HyperArt – Generating Hyperbolic patterns for Regular and Non-Regular p-gons

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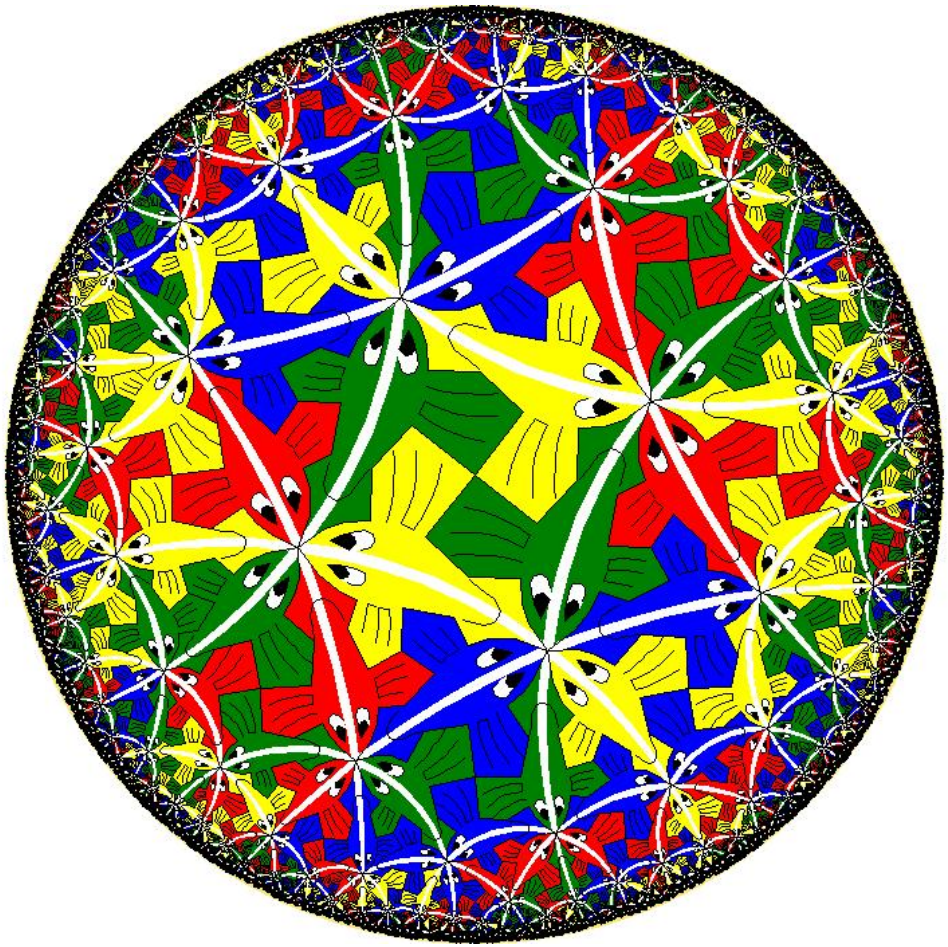
A Hyper*Art* rendition of Circle Limit III



Outline

1. Motivation
   - Hyperbolic Patterns eh?
2. Theory
   - Lets Learn Geometry
   - Definitions Galore
3. Algorithms
   - Concepts
   - Regular tessellations
   - Non-regular tessellations
4. HyperArt
   - Features
   - Design

Not *Just* Pretty Pictures

- Tessellation is a covering of the plane with (symmetrical) patterns.
- Types of tessellations :
  - Euclidean – Penrose tiling.
  - Spherical – Temari balls
  - Hyperbolic – Escher's Circle Limit Patterns.
- Challenging and aesthetically rewarding!

From Desk to Desktop

- First hyperbolic patterns - M. C. Escher – *by hand!*.
- H. S. M. Coxeter – well known geometer – non–Euclidean geometry
- Dr. Douglas Dunham – computer algorithms and programs.

Goals

- Implement and refine *Regular* and *Non-regular* hyperbolic tessellation algorithms.
- To provide a unifying extensible programming framework for these algorithms.

Absolute Geometry

- A straight line segment can be drawn joining any two points.
- Any straight line segment can be extended indefinitely in a straight line.
- Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
- All right angle are congruent.

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The parallel postulate

Euclidean Parallel postulate

If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

Types of geometries

Reinterpret the Euclidean parallel postulate as . . .

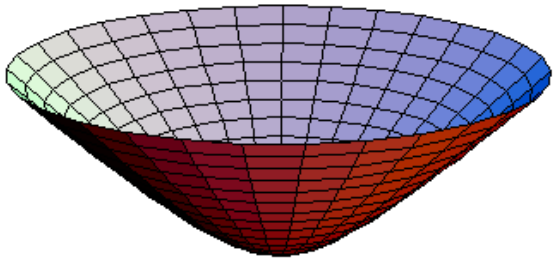
Elliptical/Spherical Parallel postulate

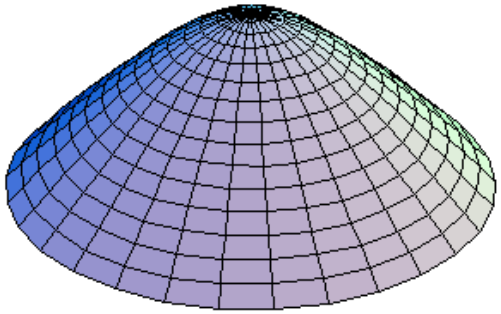
Through any point in the plane, there exists no lines parallel to the given line.

Hyperbolic parallel postulate

For any infinite straight line L and a point P not on it, there are many infinitely extending straight lines that pass through P and do not intersect L.

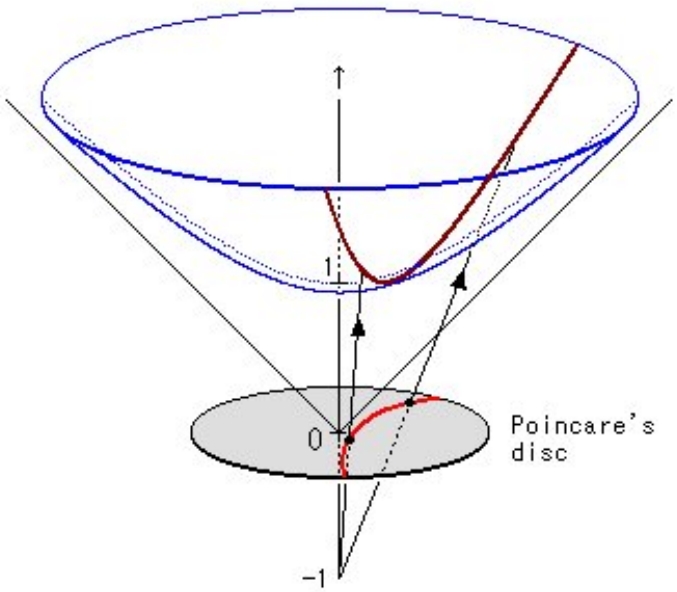
Weierstrass model





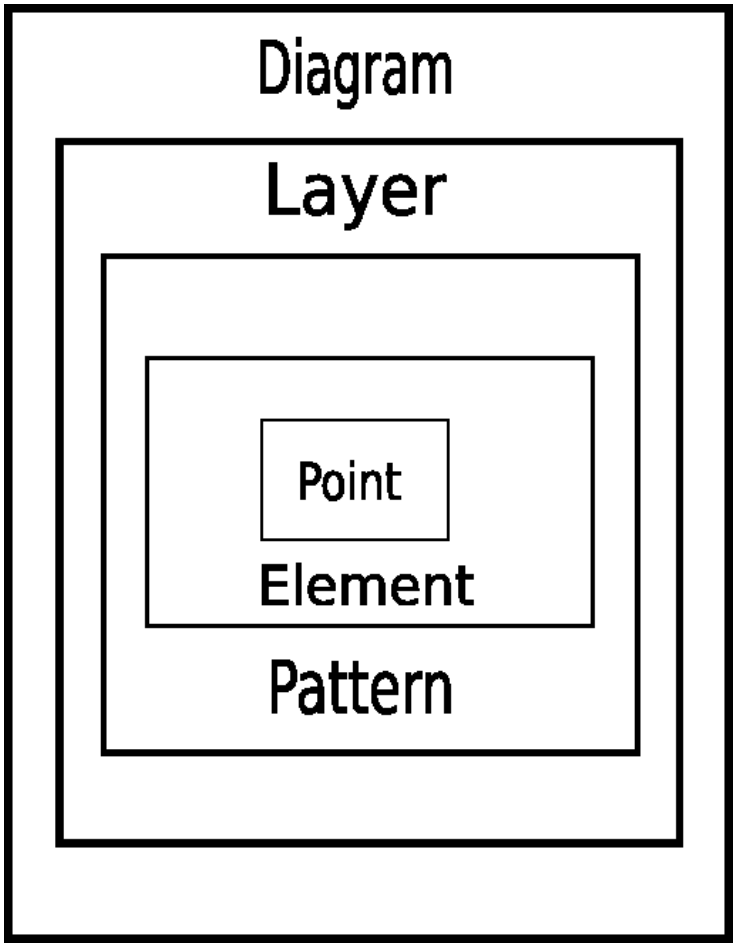
- Upper sheet of the hyperboloid of revolution $z = \sqrt{1 + x^2 + y^2}$.
- Hyperbolic lines are the arcs lying in this plane

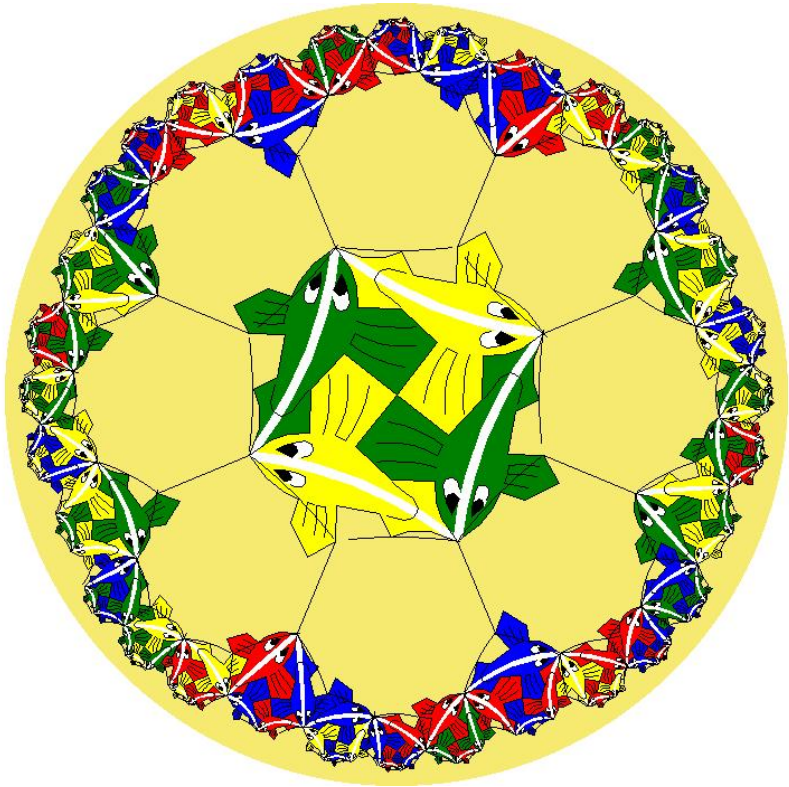
Poincaré model



- Projection of Weierstrass on a unit circle at origin towards $(0, -1, 0)$
- Hyperbolic lines are
  - arcs orthogonal to boundary
  - diameters of the disk
- "boundaryless" model

Parts of a Diagram





Central P-gon and Fundamental region

Definition

Central p-gon pattern is a pattern which remains invariant under certain transformations of the hyperbolic plane.

Definition

The fundamental region is a region in the hyperbolic plane which when transformed by all the transformations in the symmetry group, will cover the hyperbolic plane. *A Fundamental Pattern* is a pattern in the Fundamental region.

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Symmetry groups

p is the number of sides/vertexes of the p-gon.

q is the number of p-gons meeting at a vertex.

- *Symmetry group* is a group of transformations of the hyperbolic plane that preserve a pattern.
- Infinitely many symmetry groups.
- eg Symmetry groups of Regular tessellations p,q
  - $[p, q]$ – 3 reflections
  - $[p, q]+$ – 3 rotations
  - $[p+, q]$ – p rotations about origin and a reflection across p-gon edge
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Classification of Algorithms

- According to p-gon type
  - Regular p-gon algorithms
  - Non-regular p-gon algorithms
- According to replication order
  - Hamiltonian methods.
  - Spanning-tree methods
- According to *central* p-gon
  - P-gon center at the origin
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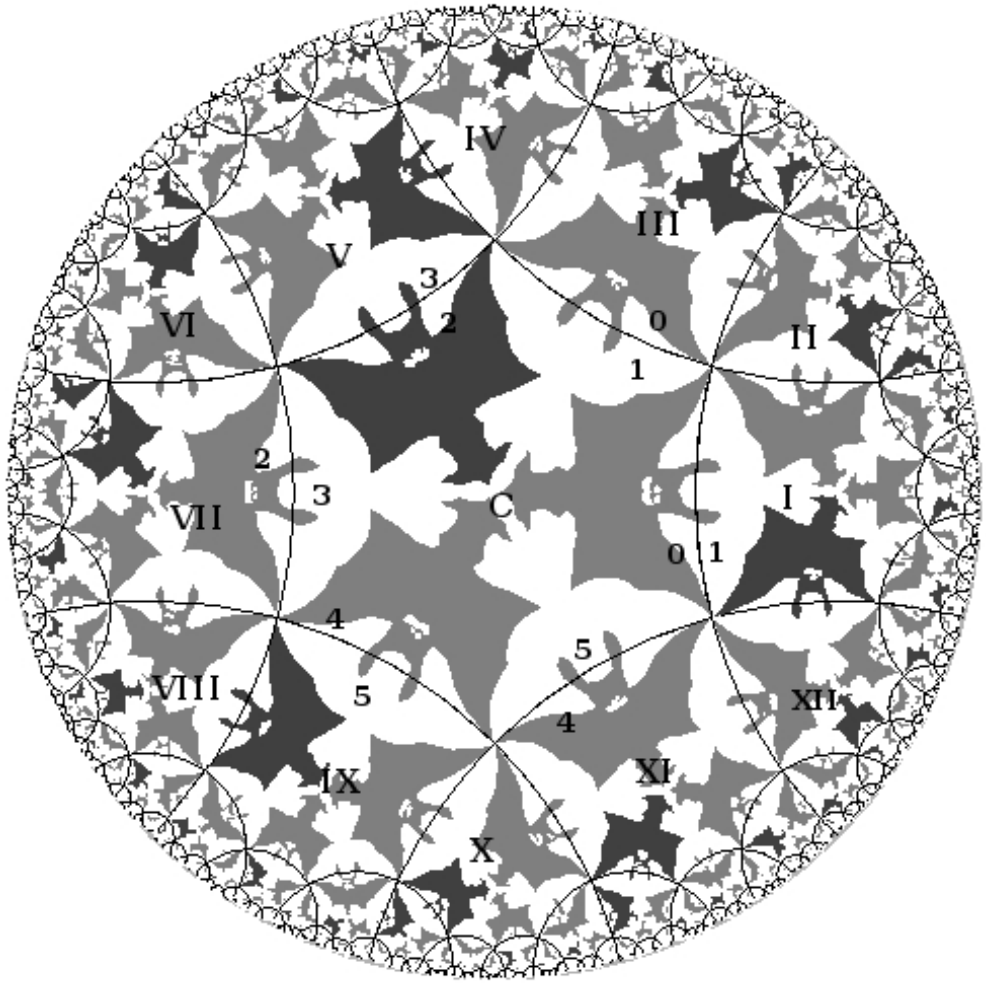
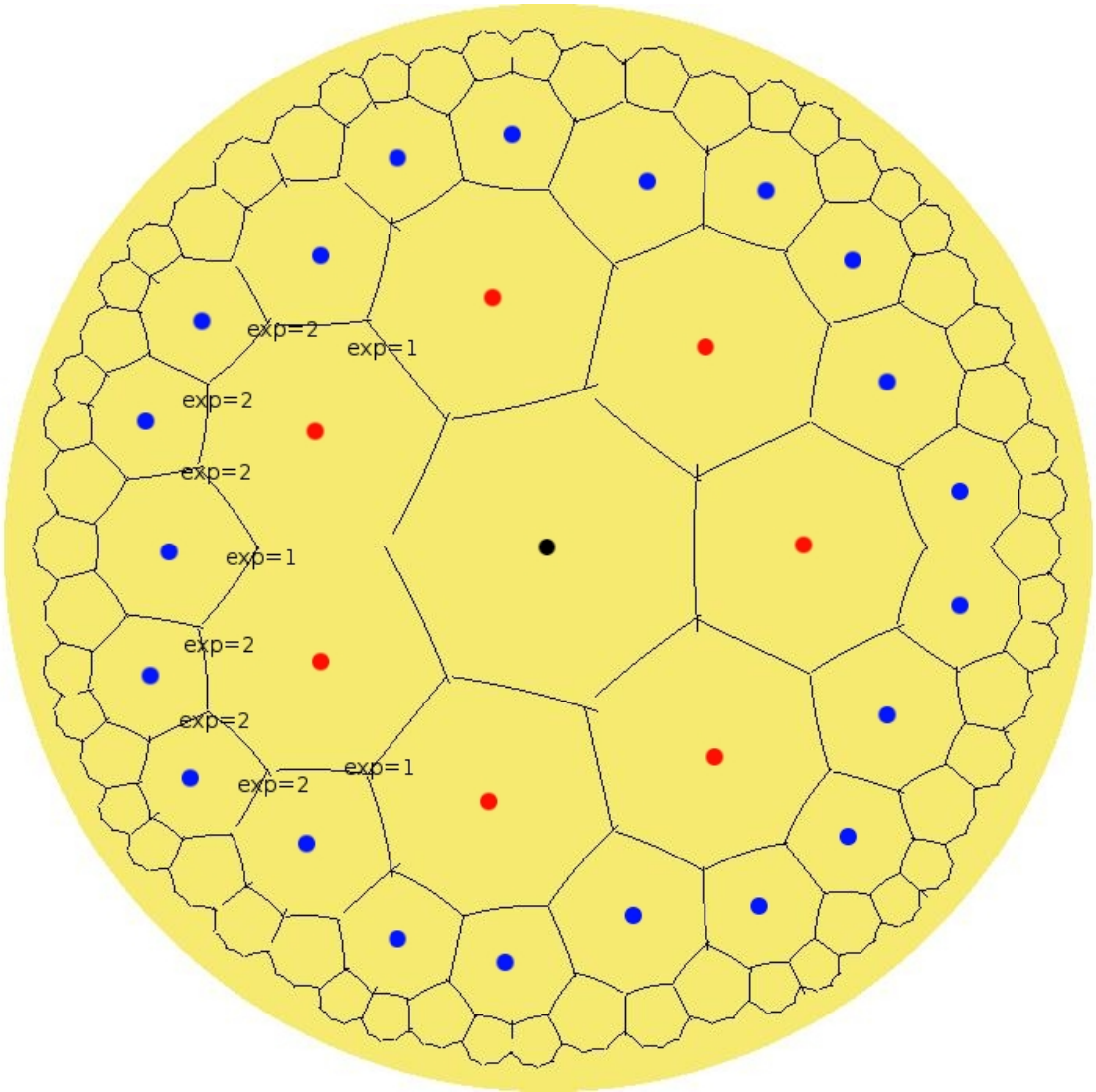
Edge Adjacency and Orientation



Exposure



Exposure of a vertex in the k^{th} layer : the number of p-gons in the $(k+1)^{st}$ layer sharing that vertex.

Generating the central p-gon

- Needed for Regular p-gon algorithms.
- Fundamental pattern needs to be replicated to fill up central p-gon
- which is then replicated to fill up the disk
- Transformations used :
  - Reflection across p-gon radius
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Regular P-gon Pattern Replication

- Spanning tree method of replication.
- The idea is to
  - Start with the central p-gon
  - Visit each shared vertex.
  - And recursively process the optimal number of p-gons around it.

Lets look at an example in HyperArt

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- P-gon *is* the Fundamental region.
- But now we have different number of p-gons meeting at each vertex.
- Fundamental region moved to put vertex at center.
- ∴ the concept of layers is slightly changed.
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Hyper*Art* Features From User's Perspective

- XML data file format
- Crossplatform – Written in Qt from trolltech
- Single unified interface for different types of algorithms
- Layer and Frame toggling
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- Modern UI with zooming, panning and printing support.
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- Document view architecture.
- Abstraction of Diagram and View classes.
- Importing old dat files to the new XML format.
- Element based patterns instead of pen based.
- Source-code available under GPL license on sourceforge.net.

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Design of HyperArt

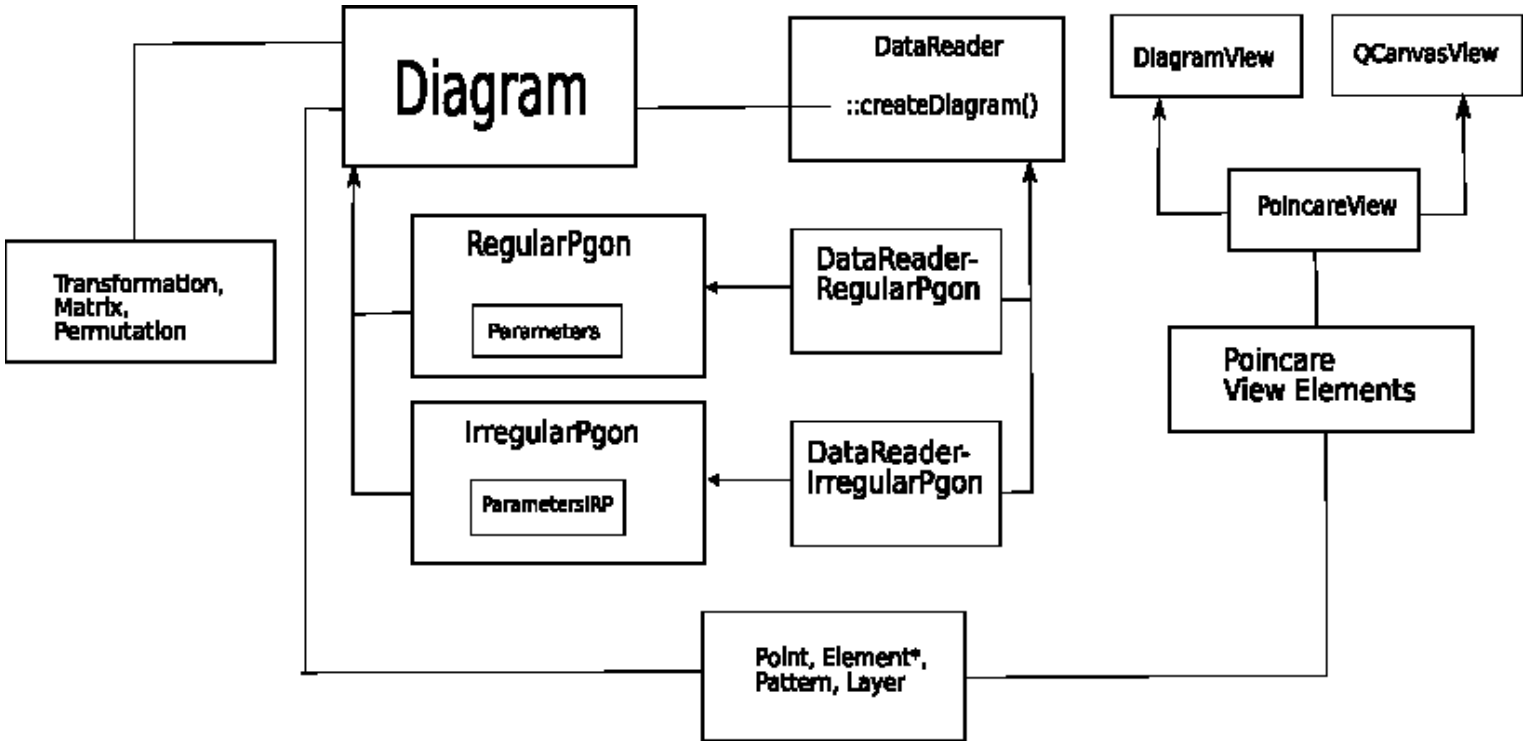

Diagram classes

- Diagram – abstract base class
  - RegularPgon
  - IrregularPgon
- Parameters – For RegularPgon
  - ParametersIRP – For IrregularPgon
- Element
  - Circle
  - EuclidPoly, EuclidPolyLine
  - HyperPoly, HyperPolyLine
- Other utility classes such as Matrix, Permutation, Transformation

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Presentation classes

- DiagramView – abstract base view class
  - PoincareView – derives from DiagramView and QCanvasView
- Canvas counterparts of Element classes such as CanvasHyperPoly

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Reader classes

- DataReader – abstract base class for readers, also a Diagram Factory
  - DataReaderRegularPgon
  - DataReaderIrregularPgon

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- Implemented and refined Regular and Non-Regular P-gon algorithms

- Future Work
  - Diagram Designer for HyperArt
  - Implement other algorithms.

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Lets see some nice designs . . .

For Further Reading I

HSM. Coxeter

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For Further Reading II

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