho(A')$ has positive measure. [M]

Ch 2.2. Relationships between definitions and examples.

The following section will present the relationships between the definitions when the relationship was readily known. It should be clearly noted that not all relationships were known to the author at this time; however, as will be seen these relationships are not critical to the rest of this paper. The proof of the Lorenz Conjecture in Ch 5.1 will only require that A be invariant, which is common to all four of the above definitions of attractor [M].

Example of Attractor under Definition 1 but not Definition 2: A circle, S, where $S \subseteq \mathbb{R}^2$, that attracts all points in a neighborhood of itself and has an attracting and repelling fixed point within the circle is an example of an attractor according to the first definition since any annulus containing the circle, N, is a trapping region and clearly $\bigcap_{n\geq 0} f^n(N)=S$. However, the presence of an attracting fixed point within the circle forces the circle to not be a part of the non-wandering set, thus not an attractor by definition 2.

Example of Attractor under Definition 4 but not Definition 2: In [M] it was shown that for a map of the interval, $f: I \to I$ where $x \mapsto x^2 - c$ and $c = 1.401155189...$, almost every initial value $x_0 \in I$ converges toward a cantor set denoted A. However, there is a countable infinity of points that fails to converge to A, where these exceptional points are dense in I. Being these exceptional points are dense, any open $U \subset I$ contains some exceptional point and although A is considered an attractor by definition 4, it fails to be an attractor by definition 2.

Definition 3 implies Definition 1: This implication is trivial and is an immediate consequence of the definitions. Furthermore, any example where $f|_A$ is not transitive, such as the identity on a compact set, would show that the reverse implication is not true in general.

Ch 3. Chaos

Similar to the intention of the previous section on the definitions of attractor, this chapter will present a sample of the more widely used notions of chaos and attempt to present how the definitions are related to each other.

Ch 3.1. Dense orbit versus topological transitivity

Before proceeding to the section dealing with chaos, it will be beneficial to the reader to be aware of when the existence of a dense orbit is equivalent to topological transitivity. In

[KS] it was shown that for compact metric spaces with no isolated points, topological transitivity is equivalent to the existence of a dense orbit. So for example, for any compact, connected subset of $\mathbb{R}^n$ the two definitions are equivalent.

Ch 3.2. Definitions

Chaos Definition 1 (Devaney Chaos or DC): A system (X, f) is *Devaney chaotic* if f on X is topologically transitive, has a dense set of periodic points and exhibits sensitive dependence on initial conditions. [D]

Chaos Definition 2 (Li Yorke Chaos or LYC): A system (X, f) is considered *Li-Yorke chaotic* if X contains an uncountable scrambled set. [K]

Chaos Definition 3 (Lorenz Chaos or LC): A system (X, f) will be *Lorenz chaotic* if f on X is topologically transitive and exhibits sensitive dependence on initial conditions over the set.

Note: Definition 3 is equivalent to that used by Wiggins [W] and Robinson [Ro].

Ch 3.3. Relationships between definitions and examples.

It should be pointed out that the above definitions of chaos are by no means exhaustive. The existence of a positive Lyapunov exponent and positive topological entropy are two other widely used notions of a system's chaoticity. Scattering and 2-scattering also appear to be frequently used notions of chaos.

Before continuing to the next section it will be worth pointing out that [S] showed there were redundancies in the three conditions that define DC. First it was shown that, in general, topological transitivity and dense periodic points imply sensitive dependence on initial conditions. It was then shown that for maps of the interval transitivity implied dense periodic points which in turn implied sensitive dependence on initial conditions. These results were also shown in two Math monthly articles in 1992 and 1994 by Banks et all [BBC] and Vellekoop and Berglund [VB], respectively.

Definition 1 implies Definition 2: In order to show that Devaney chaos is a stronger definition of chaos than Li-Yorke chaos, Huang and Ye [HY] showed that if $f: X \to X$ is transitive, X not being finite, and containing a periodic point, then f had an uncountable scrambled set. They started by assuming f had a fixed point denoted by p. Then it was noticed that for each point x in X, the set of points proximal to x is a G_δ subset. If x has a dense orbit then there is an ordering n_i such that $\lim_{n_i \to 0} f^{n_i}(x) = p$ which implies that x and $f^i(x)$, $i = 1, 2, \ldots$, are proximal. Thus the points proximal to each point x with a dense orbit forms a dense G_δ set. Next denote R to be the set of Li-Yorke pairs and A to be the set of points with dense orbits. Then each point that is in A and R contains a dense G_δ subset. Then by Lemma 3.1 in [HY], there is a uncountable subset of X denoted B where $B \times B \setminus \{(x,x) : x \in X\} \subset R$ and B is a scrambled set of f. Next assume that f has a periodic point with period $n > 1$. Let x be a point with a dense orbit. Then $\omega(x, f) = X$. Next set $D_i = \omega(f^i(x), f^n)$ where $0 \le i \le n-1$. Since $f(D_i) = D_{i+1(\mathrm{mod}\, n)}$ it is true that each D_i is uncountable and contains an n-periodic point. Since f^n restricted to D_0 is transitive and contains a fixed point there is an uncountable scrambled set B for f^n. Yet B is also a scrambled set for f. Thus if f contains a periodic point and is transitive it is LYC.

Since [HY] showed that a map that is transitive and contains a single periodic point is LYC, then certainly a map that is DC is LYC being a map that is DC contains a dense set of periodic points. Therefore DC implies LYC.

Definition 1 implies Definition 3: It should be clear from the definitions that DC implies LC since the only variation in the two definitions is that Devaney required the extra condition of the existence of a dense set of periodic points.

Definition 2 does not imply Definition 1: An example presented in [Si] for other purposes shows that LYC does not imply DC and simultaneously shows that LC does not imply DC. To show that LYC does not imply DC, let

$G(r,\theta) = (f(r), R_\alpha(\theta)) = (4r(1-r), \theta+\alpha)$ where α is irrational, $r \in [0,1]$, and $\theta \in S$. Let T be the scrambled set of $G(r,\theta)$. f is known to be Devaney chaotic on $[0,1] \Rightarrow f$ is Li-Yorke chaotic on [0,1] by [HY] $\Leftrightarrow \exists$ an uncountable scrambled set, $S \subset [0,1]$

$\Leftrightarrow \forall r_1, r_2 \in S \ \liminf_{n\to\infty} d(f^n(r_1), f^n(r_2)) = 0$ and $\limsup_{n\to\infty} d(f^n(r_1), f^n(r_2)) > 0$. Since $R_\alpha(\theta)$ is independent of $r \Rightarrow d(G^k(r_1,\theta), G^k(r_2,\theta)) = d(f^k(r_1), f^k(r_2))$. So if

$\liminf_{n\to\infty} d(f^n(r_1), f^n(r_2)) = 0$ then $\liminf_{n\to\infty} d(G^n(r_1,\theta), G^n(r_2,\theta)) = 0$ and similarly if

$\limsup_{n\to\infty} d(f^n(r_1), f^n(r_2)) > 0$ then $\limsup_{n\to\infty} d(G^n(r_1,\theta), G^n(r_2,\theta)) > 0$. Thus if $\{r_i, r_j\}$ is a Li-Yorke pair for f then $\{(r_i,\theta),(r_j,\theta)\}$ is a Li-Yorke pair for $G \Rightarrow S \times \theta \subset T \Rightarrow T$ is uncountable

$\Rightarrow G$ is LYC.

However, since $(0,0)$ is the only periodic point of G it cannot be DC. Thus LYC does not imply DC.

Definition 3 does not imply Definition 1: In [Si] it was shown that $G(r,\theta)=(f(r),R_\alpha(\theta))=(4r(1-r),\theta+\alpha)$ has sensitive dependence on initial conditions and a dense orbit, yet since (0,0) is the only periodic point, fails to have dense periodic points and is therefore LC but not DC.

Neither Definition 2 or Definition 3 is implied by the other: So now we have that DC implies LYC (but not conversely) and DC implies LC (but not conversely). What needs to be addressed is how LYC relates to LC. From [K] it is found that neither LYC implies LC nor LC implies LYC. Kolyada [K] showed that what he called Sturm systems are an example of an almost distal system that has sensitivity but is not LYC. Since all invertible, distinct points are distal, they cannot form Li-Yorke pairs and thus the system cannot be LYC. The points being distal guarantees that they are sensitive. However, as stated by Kolyada [K], 2-scattering systems are LYC and transitive, yet not necessarily sensitive.

Before continuing it should be pointed out that since $G(r,\theta)=(f(r),R_\alpha(\theta))$ $=(4r(1-r),\theta+\alpha)$ is Lorenz chaotic but not Devaney chaotic, as stated above, where $f(r)$ is not one-to-one, then $G(r,\theta)$ provides an example that the Lorenz Conjecture is not true if chaos is considered in the sense of Devaney. Since Lorenz wrote in [L] that he considered chaos to be sensitive dependence on initial conditions, and topological transitivity was implied in [L], chaos will be defined as Lorenz chaos for the continuation of the thesis.

Ch 4. Work already done concerning the Lorenz Conjecture

This section's goal is to illuminate some of the work that had already been done that related to the Lorenz conjecture, which says, given a system (A, f), where f is continuous and $f(A) = A$, if $f|_A$ is transitive and not one-to-one, then f is sensitive on A. All the results below illuminate the important role of transitivity in chaos.

Result 1 (TT + DPP implies SIC): The first result does not have any direct connection to the Lorenz conjecture, although it was one of the earlier results that showed transitivity with some other condition implied sensitive dependence on initial conditions. This perhaps drove others to begin looking at how role transitivity played in producing chaos. In [Si] and [BBC] it was shown that transitivity along with dense periodic points implied sensitive dependence on initial conditions.

Result 2 (On intervals, TT implies DPP and SIC) : The second result relates more to the Lorenz conjecture, but is restricted to an interval of the reals. However, in this special case transitivity is enough to give chaos. In [Si] and [T] it was shown that transitive maps of the interval were chaotic. The idea of the proof from [Si] was to first show that there was a dense set of periodic points, then by the first result the proof would be done. For contradiction, it was assumed there existed an interval containing no periodic points. This gives some maximal interval free of periodic points, implying that all forward images of that interval were also free of periodic points. This ended up forcing a contradiction to the existence of a dense orbit, guaranteeing that transitive maps of the interval are chaotic.

Result 3 (On circles, TT + not 1-1 implies SIC): Along with his proof that a map of the interval being sensitive if it is topologically transitive, Silverman [Si] also proved that if

$f:S^1 \to S^1$ has a dense orbit and f is not one-to-one, then f is chaotic. To do this he began by using the idea of lifts to show that there was a periodic point in the map of the circle given the existence of a dense orbit. Then using arguments similar to the ones from his proof of transitive maps of the interval, he showed that one periodic point implied dense periodic points by a contradiction argument stemming from the assumption that non-dense periodic points led to the contradiction of the assumption of a dense orbit.

Result 4 (For minimal systems, TT + not 1-1 implies SIC): In [AAB] it was shown that given a minimal system, transitivity in conjunction with noninvertibility implied sensitive dependence on initial conditions. Since the conjecture as stated by Lorenz gave no reason to assume there were no periodic points, the minimality condition seemed a little severe, and thus cannot be considered as a proof of the Lorenz conjecture in the most general setting.

Ch 5. My work

The first section of the following chapter will present this author's proof of the Lorenz conjecture. The second section will offer some partial results that attempt to answer what will be known as the SIC conjecture, or in other words, a function that is transitive over a set X and sensitive at a point $x \in X$, then it is sensitive over X.

Ch 5.1. Proof of Lorenz conjecture

The motivation for the work done in this section was to try to extend the results from Ch 4 to prove the Lorenz conjecture in a general metric space setting without assuming minimality of the system.

The proof of the Lorenz conjecture makes use of the following two lemmas. The first lemma is almost trivial in the sense that it will show that if $f : X \to X$ is sensitive at a point, then it is sensitive at any forward iterate of x. The second lemma will show that sensitivity at a transitive point, that is a point with a dense orbit, implies sensitivity over the whole set. The contrapositive of the second lemma will be integral in proving the Lorenz conjecture.

<u>Lemma 1:</u> For $f : X \to X$, where f is continuous, if f has sensitive dependence on initial conditions at $x \in X$, with sensitivity constant β ,then f has sensitive dependence on initial conditions at $f^m(x), m \geq 1$, with sensitivity constant β.

<u>Proof:</u> Let $\varepsilon > 0$ and m be given. Start with $B_\varepsilon(f^m(x))$, where $B_\varepsilon(f^m(x))$ is the ball of radius ε centered at $f^m(x_0)$. Next construct the following series of open neighborhoods.

$$V_1 = f^{-1}(B_\varepsilon(f^m(x))) \cap B_{\beta/2}(f^{m-1}(x))$$

$$\begin{aligned}
V_2 &= f^{-1}(V_1) \cap B_{\beta/2}(f^{m-2}(x)) \\
&\vdots \\
V_i &= f^{-1}(V_{i-1}) \cap B_{\beta/2}(f^{m-i}(x)) \\
&\vdots \\
V_m &= f^{-1}(V_{m-1}) \cap B_{\beta/2}(x)
\end{aligned}$$

Since f has sensitive dependence on initial conditions at x it implies there exists a $y \in V_m$ such that for some n, $d(f^n(x), f^n(y)) > \beta$. By construction, $V_i \subset B_{\beta/2}(f^{m-i}(x)),\ i = 1,...,m$, making $n > m$. Also by design $f^m(y) \in B_\varepsilon(f^m(x))$. Thus we have found a point within ε of $f^m(x)$, namely $f^m(y)$, such that $d(f^{n-m}(f^m(x)), f^{n-m}(f^m(y))) > \beta$. Therefore f has sensitive dependence on initial conditions at $f^m(x), m \geq 1$. □

Lemma 2: For $f: X \to X$, where f is continuous and transitive on A, a closed invariant set, if f has sensitive dependence on initial conditions at $x \in A$, with sensitivity constant β, where the orbit of x, $O(x)$, is dense in A, then f has sensitive dependence on initial conditions on A, with sensitivity constant $\beta / 2$.

Proof: Start by assuming $x \in A$ and $O(x)$ is dense in X. Let $\varepsilon > 0$ and some $y \in A$ be given. Since $O(x)$ is dense it implies there exists an m such that $f^m(x) \in B_{\varepsilon/2}(y)$. By Lemma 1, since f has sensitive dependence on initial conditions at x then f has sensitive dependence on initial conditions at $f^m(x)$ which implies there exists a $z \in B_{\varepsilon/2}(f^m(x))$ (implying $z \in B_\varepsilon(y)$) such that for some n, $d(f^{m+n}(x), f^n(z)) > \beta$. Since $O(y)$ cannot simultaneously stay close to $O(f^m(x))$ and $O(z)$, this implies that the orbit of y separates from either the orbit of $f^m(x)$ or the orbit of z. More formally, the triangle inequality implies $d(f^{m+n}(x), f^n(y)) > \beta / 2$ or $d(f^n(z), f^n(y)) > \beta / 2$. Either way we have found a point within ε of y whose orbit eventually diverges by $\beta / 2$ from $O(y)$. Therefore, since y was arbitrary, f has sensitive dependence on initial conditions at every point in A with the uniform sensitivity constant $\beta / 2$. Thus f is sensitive on A □

Contrapositive of Lemma 2: Let $f: X \to X$, where f is continuous and transitive on A, a closed invariant set. If f is not sensitive on A, then f is not sensitive at any of its transitive points. In other words, if f is not sensitive, then for each transitive point $x \in X$ and any $\varepsilon > 0$ there exists a neighborhood $U(x,\varepsilon)$ of x such that for all $y \in U(x,\varepsilon)$ and all $n \in \mathbb{N}$ $d(f^n(x), f^n(y)) \le \varepsilon$.

<u>Lorenz Conjecture:</u> Given a system (A, f), where f is continuous and $f(A) = A$, if $f|_A$ is transitive and not one-to-one, then f is sensitive on A.

<u>Proof of Lorenz Conjecture:</u> Let $p_1 \ne p_2$ be given such that $f(p_1) = f(p_2) = p$. Choose ε such that $d(p_1, p_2)/2 > \varepsilon$. For the sake of contradiction, assume f is not sensitive on A, then by the contrapositive of Lemma 2 there exists a transitive point $x_0 \in A$ and a neighborhood $U(x_0, \varepsilon/3)$ of x_0 such that $d(f^n(x_0), f^n(y)) \le \varepsilon/3$ for all $y \in U(x_0, \varepsilon/3)$ and all $n \in \mathbb{N}$. Since x_0 has a dense orbit, there exists an $m \ge 1$ such that $f^m(x_0) \in U(x, \varepsilon/3)$. This implies $d(f^n(x_0), f^{n+m}(x_0)) \le \varepsilon/3$ for all $n \in \mathbb{N}$.

<u>Claim:</u> For this m and any $z \in A$, $d(z, f^m(z)) \le \varepsilon$.

<u>Proof:</u> Assume not. Label the following distances:

$$\begin{aligned} a &= d(z, f^{n_z}(x_0)) \\ b &= d(f^{n_z}(x_0), f^{n_z+m}(x_0)) \\ c &= d(f^{n_z+m}(x_0), f^m(z)) \\ d &= d(z, f^m(z)) \end{aligned}$$

for some n_z that will be assigned below.

First note that the triangle inequality implies $d \le a+b+c$, or equivalently, $d-a-c \le b$, where it is known that $b \le \varepsilon/3$ and it is assumed that $d > \varepsilon$. To force the contradiction it is sufficient that $a, c \le \varepsilon/3$, so that $d-a-c > \varepsilon/3$ contradicting $d-a-c \le b \le \varepsilon/3$. First by continuity there exists a $\delta > 0$ such that if $x \in B_\delta(z)$, then $f^m(x) \in B_{\varepsilon/3}(f^m(z))$ which would assure that $c \le \varepsilon/3$ once it is shown that $f^{n_z}(x_0) \in B_\delta(z)$. To have $a \le \varepsilon/3$, n_z would need to be chosen such that $f^{n_z}(x_0) \in B_\gamma(z)$ where $\gamma = \min\{\varepsilon/3, \delta\}$. Since x_0 has a dense orbit, there is such a n_z, where $f^{n_z}(x_0) \in B_\gamma(z)$. Since $\gamma \le \varepsilon/3$, then $f^{n_z+m}(x_0) \in B_{\varepsilon/3}(f^m(z))$. Therefore, if $d > \varepsilon$, then $d - a - c > \varepsilon/3$ which contradicts $d - a - c \le b \le \varepsilon/3$. □

Since, by the above claim, $d(f^m(z), z) \le \varepsilon$ for any $z \in A$, then $d(f^m(p_1), p_1) \le \varepsilon$ and $d(f^m(p_2), p_2) \le \varepsilon$. Yet since $f(p_1) = f(p_2)$, then $f^m(p_1) = f^m(p_2)$ which implies $d(p_1, p_2) \le 2\varepsilon$. This contradicts the fact that ε was chosen such that $d(p_1, p_2) > 2\varepsilon$. Therefore f must be sensitive on A. □

Before proceeding to the next section, it was deemed important to point out that the contrapositive of Lemma 2 in this paper is similar to the Lemma 1.1 in [GW], yet is in fact a stronger result. [GW] showed that a system being not sensitive was equivalent to the existence of some transitive point that was not sensitive; whereas, the contrapositive to Lemma 2 shows that a system being not sensitive is equivalent to all transitive points being not sensitive. Lemma 1.2 from [GW] would have been sufficient to complete the proof of the Lorenz conjecture, but it was decided that a more direct proof from this author was to be desired over merely copying the results of another author. Furthermore, [GW] used Lemma 1.1 to prove Lemma 1.2 that showed if a system was not sensitive then it was uniformly rigid. Within their proof of Lemma 1.2 one can find the result of the Claim made within the proof of the Lorenz conjecture, although this author added details that were missing from [GW].

Ch 5.2. Partial results on SIC Conjecture

One question that frequently surfaced during the research process into the Lorenz conjecture was whether, given a topologically transitive system (X, f), sensitive dependence on initial conditions at a point $x \in X$ meant that f was sensitive for every point in X. Since there was no reference to this that could be found in the literature, the author has taken it upon himself to try and address what is now being called the SIC Conjecture, or transitivity and point sensitivity imply set sensitivity . The following two theorems make some progress toward answering the SIC Conjecture. The first theorem will show that all intransitive points are sensitive and the second theorem will show that any accumulation point of periodic points is sensitive.

Theorem 1: If $f : A \rightarrow A$ is continuous and transitive on A (a closed, invariant subset of $\mathbb{R}^n$), then any point $x \notin Trans_f(A)$, where $Trans_f(A)$ is the set of points in A whose orbits are dense, then f has sensitive dependence on initial conditions at x.

Proof: Let $x \in A$ such that $x \notin Trans_f(A)$ be given. Let $\varepsilon > 0$ be given. Since f is transitive on A there exists $y \in Trans_f(A)$ such that $y \in B_\varepsilon(x)$. Since $x \notin Trans_f(A)$ this implies there exists a $\beta_0 > 0$ and a $z \in A$ such that $O(x) \cap B_{\beta_0}(z) = \varnothing$, $O(x)$ being the forward orbit of x. Yet $y \in Trans_f(A)$ implies there exists an $n \in \mathbb{N}$ such that $d(f^n(y), z) \leq \beta_0 / 50$. However, $O(x) \cap B_{\beta_0}(z) = \varnothing$ implies $d(f^n(y), f^n(x)) > 49\beta_0 / 50 \equiv \beta$. Therefore, given $\varepsilon > 0$, y and n have been found such that $d(x, y) < \varepsilon$ yet $d(f^n(x), f^n(y)) > \beta$. So f has sensitive dependence on initial conditions at x. □

The proof of the next theorem mimics the proof used in [BBC] to show that transitivity and dense periodic points imply sensitive dependence on initial conditions to show that transitivity and an accumulation point of periodic points implies sensitive dependence on initial conditions at the accumulation point. The following theorem could prove useful if it were known that a transitive point was an accumulation point of periodic points.

Theorem 2: If $f : A \to A$ is continuous and transitive on A (a closed, invariant subset of $\mathbb{R}^n$), and p is an accumulation point of periodic points, then f has sensitive dependence on initial conditions at p.

Proof: First notice that there is a $\beta_0 > 0$ such that there is a periodic point q whose orbit O(q) is at least $\beta_0 / 2$ from p. In fact, arbitrarily pick two periodic points q_1, q_2 with disjoint orbits $O(q_1), O(q_2)$ and letting β_0 denote the "distance between the orbits", then by the triangle inequality, p is at least a distance of $\beta_0 / 2$ from one of the two orbits. Next it will be shown that f has sensitive dependence on initial conditions at p with sensitivity constant $\beta = \beta_0 / 8$.

Let N be some neighborhood of p. Since p is an accumulations point of periodic points, there exists a periodic point q^* in the intersection $U = N \cap B_\beta(p)$. Let the period of q^* be denoted as n. As was shown above, there exists a point q whose orbit O(q) is of distance at least $4\beta = \beta_0 / 2$ from p_1. Set $V = \bigcap_{i=0}^{n} f^{-i}(B_\beta(f^i(q)))$. It should be clear that V is both open and nonempty. Since f is topologically transitive, there exists an x in U and a natural number k such that $f^k(x) \in V$. Let j be the integer part of $k / n + 1$ so that

$1 \le nj - k \le n$ so by design one has $f^{nj}(x) = f^{nj-k}(f^k(x)) \in f^{nj-k}(V) \in B_\beta(f^{nj-k}(q))$. Now since $f^{nj}(q^*) = q^*$, one has by the triangle inequality, $d(f^{nj}(q^*), f^{nj}(x)) =$ $d(q^*, f^{nj}(x)) \ge d(p, f^{nj-k}(q)) - d(f^{nj}(x), f^{nj-k}(q)) - d(p, q^*)$, given that d is some distance function on A. Since $q^* \in B_\beta(p_1)$ and $f^{nj}(x) \in B_\beta(f^{nj-k}(q))$ one has $d(f^{nj}(q^*), f^{nj}(x)) >$ $4\beta - \beta - \beta = 2\beta$. Therefore, using the triangle inequality again, either $d(f^{nj}(p), f^{nj}(x)) > \beta$ or $d(f^{nj}(p), f^{nj}(q^*)) > \beta$. Either way this has shown there exists a point in N such that the distance of its nj^{th} iterate is more than β from the nj^{th} iterate of p. Thus f has sensitive dependence on initial conditions at p with sensitivity constant $\beta = \beta_0 / 8$. □

Before moving on to the summary, it should be stated what has been accomplished by the previous two theorems. The first theorem might prove to be the most useful along with Lemma 2 from Ch 5.1. In [KS] it was shown that either a system will have no intransitive points, meaning the system is minimal, or there will be a dense set of intransitive points. If the system is indeed minimal then every point is a transitive point which means that a sensitive point implies sensitivity over the set. If not, then Theorem 1 guarantees there is a dense set of sensitive points. It seems likely that a dense set of intransitive points, along with topological transitivity, will force the system to be sensitive using a similar approach that showed a dense set of periodic points, along with topological transitivity, forced the system to be chaotic. Unfortunately this is not completely resolved.

Ch 6. Summary

In summary, Lorenz [L] looked at Euler approximations of systems of ODEs with large time steps. His numerics suggested that continuous, noninvertible maps of attractors were chaotic. Lorenz's use of the words attractor and chaos prompted this author to investigate various common definitions from the literature of those two terms to determine whether certain definitions would allow for the proof of the Lorenz conjecture as opposed to others. Although the relationships between the four definitions of attractor remain a little muddled, the truly important feature of attractor that was required for the proof of the Lorenz Conjecture was that the set be f -invariant. Since all four definitions contain this property, it did not matter which definition one considered when looking at the Lorenz Conjecture.

It was found that *Lorenz chaos* and *Li-Yorke chaos* were both implied by *Devaney chaos* with no reverse implication, while neither LYC or LC were implied by the other. As mentioned at the end of Ch 3.3, the example that showed that LC and LYC does not imply DC further showed that the Lorenz Conjecture was not true when using *Devaney chaos* as the definition of chaos. Since Lorenz stated that by chaos he meant SIC and TT was implied, LC was adopted as the definition used for proving the Lorenz Conjecture. Although the SIC conjecture has not been proved, it true in the case that the system is minimal, and is likely to be true in general, since one intransitive point implies the existence of a dense set of intransitive points, all of which are sensitive.

It was shown that there were numerous results from the literature ([Si,][BBC],[T],[AAB]) that ended up being special cases of the Lorenz Conjecture. Furthermore, it was stated that the Lorenz Conjecture followed from the addition of a few lines to the results of [GW]. The Lorenz Conjecture was thus shown to be true in a general metric space.

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Appendix A - List of Results from Kolyada (2004)

Theorem 1. Let (X,T) be a topologically transitive system. Then exactly one of the following cases takes place:

1. There exists an equicontinuity point for the system. In this case, the equicontinuity points are transitivity points, i.e., $Eq(T) = Trans(T)$, and the system is almost equicontinuous. The map T is a homeomorphism, and the inverse system (X,T^{-1}) is almost equicontinuous. Moreover, the system is uniformly rigid, i.e., there exists a subsequence $\{T^n : n = 0,1,...\}$ that converges to the identity map.
2. The system does not have equicontinuity points. In this case, the system is sensitive, i.e., there exists $\varepsilon > 0$ such that $Eq_\varepsilon(T) = \varnothing$.

Corollary 1. If (X,T) is a minimal dynamical system, then it is either sensitive or equicontinuous.

Corollary 2. If (X,T) is an almost equicontinuous transitive system, then all its asymptotic pairs are diagonal, i.e., $Asym(T) = \Delta$.

Theorem 2. If a topological dynamical system (X,T) has positive topological entropy, then it is Li-Yorke chaotic.

Theorem 3. For any dynamical system (X,T) the following conditions are equivalent:

1. The system is sensitive
2. There exists a positive ε such that $Asym_\varepsilon(T)$ is a set of the first category in $X \times X$.
3. There exists a positive ε such that, for any $x \in X$, $Asym_\varepsilon(T)(x)$ is a set of the first category in X.
4. There exists a positive ε such that any $x \in X$ is a limit point of the complement of $Asym_\varepsilon(T)(x)$, i.e., $x \in \overline{X \setminus Asym_\varepsilon(T)(x)}$.
5. There exists a positive ε such that the set of pairs $\{(x,y) \in X \times X : \limsup_{n\to\infty} \rho(T^n(x),T^n(y)) > \varepsilon\}$ is everywhere dense in $X \times X$.

Theorem 4. If a dynamical system (X,T) is Li-Yorke sensitive, then it is sensitive. If (X,T) is sensitive and, for any point $x \in X$, the proximal cell $\mathrm{Prox}(T)(x)$ is everywhere dense in X, then (X,T) is Li-Yorke sensitive.

Theorem 5. If (X,T) is a weakly mixing dynamical system, then, for all $x \in X$, the proximal cell $\mathrm{Prox}(T)(x)$ is an everywhere dense (residual) subset in X.

Corollary 3. Any nontrivial weakly mixing system (X,T) is Li-Yorke sensitive.

Theorem 6. For a minimal dynamical system (X,T), the following conditions are equivalent:

1. The system (X,T) is weakly mixing.
2. For any $x \in X$, the proximal cell $\operatorname{Prox}(T)(x)$ is everywhere dense in X.
3. For some $x \in X$, the proximal cell $\operatorname{Prox}(T)(x)$ is everywhere dense in X.
4. $\operatorname{Prox}(T)$ is everywhere dense in $X \times X$.

Theorem 7. If a system (X,T) is sensitive (or Li-Yorke sensitive), then the product system $(X \times Y, T \times S)$ is sensitive (respectively, Li-Yorke sensitive) for any dynamical system (Y,S).

Theorem 8. Suppose that a dynamical system (X,T) satisfies the following conditions:

1. the system is infinite and transitive;
2. any point is recurrent;
3. any minimal point is periodic.

Then the system is space-time chaotic.

Corollary 4. Suppose that, for a nonminimal transitive dynamical system (X,T), any point is either transitive or periodic. Then the system is space-time chaotic.

Corollary 5. Suppose that a nonminimal transitive dynamical system (X,T) is almost equicontinuous and all its minimal points are periodic. Then the system is space-time chaotic but not Li-Yorke sensitive.

Appendix B - List of results from Silverman (1992)

Proposition (1.1). Let M be a perfect (has no isolated points). Then DO implies TT. Furthermore, if M is separable and second category, then TT implies DO.

Theorem (2.1). If M is infinite, then DO and DPP imply SIC.

Corollary (2.2). If $f \sim g$ and f is chaotic, then g is chaotic.

Corollary (2.3). The determination of whether f is chaotic or not depends only on the topology of M and not on the metric.

Lemma (3.1). Let M be an arbitrary subinterval of $\mathbb{R}$ and let $f : M \to M$ have a dense orbit. If $(a,b) \subset M$ is free of periodic points, then so is $f^j(a,b)$ for all $j \geq 0$.

Theorem (3.2). If M is a subinterval of $\mathbb{R}$, then DO implies DPP and SIC.

Lemma (6.1). Let $f : S^1 \to S^1$ have a dense orbit. If $(a,b) \subset S^1$ is free of periodic points, then so is $f^j(a,b)$ for all $j \geq 0$.

Lemma (6.2). If $f : S^1 \to S^1$ has a dense orbit and F is a lift of f with $F(c) = F(d), 0 \leq c < d < 1$, then f has a periodic point in $[c,d]$.

Theorem (7.1). If $f : S^1 \to S^1$ has a dense orbit, then any of the following are equivalent to f being chaotic:

a) f has a periodic point
b) f is not one-to-one
c) f is sensitive to initial conditions
d) f has a non-dense orbit

Proposition (7.2). A homeomorphism with a dense orbit and no periodic points is conjugate to R_α.

(8.4) A map with a dense orbit and no periodic points is conjugate to R_α.

Appendix C - Other useful results

Results from [AAB] (1996)

Theorem 1.4. If a compact dynamical system is transitive, but not minimal, then the set of intransitive points is dense.

Theorem 2.4. Let (X, f) be topologically transitive, if (X, f) is almost equicontinuous then the set of equicontinuity points coincides with the set of transitive points (and so the set of equicontinuity points is a dense G_δ. In particular, a minimal almost equicontinuous dynamical system is equicontinuous. If (X, f) has no equicontinuity points then it is sensitive. In particular, a minimal system is either equicontinuous or sensitive.

Results from [GW] (1993)

Lemma 1.1. For a topologically transitive system (X, T) with no isolated points, being ~S is equivalent to the following property: For every $\varepsilon > 0$ there exists a transitive point $x_0 \in X$ and a neighborhood U of x_0 such that for every $y \in U$ and every $n \in \mathbb{N}$, $d(T^n x_0, T^n y) \le \varepsilon$.

Lemma 1.2. A topologically transitive system without isolated points which is not sensitive is uniformly rigid.

Results from [HY] (2002)

Theorem 4.1. Assume that $f : X \to X$ is transitive with X infinite and contains a periodic point. Then there is an uncountable scrambled set for f . Moreover, if f is totally transitive, then f is densely Li-Yorke chaotic. Particularly, chaos in the sense of Devaney is stronger than that in the sense of Li-Yorke.

Results from [KS] (1997)

Theorem 4.3.1. Let (X, f) have no isolated point. Then the set $\mathrm{intr}_f(X)$, the set of intransitive points, is either empty or dense in X .