The Cusp-cusp Bifurcation
for noninvertible maps of the plane

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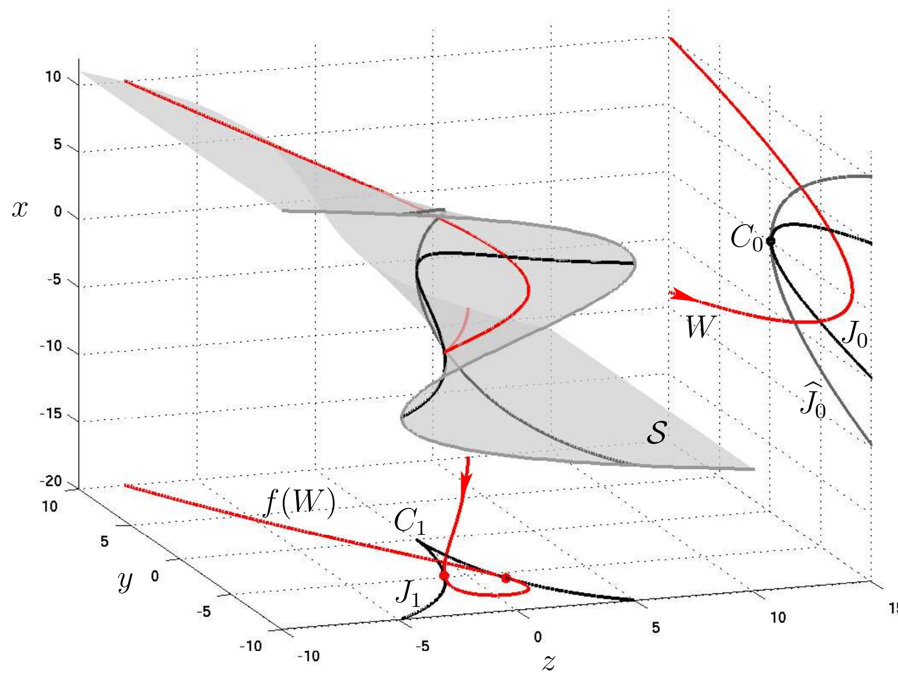


Midwest Dynamical Systems Conference, October 21, 2007, The University of Michigan, Ann Arbor

Talk Outline

1. Background on noninvertible maps of the plane: some features and bifurcations
2. **The codimension-two cusp-cusp bifurcation**
   (a) The two cusps
   (b) The normal form of the model
   (c) Definition of equivalence
   (d) Analysis of the model: five codimension-one bifurcations, eight generic cases
3. An adaptive control application
4. Summary

Some research groups in noninvertible maps of the plane

- Gumowski and Mira 1980 book *Recurrences and Discrete Dynamic Systems*

$$(x, y) \mapsto (ax + y, b + x^2)$$

- Christian Mira "group": Laura Gardini, Millerioux, Barugola, Cathala, Carcasses, Ralph Abraham. Several books and lots of articles.
- Yannis Kevrekidis "group": Christos Frouzakis, Ray Adomaitis, Rafael de Llave, Rico-Martinez, BP
- Dick McGehee group: Evelyn Sander, Josh Nien, Rick Wicklin, ...
- Lorenz (1989 paper):

$$(x, y) \mapsto ((1 - a\tau)x - \tau xy, (1 - \tau)y + \tau x^2)$$

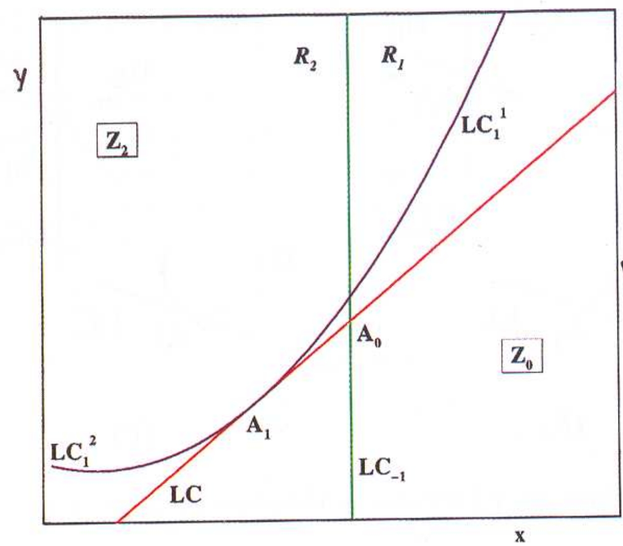
- Bristol Group: Bernd Krauskopf, Hinke Osinge, James England, BP, ...
- V. Maistrenko and Y. Maistrenko

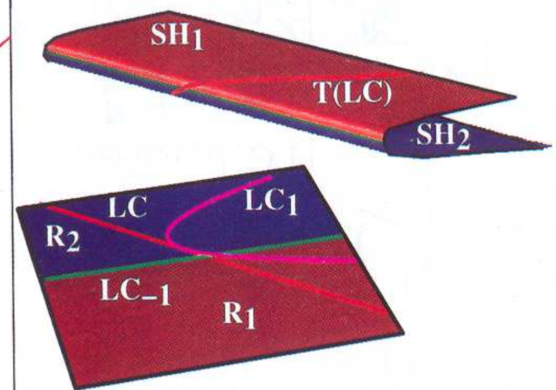
A good introductory reference to Noninvertible maps of the plane: *On some properties of invariant sets of two-dimensional noninvertible maps*, Frouzakis, Gardini, Kevrekidis, Millerioux, and Mira, IJBC, Vol 7, No. 6, (1997), 1167-1194.

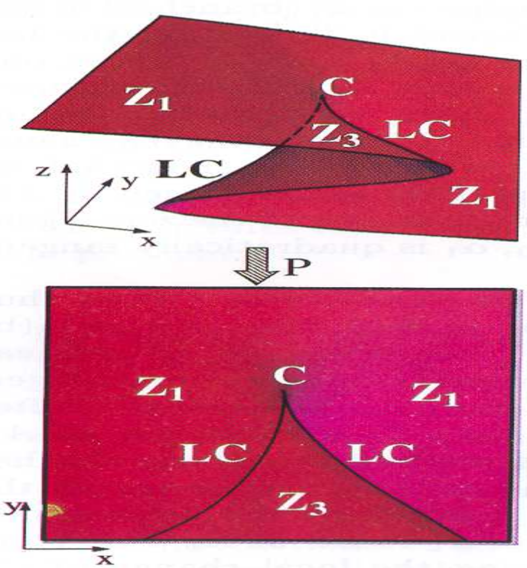
Features and bifurcations unique to NONINVERTIBLE maps of the plane

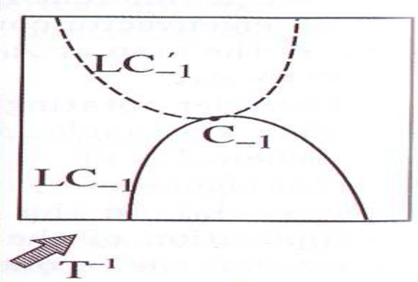
Some features unique to *noninvertible* maps of the plane

- Critical curves: $J_0(\approx LC_{-1})$, $J_1(\approx LC)$
  $J_0 = \{x : \det(DF(x)) = 0\}$; $J_1 = F(J_0)$
- Folding of phase space: $Z_0 - Z_2$, $Z_1 < Z_3$, ... Figs from FGKMM (1997).







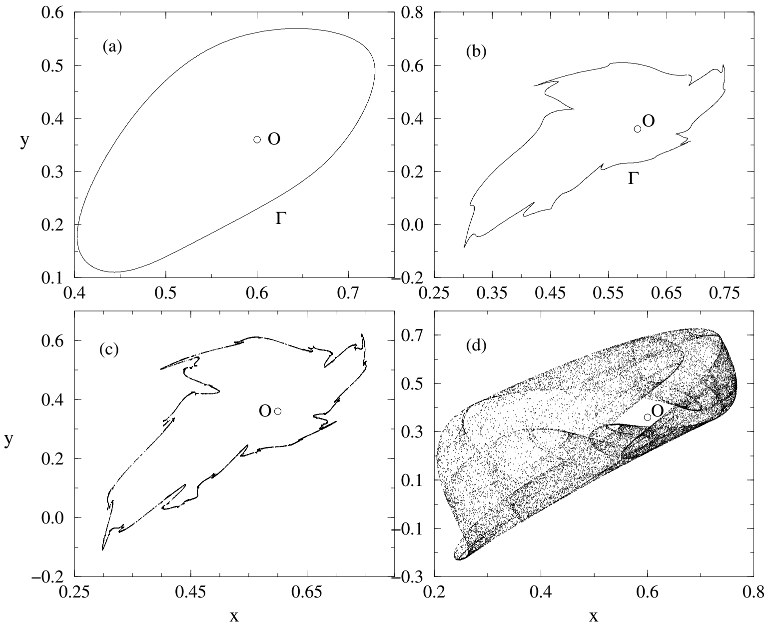


Some features unique to *noninvertible* maps of the plane (cont.)

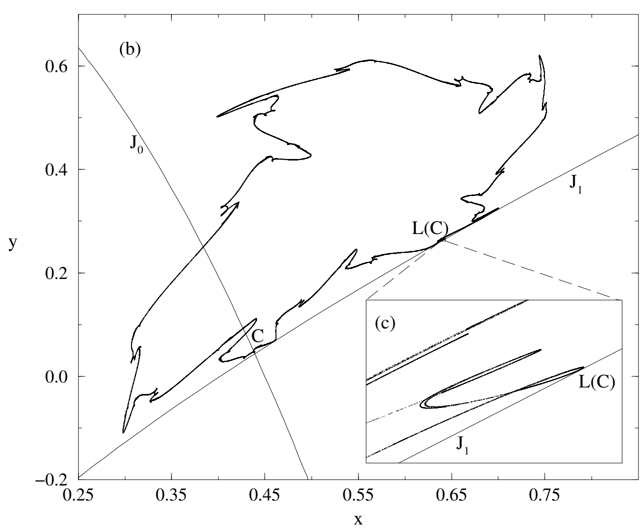
- Interaction of critical curves with
  - fixed/periodic points (eigenvalue zero)
  - Unstable manifolds (*outsets*): self-intersections, loops, cusps
  - Stable manifolds (*insets*): disconnected, allowing disconnected and multiply connected basins of attraction
  - Invariant circles
- Chaotic attractors

Example: Lorenz (1989)

$$(x, y) \mapsto ((1 - a\tau)x - \tau xy, (1 - \tau)y + \tau x^2)$$



Frouzakis, Kevrekidis, P, 2003



Some bifurcations unique to *noninvertible* maps of the plane

- Codimension-one Interactions of J_0 with

  1. fixed/periodic points - transition from orientation preserving to orientation reversing,
  2. creating self-intersections of W^u via tangency (Mira: *contact bifurcation*) or loop formation
  3. creating disconnected or multiply connected basin boundaries
  4. breakup of an invariant circle (loops on what "used to be" the smooth invariant curve - Lorenz)
  5. creating intersections of basin boundaries with J_0

- Codimension-two

  1. (0,1) eigenvalues (Josh Nien thesis with Dick McGehee - 1997) (Unfoldings not finitely determined.)
  2. Sander (2000) A transverse homoclinic point with no tangle
  3. The Cusp-cusp bifurcation

Global bifurcation diagrams. Ex. Mira 1991. Note to Kevrekidis.

Zero-order draft for preparing a study of the bifurcation plane of a dim2 (z_0-z_2) map [see p.4]
The study is made from the bif. curves obtained numerically

BIFURCATIONS STRUCTURE FOR QUADRATIC SECOND ORDER ENDOMORPHISMS

Due to other tasks this study was abandoned in 1992

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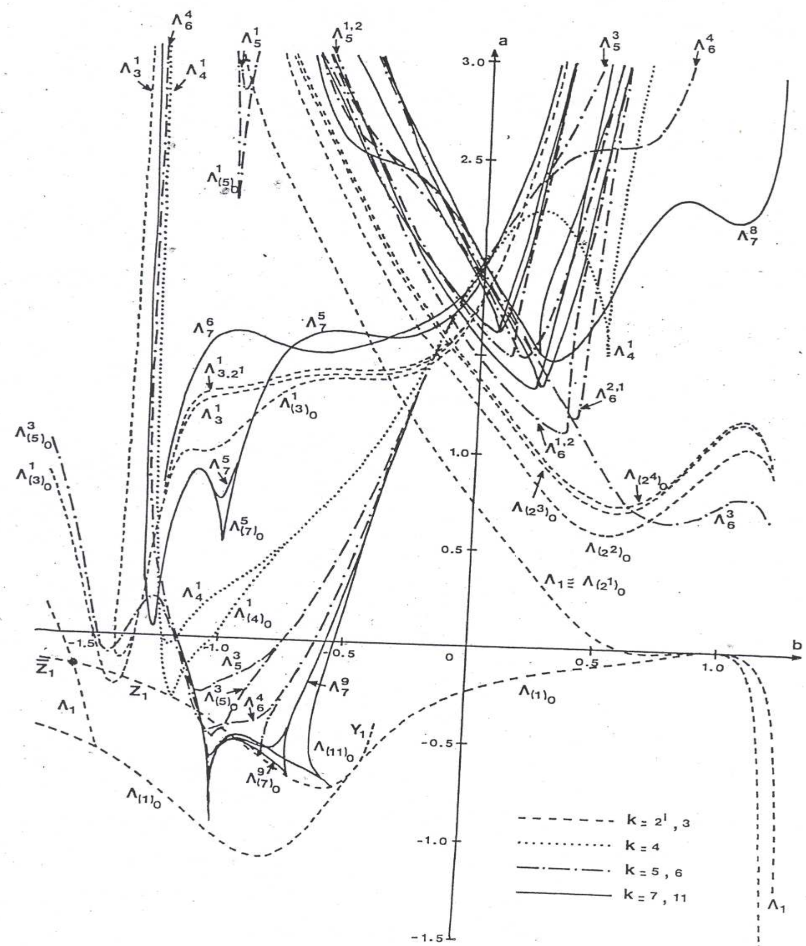
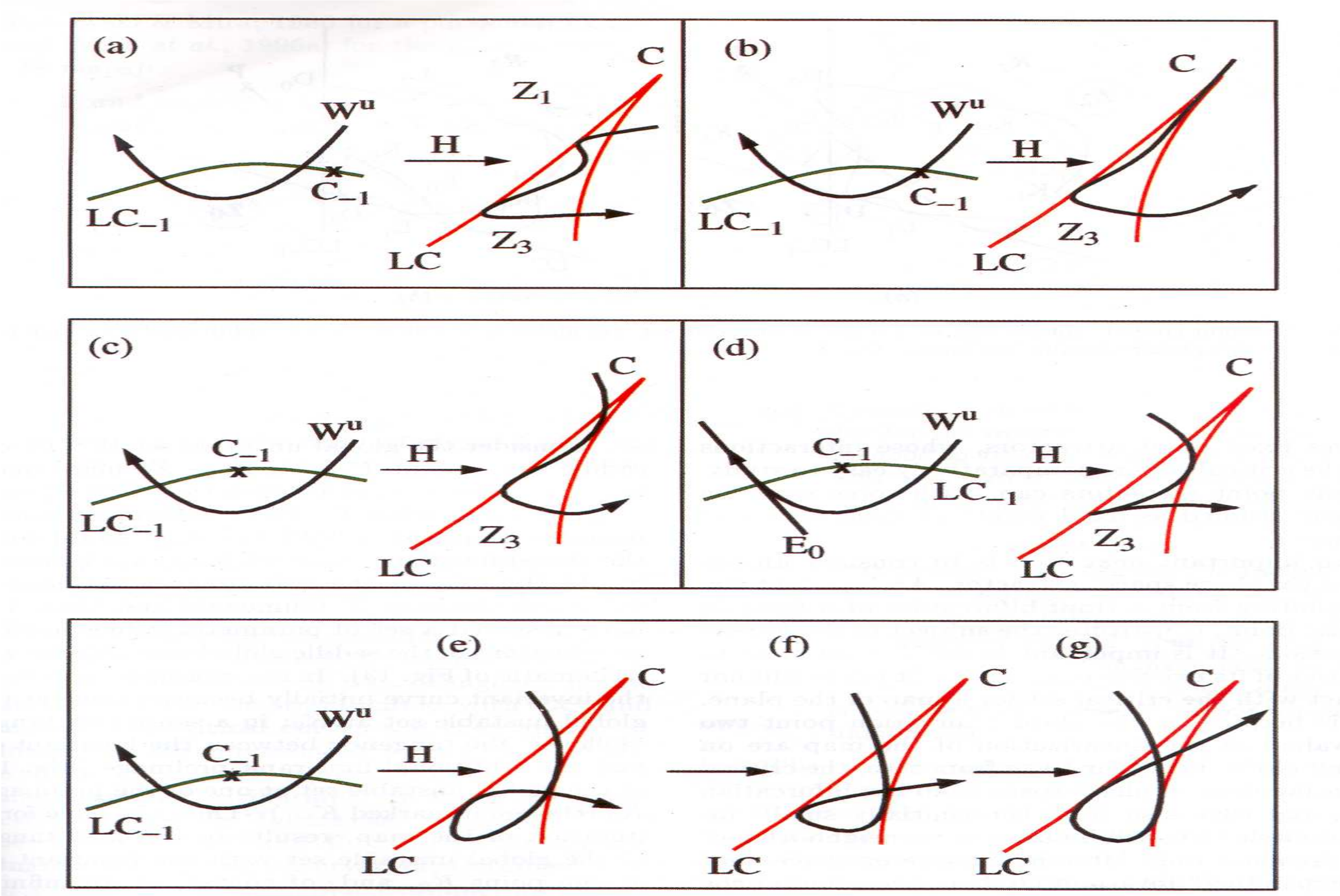


Fig . 1

The CUSP-CUSP bifurcation

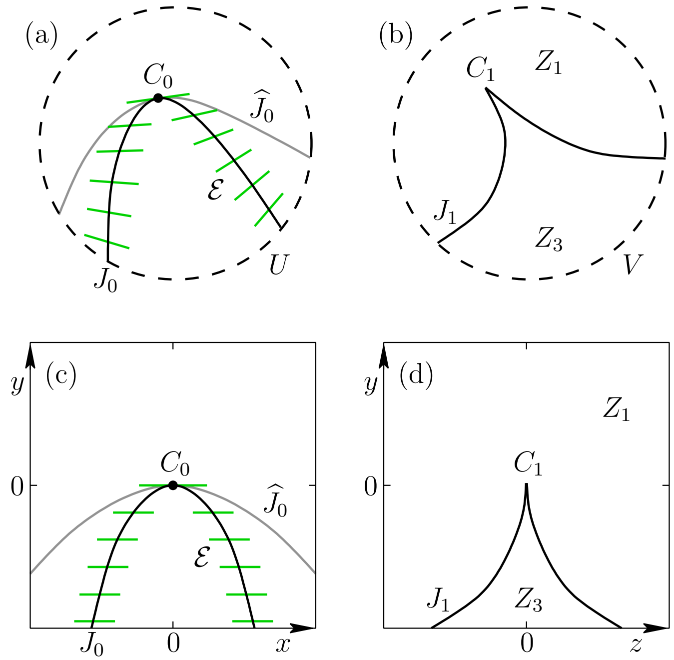
Motivation from FGKMM (1997): "Interactions in the neighborhood of a cusp point"



The first cusp - along the phase space fold

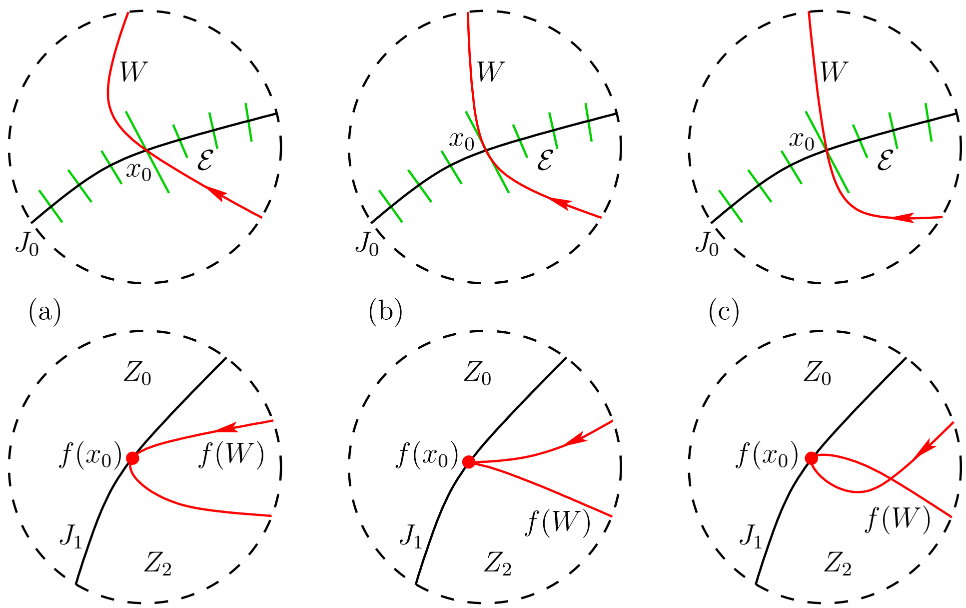
The Cusp point C_0 and its image C_1. Persists under perturbation. The green *line field* denotes the direction of the zero eigenspaces at points of J_0.

(a) and (b) General position; (c) and (d) normal form position



The second cusp - on the image of W

Cusps on images of curves which cross J_0 tangent to the line field. Codimension-one occurrence.



Assume W is described locally by $\alpha(t)$ with $\alpha(t_0) = x_0$ and tangent vector $\alpha'(t)$. At $f(x_0)$, W is described locally by $f(\alpha(t))$, and has tangent vector $Df(x_0).\alpha'(t)$, which implies W is still smooth at $f(x_0)$ unlees $\alpha'(t)$ is an eigenvector for eigenvalue zero at x_0.

The normal forms of the cusp map and W (cont)

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} -x^3 - 3xy \\ y \end{pmatrix}. \tag{1}$$

$$DF \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -3x^2 - 3y & -3x \\ 0 & 1 \end{bmatrix}$$

is singular along the particularly simple critical curve

$$J_0 := \{y = -x^2\}. \tag{2}$$

The image of the parabola J_0 is the standard cusp

$$J_1 := \{z = \pm 2(\sqrt{-y})^3 \mid y \leq 0\}. \tag{3}$$

A straightforward calculation shows that it has the second pre-image

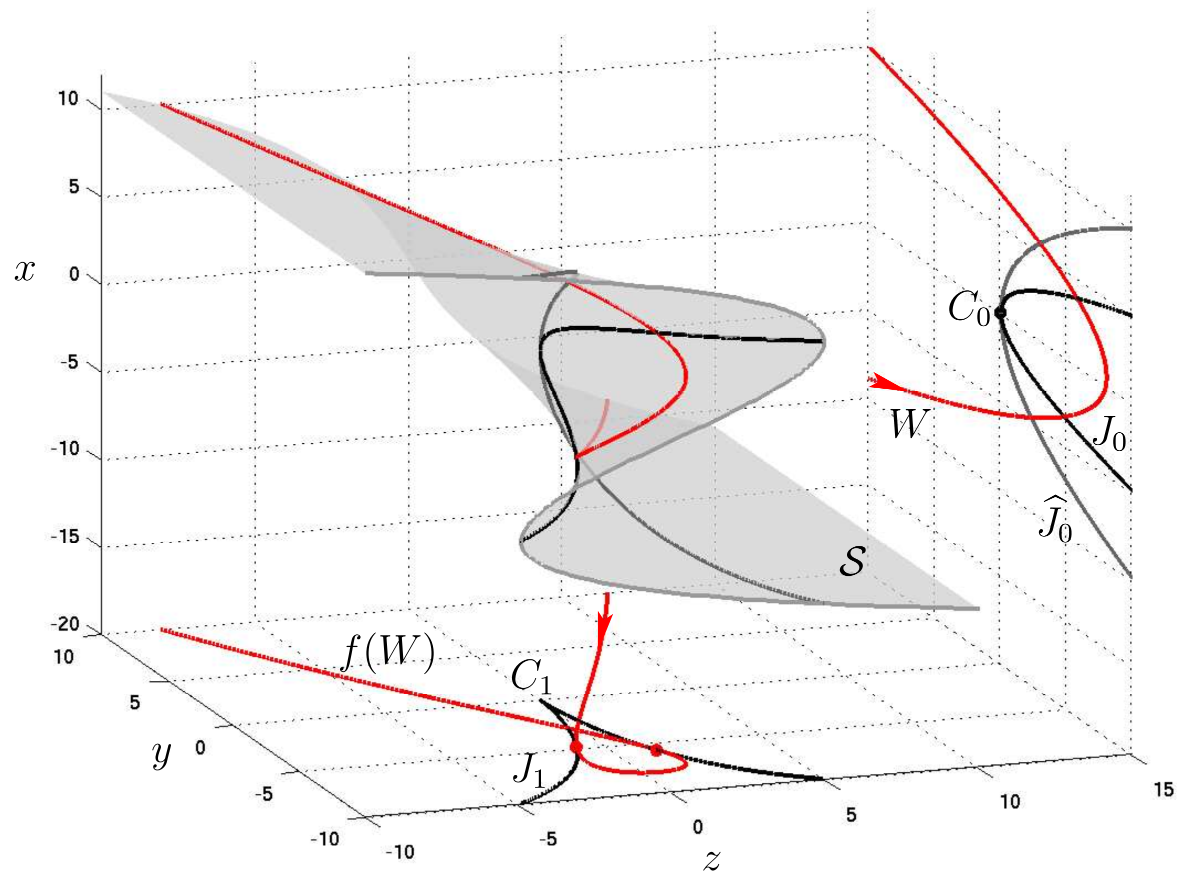
$$\widehat{J}_0 := \{y = -\tfrac{1}{4}x^2\}. \tag{4}$$

The zero-eigenvector line field is horizontal (independent of $x \in J_0$).

Normal form (truncated) of W:

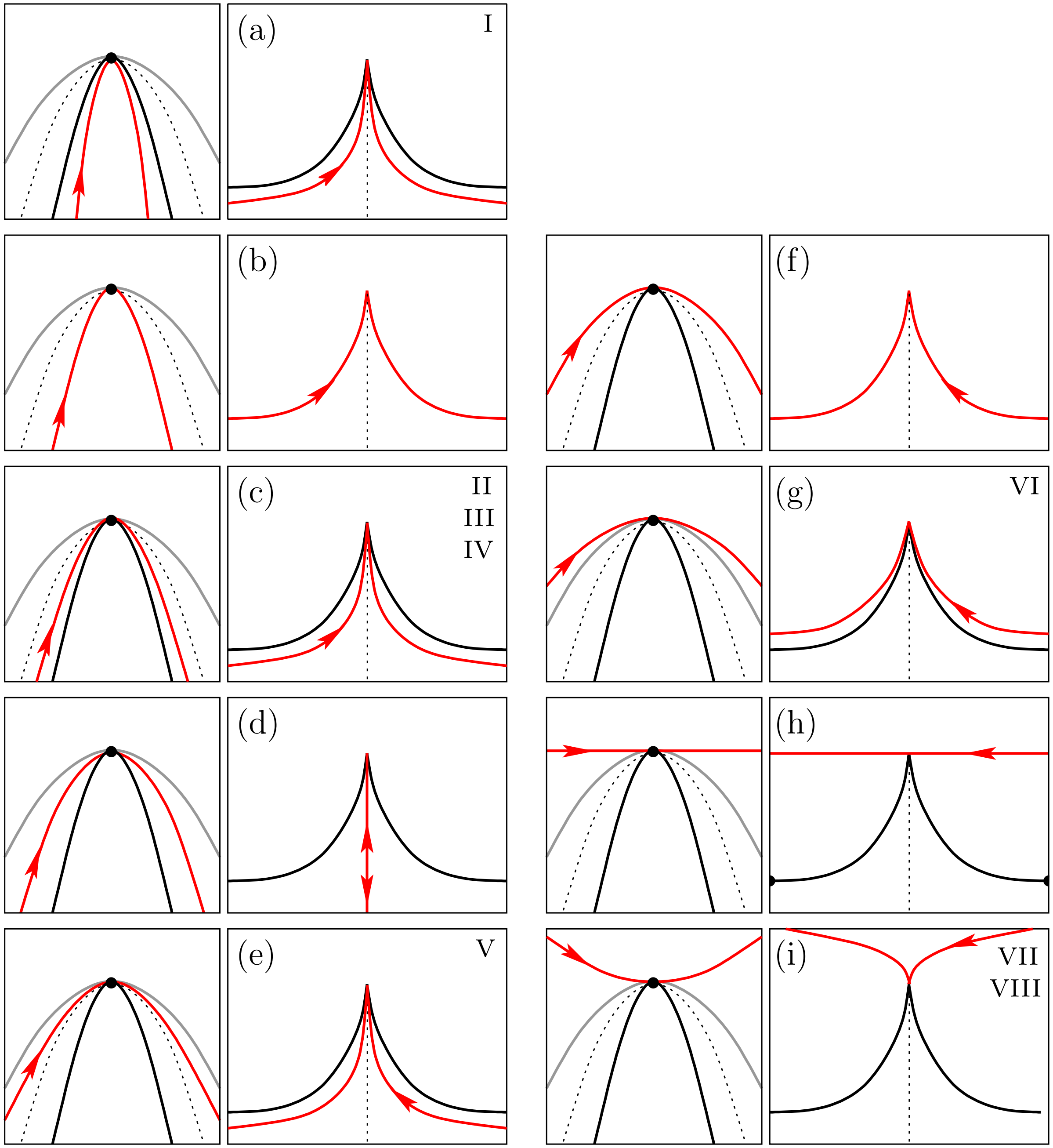
$$W := \{y = \gamma(x - a)^2 + b\}. \tag{5}$$

The projection of the graph of the normal forms. The cusp maps is fixed; $W = \{\gamma(x-a)^2 + b\}$ varies. The parameters a and b are the pimary parameters while γ is a secondary parameter. We construct bifurcation diagrams in the (a, b) plane for fixed γ. There turn out to be eight generic cases, depending on γ.

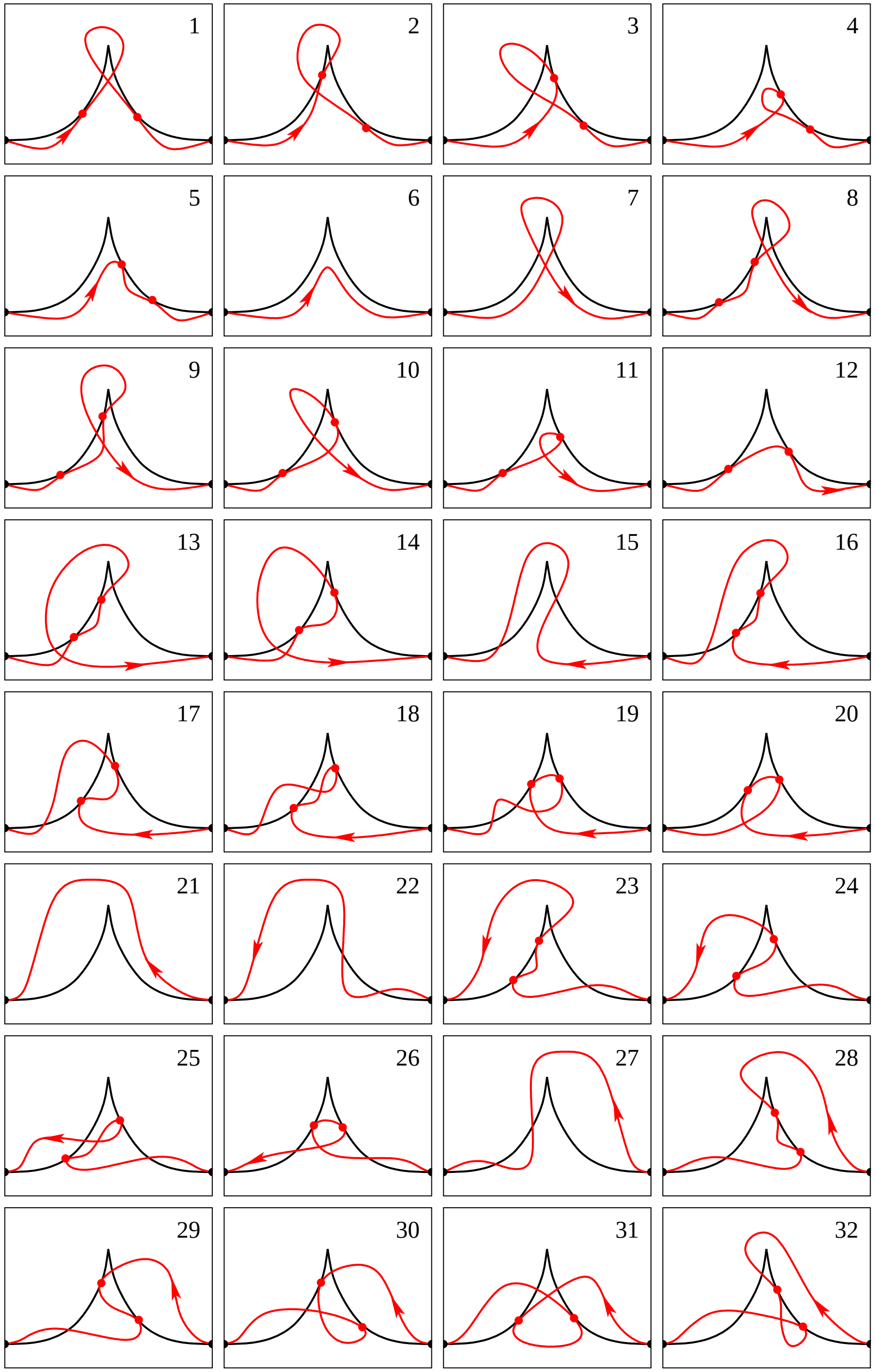


The organizing center for different values of γ

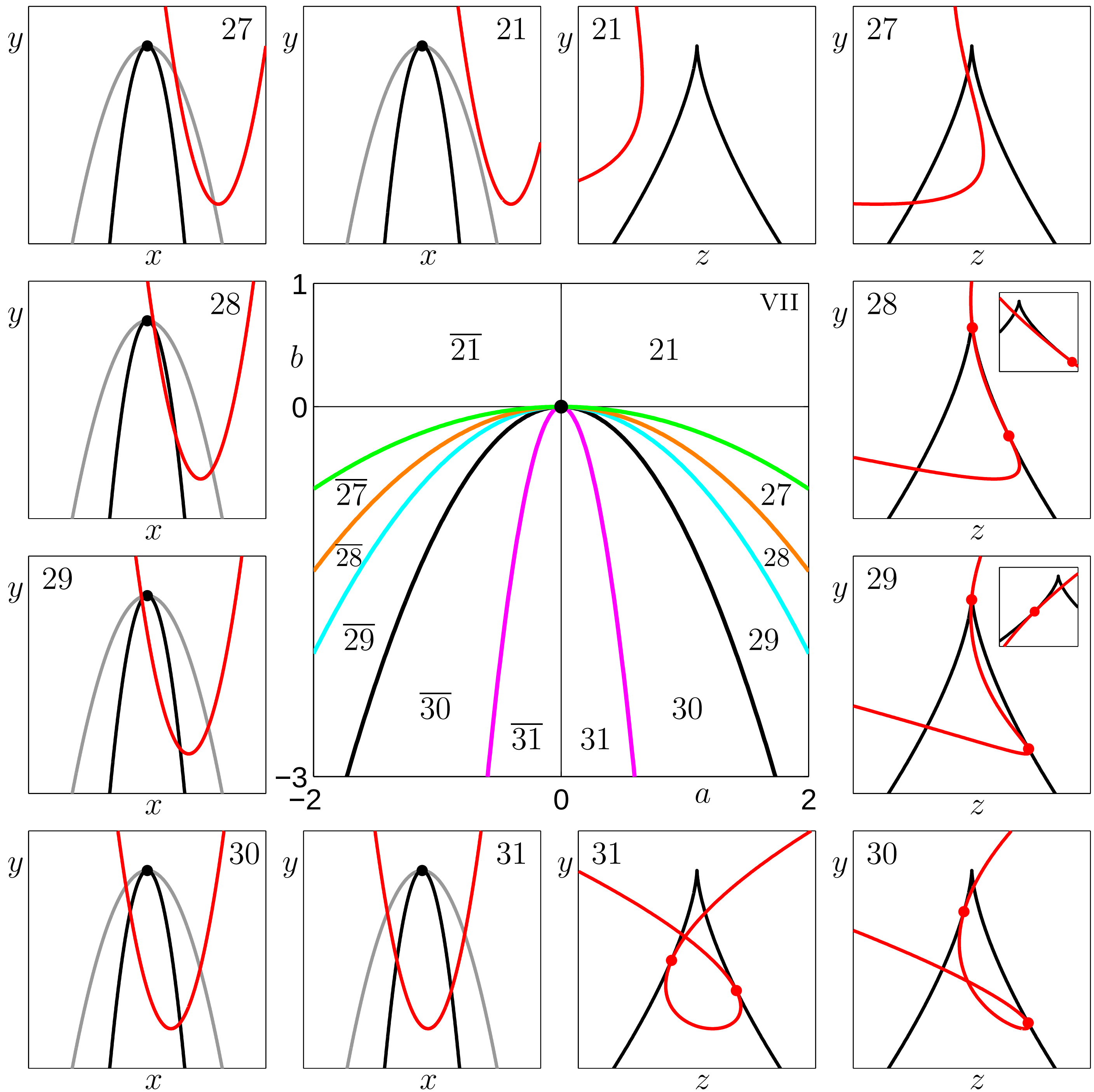
Focus on case (i): $\gamma = 0.5$.



All possible normal form configurations

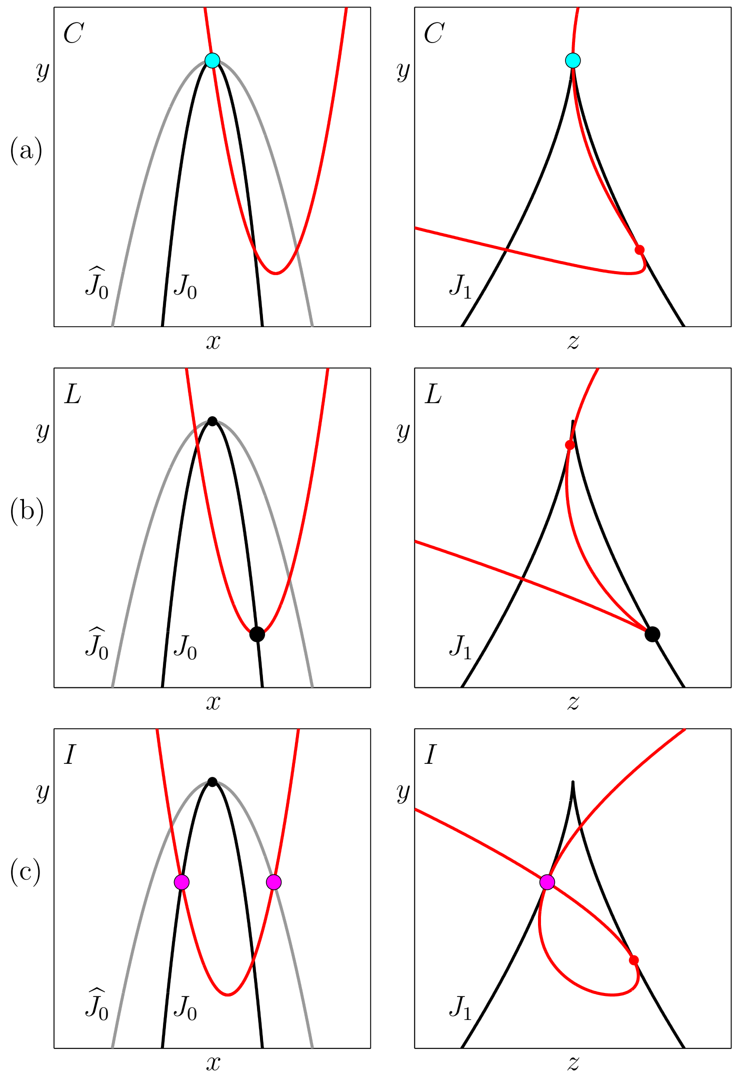


The cusp-cusp unfolding for Case VII ($\gamma = 0.5$)



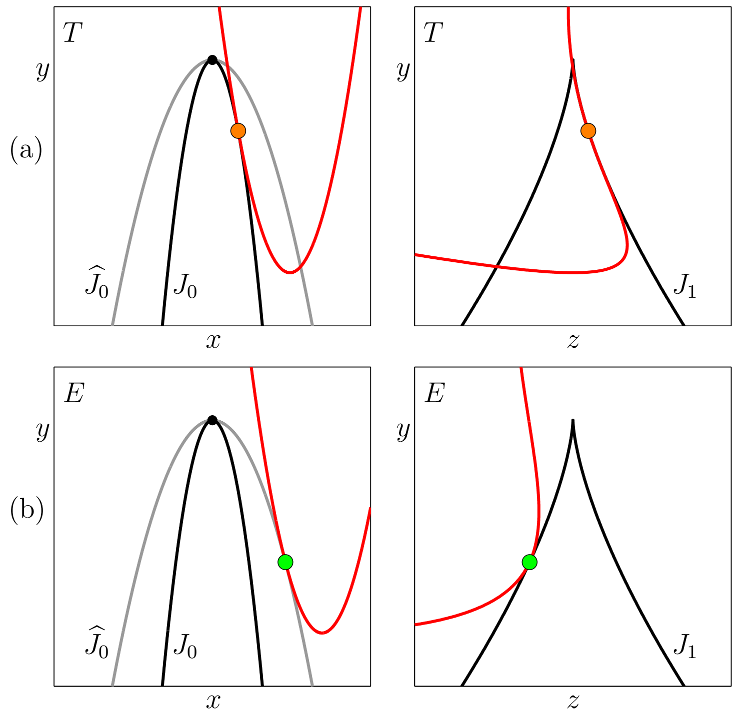
The codimension-one bifurcations:

(a) Cusp transition (pass thru the cusp point C) (b) Loop creation (cusp on W at critical parameter value) (c) Intersecion at tangency (topological change in intersections in range)



The codimension-one bifurcations (continued)

(a) tangency-creation (W tangent to J_0) (b) Enter-exit (W tangent to $\widehat{J}_0$)



Definition of equivalence:

Toplogical equivalence of the two curves in the image.

Explicit computation of bifurcation curves: all parabola in (a,b)

1. The *cusp transition*, denoted by C, where the curve W passes through J_0 at the pre-cusp point C_0, which means that $F(W)$ passes exactly through the cusp point C_1 on J_1; The locus of this bifurcation in the (a,b)-plane is the parabola

$$b = c_C(\gamma)\, a^2 = -\gamma\, a^2. \tag{6}$$

2. The *loop-creation bifurcation*, denoted by L, where W crosses J_0 tangent to the (horizontal) line field $\mathcal{E}$;

$$b = c_L(\gamma)\, a^2 = -a^2. \tag{7}$$

3. The *intersection-at-tangency bifurcation*, denoted by I, where $F(W)$ self-intersects at a tangency point with J_1;

$$b = c_I(\gamma)\, a^2 = -(9\gamma + 4)\, a^2. \tag{8}$$

4. The *tangency-creation bifurcation*, denoted by T, where W is tangent to J_0;

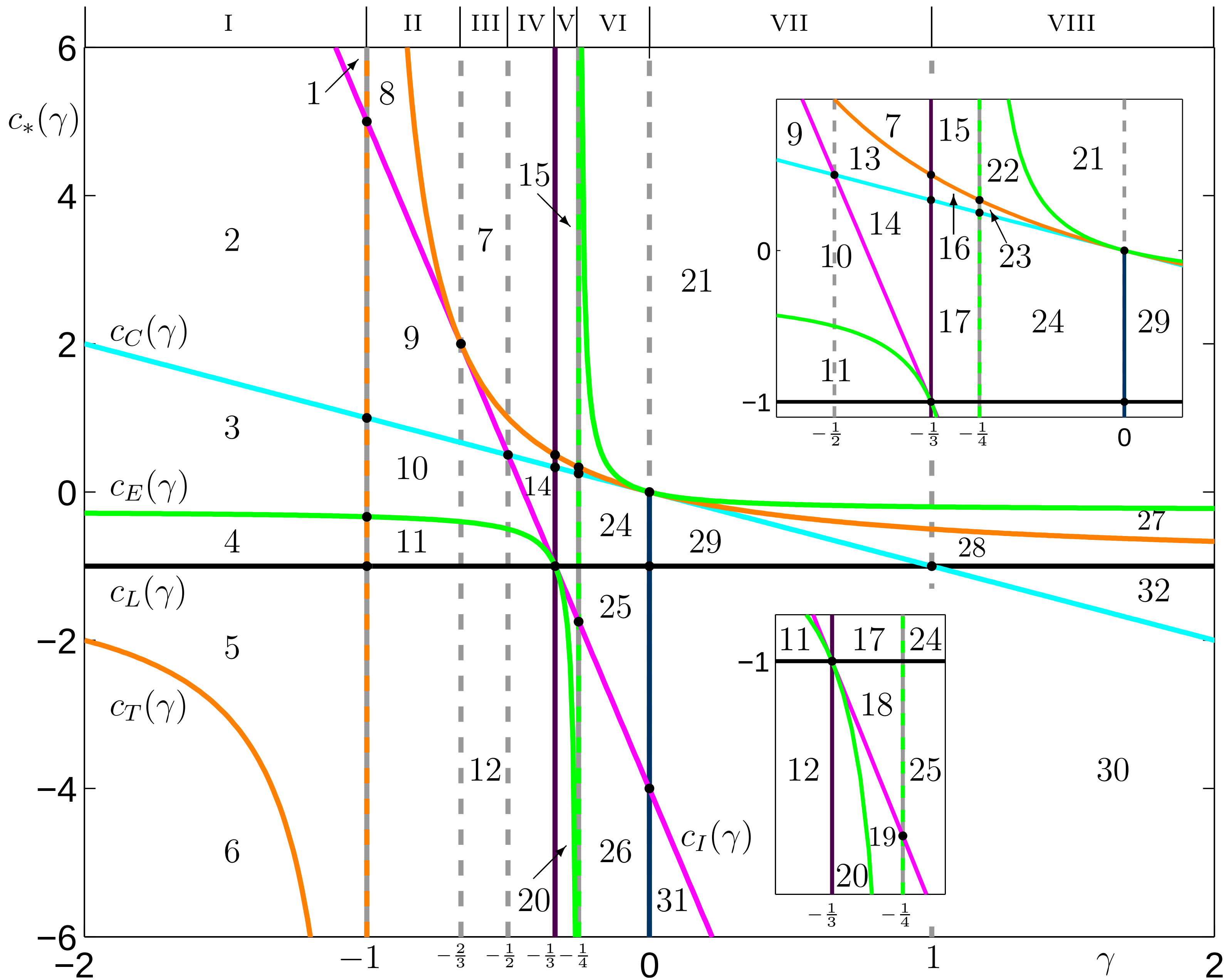
$$b = c_T(\gamma)\, a^2 = -\frac{\gamma}{1+\gamma}\, a^2. \tag{9}$$

5. The *enter-exit bifurcation*, denoted by E, where W is tangent to $\widehat{J}_0$;

$$b = c_E(\gamma)\, a^2 = -\frac{\gamma}{1+4\gamma}\, a^2. \tag{10}$$

Classification in the γ-Coefficient space

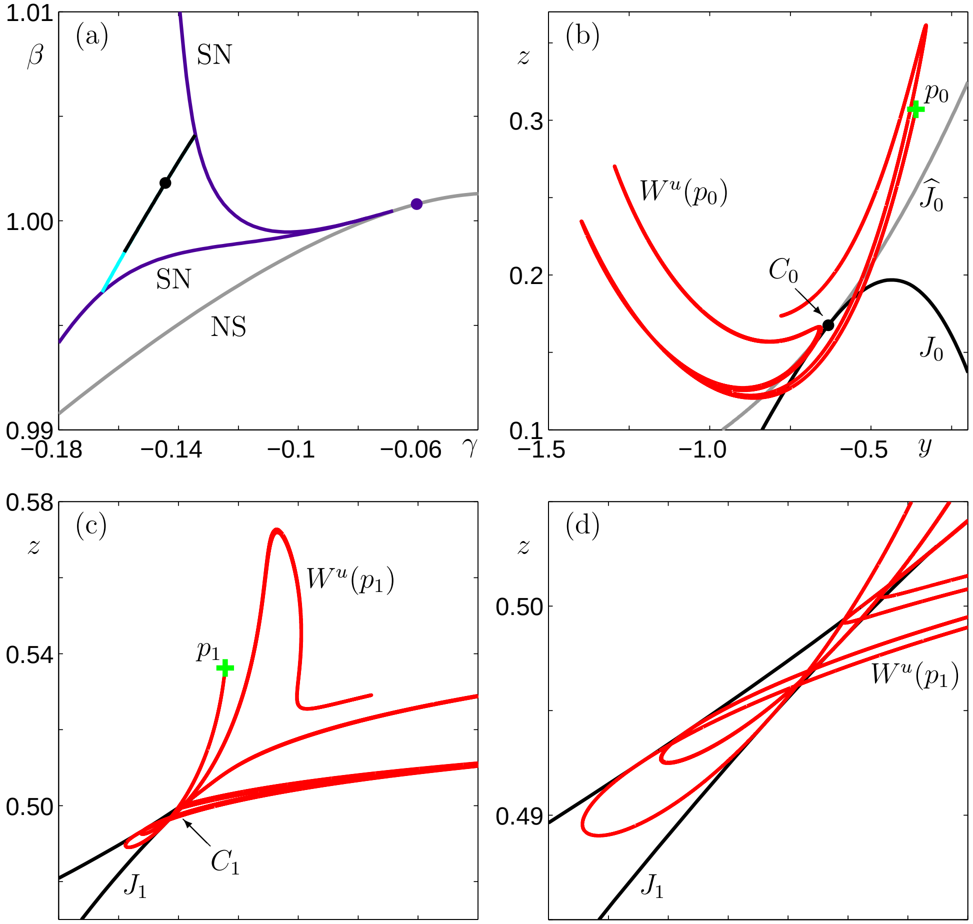
The eight cases, depending on γ, the quadratic coefficient in the normal form of the curve $W := \{y = \gamma(x-a)^2 + b\}$:



Adaptive Control Application

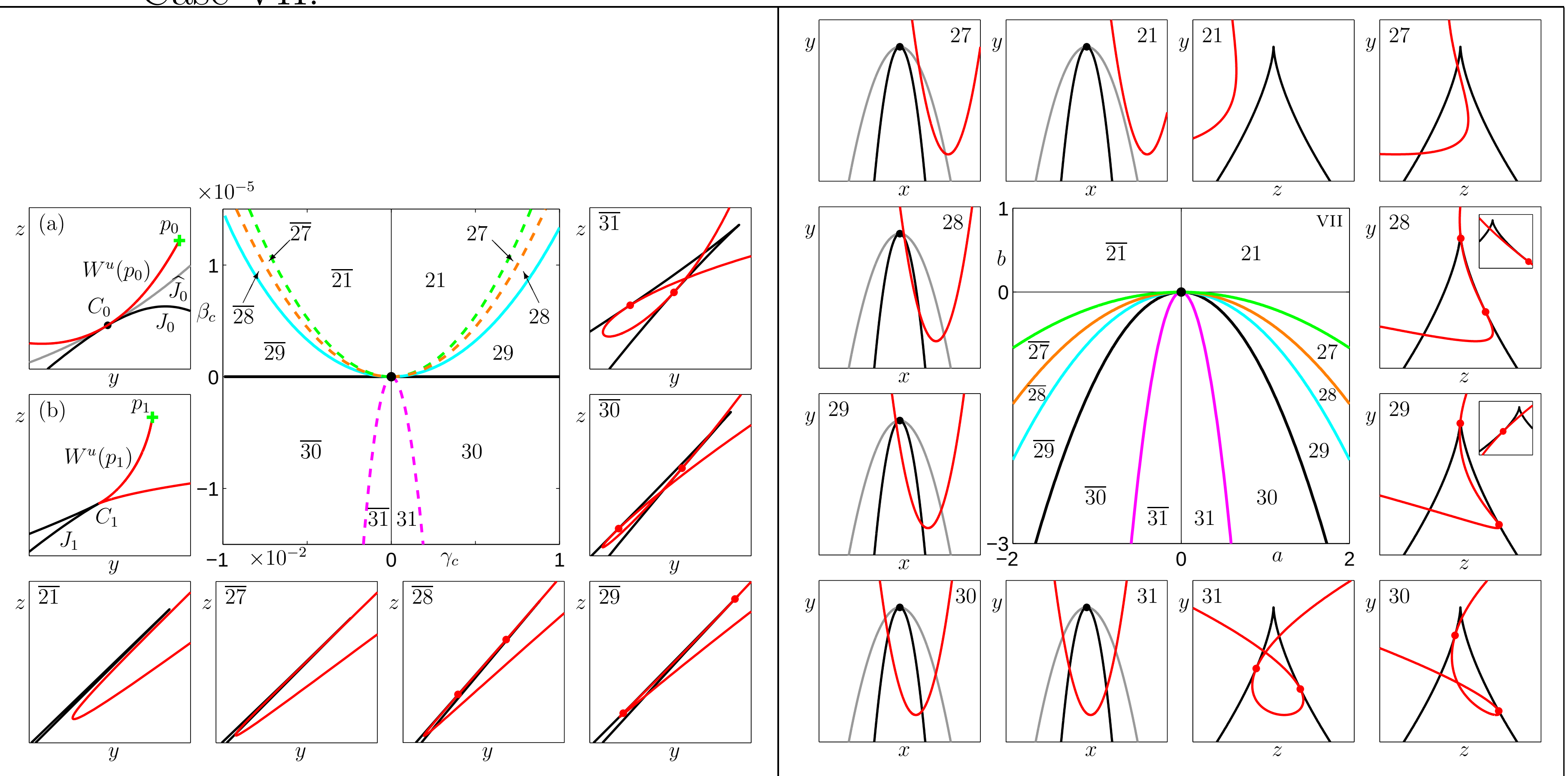
$$g : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -xy + \eta \\ \beta y + \frac{px(-yx+\eta-1)}{c+x^2} \end{pmatrix}, \quad (11)$$

Frouzakis, Adomaitis, Kevrekidis, Golden and Ydstie 1992. ($\beta = 1.0$)
Searched for cusp-cusp point numerically in (p, η) plane for $c = 1.2, \beta = 1.0$
???
In KOP: Fixed $p = 0.81$, searched in (η, β) plane. !!!



Adaptive Control Application (continued)

Case VII!



Application

Normal form

Future plans

- Find or create a better example - with period less than 30
- Implement algorithms for all codimension-one points, as well as codimension-two points
- Identify other codimension-two points
- Connect behavior inside Arnold tongues to behavior outside (irrational rotation arcs for invertible maps)

Summary:

- Systematic analysis of a codimension-two point using singularity theory
- Definition of equivalence class allowed identification of five codimension-one bifurcations
- Key transitions can be studied with only the first iterate of the map under investigation.
- Normal form allowed explicit computations of bifucation curves in (a, b) parameter space, and classification diagram for cases I - VIII in $(\gamma, coefficient)$ space.
- Allows classification of type I - VIII in applications
- Might provide a framework for analyzing other noninvertible bifurcations