A Property of Area and Perimeter

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Background

Our original goal was to randomly fill a region $R$ of area $A$ with successively smaller copies of a motif, creating a fractal pattern.

We achieved that goal with our original algorithm, which we describe below. For that algorithm, the size of the motifs was specified by an inverse power law.

Recently we used a modified version of the algorithm in which the size of the motifs was determined using an estimate of the amount of “room” left after each placement.

We have evidence that the size of the motifs in the new algorithm also obeys an inverse power law.
The Original Algorithm

We found experimentally that we could fill the region $R$ if for $i = 0, 1, 2, \ldots$, the area $A_i$ of $i$-th motif obeyed an inverse power law:

$$A_i = \frac{A}{\zeta(c, N)(N + i)^c}$$

where where $c > 1$ and $N > 0$ are parameters, and $\zeta(c, N)$ is the Hurwitz zeta function: $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$ (and thus $\sum_{k=0}^{\infty} A_i = A$).

We call this the **Area Rule**
Algorithm Details

The algorithm works by successively placing copies $m_i$ of the motif at locations inside the bounding region $R$.

This is done by repeatedly picking a random trial location $(x, y)$ inside $R$ until the motif $m_i$ placed at that location doesn’t intersect any previously placed motifs.

We call such a successful location a placement. We store that location in an array so that we can find successful locations for subsequent motifs.
A Flowchart for the Algorithm

\[ i = 0 \]

**trial** — place motif i at random position inside region R

does motif i intersect any motif with index \(< i\)?

no

**placement** — enter \((x,y)\) for motif i in the location array

yes

increment i
A Sample Pattern of Peppers
The New Algorithm

The new algorithm is the same as the original algorithm except for the determination of the size of the next motif. Rather than specifying the size by the area rule, we use information about the part of the region $R$ not yet filled to give the size of the next motif.

Now, for discussion, we will restrict both $R$ the motifs to be disks, although the algorithm works in the general case too.

We use the term **gasket** for the unfilled part of $R$ after placing the first $i$ motifs. So the gasket looks like Swiss cheese when the motifs are disks. Now we let $A_i$ be the area of the gasket and $P_i$ be its perimeter, then the radius $r_{i+1}$ of the next disk is given by:

$$ r_{i+1} = \gamma \frac{A_i}{P_i} $$

where $\gamma > 0$ is a dimensionless parameter which can be as large as 3 (but if $\gamma > 2$ the algorithm is modified as discussed below).
Notes

1. In the formula above, the size of the next motif is certainly proportional to \(A_i\) since that is the remaining area. But \(P_i\) measures the amount of “fragmentation” of that area, hence the inverse relationship.

2. To get started, the area and perimeter of the bounding circle are \(\pi R^2\) and \(2\pi R\) respectively, so the initial area/perimeter ratio is \(R/2\). Thus if \(\gamma > 2\) the first disk doesn’t fit within the bounding circle.
A Sample Pattern ($\gamma = 3/2$)
Another Pattern ($\gamma = 1/2$)
Log-log plot of trials versus placements

\( \gamma = 1.50 \)

\( \log_{10} \text{of cumulative trials} \)

\( \log_{10} \text{of placements} \)
Log-log plot of disk areas and the gasket area

\[ \log_{10} \text{of disc area (red)} \text{ or } \log_{10} \text{of gasket area (blue)} \]

\[ \log_{10} \text{of disc number} \quad \gamma = 1.50 \]
Notes

1. The plot above was again for $\gamma = 3/2$. The plot of the areas appears to be close to linear, indicating a power relationship. The slope of the last pair is -1.272727, which appears to be the beginning of a repeating decimal, and thus hints at a rational value for the exponent.

2. By running the algorithm with different values of $\gamma$, it was found that the exponent seemed to be given by

$$c = -\frac{4 + 2\gamma}{4 + \gamma}$$

3. The next plot shows how fast the estimated values of $c$ (from successive disk area values at $i = a$ power of 2) converge to the value $14/11$ given by the formula above when $\gamma = 3/2$. 
Convergence of $c$ to $\frac{14}{11}$
Variation 1

- We have noted that the new algorithm as given will not work if $\gamma > 2$. But we can start with disks of a fixed size (smaller than the bounding circle of course) and keep placing them until the disks given by the radius formula are smaller, then switch to the new algorithm. The next slide shows a sample pattern when $\gamma = 5/2$.

- It can be shown that the fractal dimension of the disk pattern is given by $D = 2/c$ in both the new algorithm and this variant.
A Pattern using the modified algorithm
Variation 2 (3 dimensions)

- Another variation is to consider spheres within a bounding sphere in 3 dimensions. Here the next radius is given by the formula

\[ r_{i+1} = \gamma \frac{V_i}{A_i} \]

where \( V_i \) is the gasket volume and \( A_i \) is the gasket surface area.

- By running several samples with different \( \gamma \) values, we discovered that the sphere volumes seemed to follow a power law whose exponent is given by:

\[ c = -\frac{36 + 3\gamma}{36 + 2\gamma} \]
Future Work

- Prove that the new algorithm always converges for all $\gamma$ less than some minimum value.
- Prove that the areas of the disks, or spheres in 3D, decrease according to a power law.
- Prove that the power laws are given by the formulas above.
- One guesses that an appropriate form of the new algorithm should also work in dimensions 1 and greater than 3. This could be investigated.
Thank You!

To the JMM organizers

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