The Cusp-cusp Bifurcation for noninvertible maps of the plane

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Talk Outline

1. Background on noninvertible maps of the plane: some features and bifurcations

2. The codimension-two cusp-cusp bifurcation
   (a) The two cusps
   (b) The normal form of the model
   (c) Definition of equivalence
   (d) Analysis of the model: five codimension-one bifurcations, eight generic cases

3. An adaptive control application

4. Summary
Some research groups in noninvertible maps of the plane

- Gumowski and Mira 1980 book *Recurrences and Discrete Dynamic Systems*
  \[(x, y) \mapsto (ax + y, b + x^2)\]

- Christian Mira “group”: Laura Gardini, Millerioux, Barugola, Cathala, Carcasses, Ralph Abraham. Several books and lots of articles.

- Yannis Kevrekidis “group”: Christos Frouzakis, Ray Adomaitis, Rafael de Llave, Rico-Martinez, BP

- Dick McGehee group: Evelyn Sander, Josh Nien, Rick Wicklin, ...

- Lorenz (1989 paper):
  \[(x, y) \mapsto ((1 - a\tau)x - \tau xy, (1 - \tau)y + \tau x^2)\]

- Bristol Group: Bernd Krauskopf, Hinke Osinga, James England, BP, ...

- V. Maistrenko and Y. Maistrenko

Features and bifurcations
unique to
NONINVERTIBLE maps of the plane
Some features unique to noninvertible maps of the plane

- Critical curves: \( J_0 (\approx LC_{-1}), J_1 (\approx LC) \)
  \( J_0 = \{ x : \det(DF(x)) = 0 \}; J_1 = F(J_0) \)

- Folding of phase space: \( Z_0 - Z_2, Z_1 < Z_3, \ldots \). Figs from FGKMM (1997).
Some features unique to *noninvertible maps of the plane* (cont.)

- Interaction of critical curves with
  - fixed/periodic points (eigenvalue zero)
  - Unstable manifolds (*outsets*): self-intersections, loops, cusps
  - Stable manifolds (*insets*): disconnected, allowing disconnected and multiply connected basins of attraction
  - Invariant circles

- Chaotic attractors
Example: Lorenz (1989)

\[(x, y) \mapsto ((1 - a\tau)x - \tau xy, (1 - \tau)y + \tau x^2)\]

Frouzakis, Kevrekidis, P, 2003
Some bifurcations unique to noninvertible maps of the plane

- Codimension-one Interactions of $J_0$ with
  1. fixed/periodic points - transition from orientation preserving to orientation reversing,
  2. creating self-intersections of $W^u$ via tangency (Mira: contact bifurcation) or loop formation
  3. creating disconnected or multiply connected basin boundaries
  4. breakup of an invariant circle (loops on what “used to be” the smooth invariant curve - Lorenz)
  5. creating intersections of basin boundaries with $J_0$

- Codimension-two
  1. (0,1) eigenvalues (Josh Nien thesis with Dick McGehee - 1997) (Unfoldings not finitely determined.)
  2. Sander (2000) A transverse homoclinic point with no tangle
  3. The Cusp-cusp bifurcation
The CUSP-CUSP bifurcation
Motivation from FGKMM (1997): “Interactions in the neighborhood of a cusp point”
The first cusp - along the phase space fold

The Cusp point $C_0$ and its image $C_1$. Persists under perturbation. The green line field denotes the direction of the zero eigenspaces at points of $J_0$.

(a) and (b) General position; (c) and (d) normal form position
The second cusp - on the image of $W$

Cusps on images of curves which cross $J_0$ tangent to the line field. Codimension-one occurrence.

Assume $W$ is described locally by $\alpha(t)$ with $\alpha(t_0) = x_0$ and tangent vector $\alpha'(t)$. At $f(x_0)$, $W$ is described locally by $f(\alpha(t))$, and has tangent vector $Df(x_0).\alpha'(t)$, which implies $W$ is still smooth at $f(x_0)$ unless $\alpha'(t)$ is an eigenvector for eigenvalue zero at $x_0$. 
The normal forms of the cusp map and $W$ (cont)

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} -x^3 - 3xy \\ y \end{pmatrix}.$$  \hfill (1)

$$DF \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -3x^2 - 3y & -3x \\ 0 & 1 \end{bmatrix}$$

is singular along the particularly simple critical curve

$$J_0 := \{ y = -x^2 \}.$$  \hfill (2)

The image of the parabola $J_0$ is the standard cusp

$$J_1 := \{ z = \pm 2(\sqrt{-y})^3 \mid y \leq 0 \}.$$  \hfill (3)

A straightforward calculation shows that it has the second pre-image

$$\tilde{J}_0 := \{ y = -\frac{1}{4}x^2 \}.$$  \hfill (4)

The zero-eigenvector line field is horizontal (independent of $x \in J_0$).

Normal form (truncated) of $W$:

$$W := \{ y = \gamma(x - a)^2 + b \}.$$  \hfill (5)
The projection of the graph of the normal forms. The cusp maps is fixed; $W = \{ \gamma(x - a)^2 + b \}$ varies. The parameters $a$ and $b$ are the primary parameters while $\gamma$ is a secondary parameter. We construct bifurcation diagrams in the $(a, b)$ plane for fixed $\gamma$. There turn out to be eight generic cases, depending on $\gamma$. 
The organizing center for different values of $\gamma$
Focus on case (i): $\gamma = 0.5$. 
All possible normal form configurations
The cusp-cusp unfolding for Case VII ($\gamma = 0.5$)
The codimension-one bifurcations:
(a) Cusp transition (pass thru the cusp point $C$) (b) Loop creation (cusp on $W$ at critical parameter value) (c) Intersecion at tangency (topological change in intersections in range)
The codimension-one bifurcations (continued)
(a) tangency-creation ($W$ tangent to $J_0$) (b) Enter-exit ($W$ tangent to $\tilde{J}_0$)

Definition of equivalence:
Topological equivalence of the two curves in the image.
Explicit computation of bifurcation curves: all parabola in \((a,b)\)

1. The *cusp transition*, denoted by \(C\), where the curve \(W\) passes through \(J_0\) at the pre-cusp point \(C_0\), which means that \(F(W)\) passes exactly through the cusp point \(C_1\) on \(J_1\); The locus of this bifurcation in the \((a,b)\)-plane is the parabola

\[
b = c_C(\gamma) a^2 = -\gamma a^2.
\]  

(6)

2. The *loop-creation bifurcation*, denoted by \(L\), where \(W\) crosses \(J_0\) tangent to the (horizontal) line field \(E\);

\[
b = c_L(\gamma) a^2 = -a^2.
\]  

(7)

3. The *intersection-at-tangency bifurcation*, denoted by \(I\), where \(F(W)\) self-intersects at a tangency point with \(J_1\);

\[
b = c_I(\gamma) a^2 = -(9\gamma + 4) a^2.
\]  

(8)

4. The *tangency-creation bifurcation*, denoted by \(T\), where \(W\) is tangent to \(J_0\);

\[
b = c_T(\gamma) a^2 = -\frac{\gamma}{1 + \gamma} a^2.
\]  

(9)

5. The *enter-exit bifurcation*, denoted by \(E\), where \(W\) is tangent to \(\hat{J}_0\);

\[
b = c_E(\gamma) a^2 = -\frac{\gamma}{1 + 4\gamma} a^2.
\]  

(10)
Classification in the $\gamma$-Coefficient space

The eight cases, depending on $\gamma$, the quadratic coefficient in the normal form of the curve $W := \{ y = \gamma(x - a)^2 + b \}$:
Adaptive Control Application

\[ g : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -xy + \eta \\ \beta y + \frac{px(-yx+\eta-1)}{c+x^2} \end{pmatrix}, \]  

(11)

Frouzakis, Adomaitis, Kevrekidis, Golden and Ydstie 1992. \((\beta = 1.0)\)

Searched for cusp-cusp point numerically in \((p, \eta)\) plane for \(c = 1.2, \beta = 1.0\)

In KOP: Fixed \(p = 0.81\), searched in \((\eta, \beta)\) plane.

![Diagram](image)

\[ W^u(p_0) \]

\[ W^u(p_1) \]
Adaptive Control Application (continued)

Case VII!

Application

Normal form
Future plans

- Find or create a better example - with period less than 30
- Implement algorithms for all codimension-one points, as well as codimension-two points
- Identify other codimension-two points
- Connect behavior inside Arnold tongues to behavior outside (irrational rotation arcs for invertible maps)
Summary:

- Systematic analysis of a codimension-two point using singularity theory
- Definition of equivalence class allowed identification of five codimension-one bifurcations
- Key transitions can be studied with only the first iterate of the map under investigation.
- Normal form allowed explicit computations of bifurcation curves in \((a, b)\) parameter space, and classification diagram for cases I - VIII in \((\gamma, coefficient)\) space.
- Allows classification of type I - VIII in applications
- Might provide a framework for analyzing other noninvertible bifurcations