Fractal Wallpaper Patterns

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Abstract

In the past we presented an algorithm that can fill a region with an infinite sequence of randomly placed and progressively smaller shapes, producing a fractal pattern. In this paper we extend that algorithm to fill rectangles and triangles that tile the plane, which yields wallpaper patterns that are locally fractal in nature. This produces artistic patterns which have a pleasing combination of global symmetry and local randomness. We show several sample patterns.

Introduction

We have described an algorithm [2, 4, 6] that can fill a planar region with a series of progressively smaller randomly-placed motifs and produce pleasing patterns. Here we extend that algorithm to fill rectangles and triangles that can fill the plane, creating "wallpaper" patterns. Figure 1 shows a randomly created circle pattern with symmetry group $p6mm$. In order to create our wallpaper patterns, we fill a fundamental region

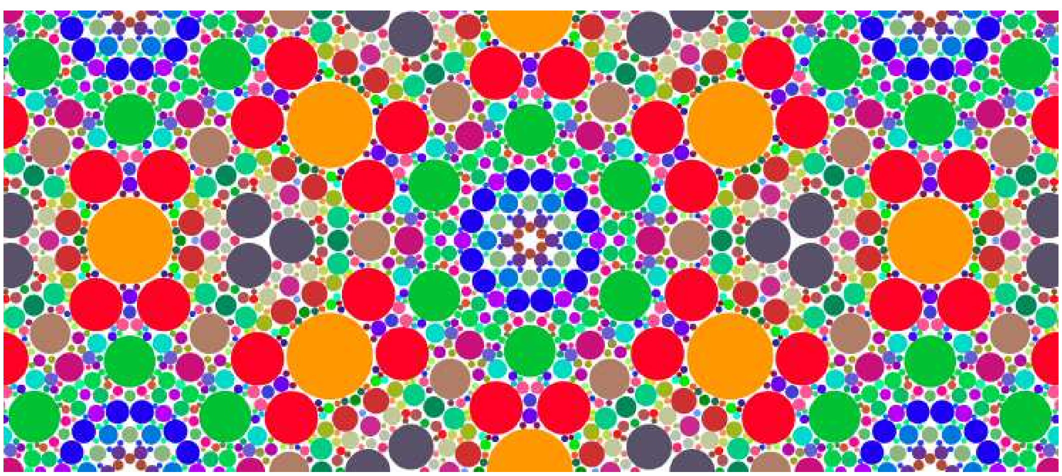

Figure 1: *A locally random circle fractal with global* $p6mm$ *symmetry.*

for one of the 17 2-dimensional crystallographic groups with randomly placed, progressively smaller copies of a motif, such as a circle in Figure 1. The randomness generates a fractal pattern. Then copies of the filled fundamental region can be used to tile the plane, yielding a locally fractal, but globally symmetric pattern.

In the next section we recall how the basic algorithm works and describe the modifications needed to create wallpaper patterns. We will also recall a few facts about wallpaper groups. Then we will exhibit sample patterns for the wallpaper groups $p2mm, p4mm,$ and $p6mm$. Finally, we draw conclusions and summarize the results.

The Algorithm and Wallpaper Groups

The idea of the algorithm is to place progressively smaller motifs m_i within a region R so that a motif does not overlap any previously placed motif. Random placements are tried until a non-overlapping one is found. As noted in [2, 4, 6], for many choices of R and motifs m_i of area A_i, the following algorithm proceeds without halting:

For each $i = 0, 1, 2, \ldots$

  Repeat:

    Randomly choose a point within R to place the i-th motif m_i.

  Until (m_i doesn't intersect any of $m_0, m_1, ..., m_{i-1}$)

  Add m_i to the list of successful *placements*

Until some stopping condition is met, such as a maximum value of i or a minimum value of A_i.

It has been found experimentally by the second author that this non-halting phenomenon is achieved by a wide range of choices of shapes of R and the motifs if the motifs obeyed an inverse power law *area rule*: if A is the area of R, then for $i = 0, 1, 2, \ldots$ the area of m_i, A_i, can be taken to be:

$$A_i = \frac{A}{\zeta(c, N)(N + i)^c} \tag{1}$$

where $c > 1$ and $N > 1$ are parameters, and $\zeta(c, N)$ is the Hurwitz zeta function: $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$. Thus $\lim_{n\to\infty} \sum_{i=0}^{n} A_i = A$, that is, the process is space-filling if the algorithm continues indefinitely. In the limit, the fractal dimension D of the placed motifs can be computed to be $D = 2/c$. Examples of the algorithm written in C code can be found at Shier's web site [7].

It is conjectured by the authors that the algorithm does not halt for non-pathological shapes of R and m_i, and "reasonable" choices of c and N (depending on the shapes of R and the m_is). In fact this has been proved for $1 < c < 1.0965...$ and $N \geq 1$ by Christopher Ennis when R is a circle and the motifs are also circles [3].

Circles make good candidates for both the enclosing region R and the motif since, by their symmetry, they play a significant role in both mathematics and decorative art. In Figure 1, mathematics provides the arrangement of the circular motifs while art provides the colors.

It has been known for over a century that there 17 different kinds of patterns that repeat in two independent directions in the Euclidean plane. Such patterns are called *wallpaper patterns* and their symmetry groups are called plane crystallographic groups or *wallpaper groups*. In 1952 the International Union of Crystallography (IUC) established a notation for these groups, and a shorthand notation soon followed. In 1978 Schattschneider wrote a paper clarifying the notation and giving an algorithm for identifying the symmetry group of a wallpaper pattern [5]. Later, Conway popularized the more general orbifold notation [1]. We create wallpaper patterns by filling a fundamental region R with motifs as above, and then extend the pattern using transformations of the wallpaper group.

In our previous paper [2], we showed examples of patterns that had $p1$ (or **o** in orbifold notation) symmetry, the simplest kind of wallpaper symmetry, with only translations in two independent directions. Figures 2 and 3 show such patterns. In Figure 3 peppers on the left edge "wrap around" and are continued on the right edge; similarly peppers on the top edge "wrap around" to the bottom.

In this paper we concentrate on symmetry groups whose fundamental regions (which becomes our region R) are bounded by mirror lines. There are four such groups: $p2mm, p3m1, p4mm$, and $p6mm$, also denoted by $pmm, p3m1, p4m$, and $p6m$ in shorthand, or $*2222$, $*333$, $*442$, and $*632$ in orbifold notation, respectively. In particular, we examine patterns whose symmetry groups are $p2mm, p4mm$, and $p6mm$.

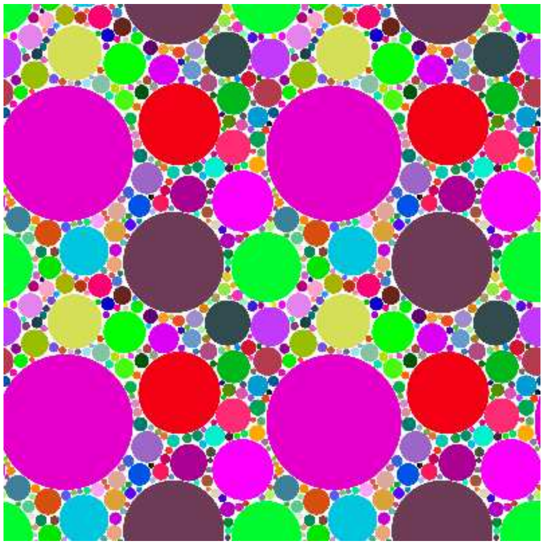

Figure 2: *A circle pattern with $p1$ symmetry.*

Figure 3: *A pattern peppers with $p1$ symmetry.*

An issue that arises for these groups, but not $p1$, is what to do when a trial placement of the motif crosses a mirror boundary of the fundamental region. The simplest solution is simply to reject motifs that cross mirror boundaries. But if a trial placement of a motif does cross a mirror boundary, we could just let that happen, as shown in Figure 4. A more satisfactory solution if the motif has mirror symmetry itself, is

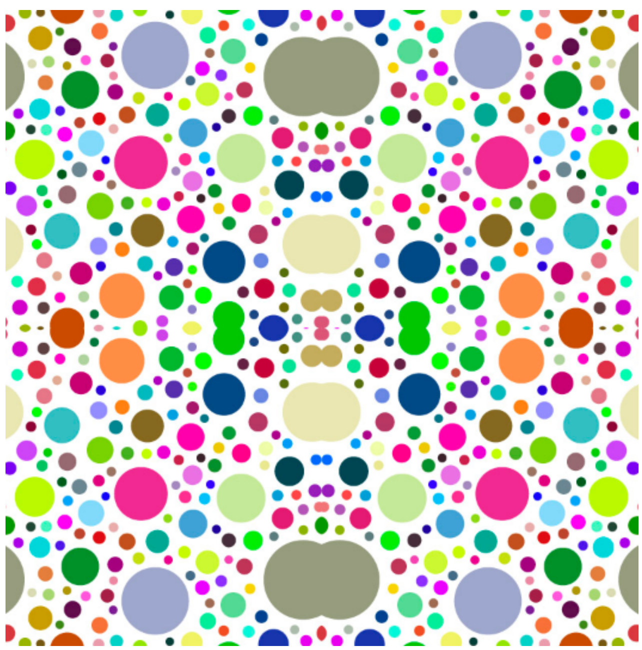

Figure 4: *A random circle pattern with $p2mm$ symmetry, and partial circles on mirror boundaries.*

that we move the motif (perpendicularly) onto the boundary so that the mirror of the motif aligns with the boundary mirror. This creates arguably more interesting patterns. Also, the area rule calculation needs to be adjusted each time this happens since only half of the motif is placed within the fundamental region. We show patterns both of this type and the simpler “reject” type in the next three sections.

Patterns with Symmetry Group p2mm

Figure 5 shows a pattern of hearts with $p2mm$ symmetry in which the hearts avoid the mirror lines of the rectangular fundamental region. Figure 6 show a pattern of circles with $p2mm$ symmetry in which some of the circles are centered on the mirror boundary lines of the fundamental region.

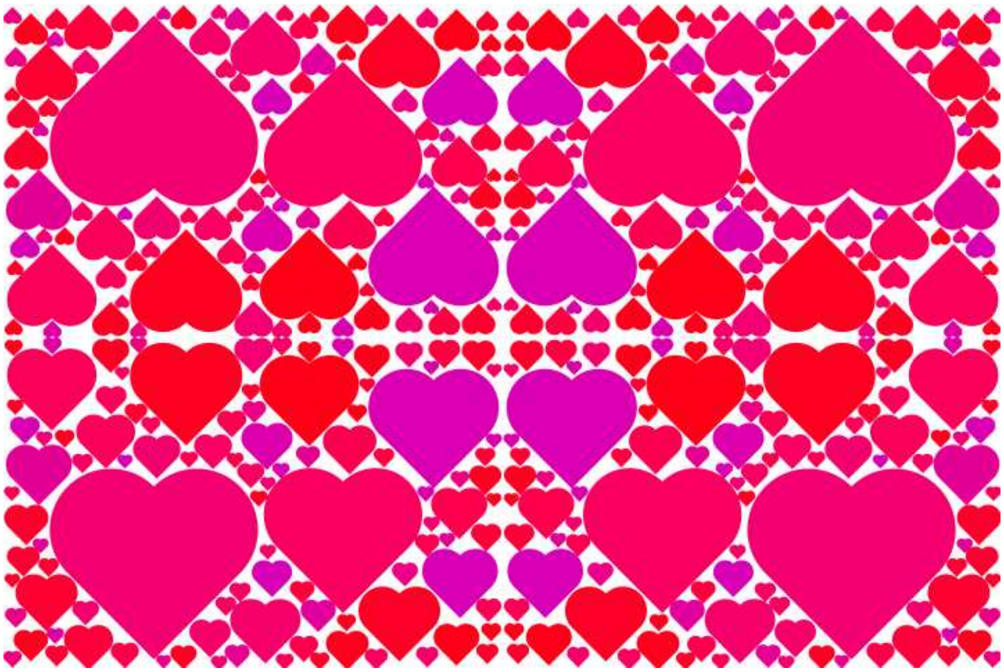

Figure 5: *A random heart pattern with $p2mm$ symmetry.*

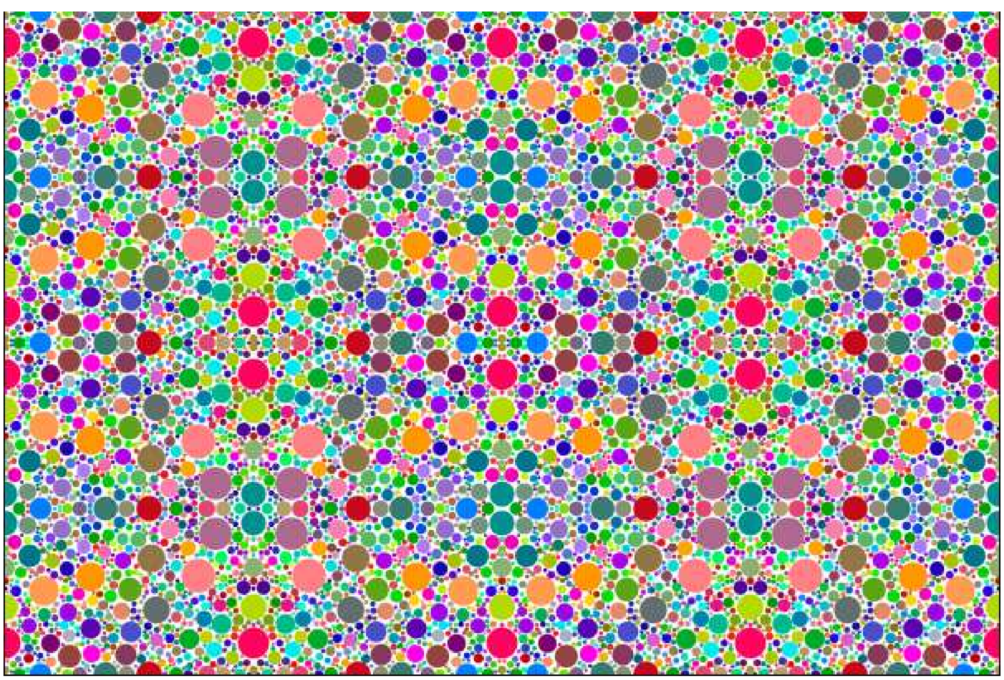

Figure 6: *A pattern of circles with global $p2mm$ symmetry, with some circles on mirror boundaries.*

Patterns with Symmetry Group p4mm

Figures 7, 8, and 9 show patterns with $p4mm$ symmetry in which the motifs avoid the mirror lines of the $45 - 45 - 90$ triangular fundamental region. Figure 7 shows the fundamental region filled with randomly placed black and white squares, creating a Rorschach test pattern. Figure 8 shows the region filled with randomly placed black and white trianles on a blue background Figure 9 shows a pattern in which the horizontal and vertical mirrors interchange two sets of four colors for the flowers. If we consider the sets of colors as single "super colors", this pattern has 2-super-color symmetry.

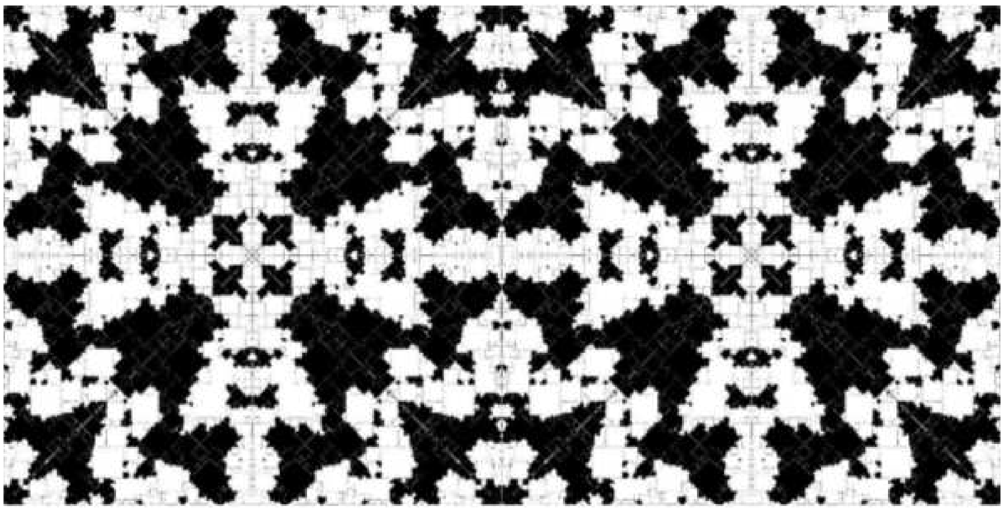

Figure 7: *A "Rorschach" pattern with $p4mm$ symmetry.*

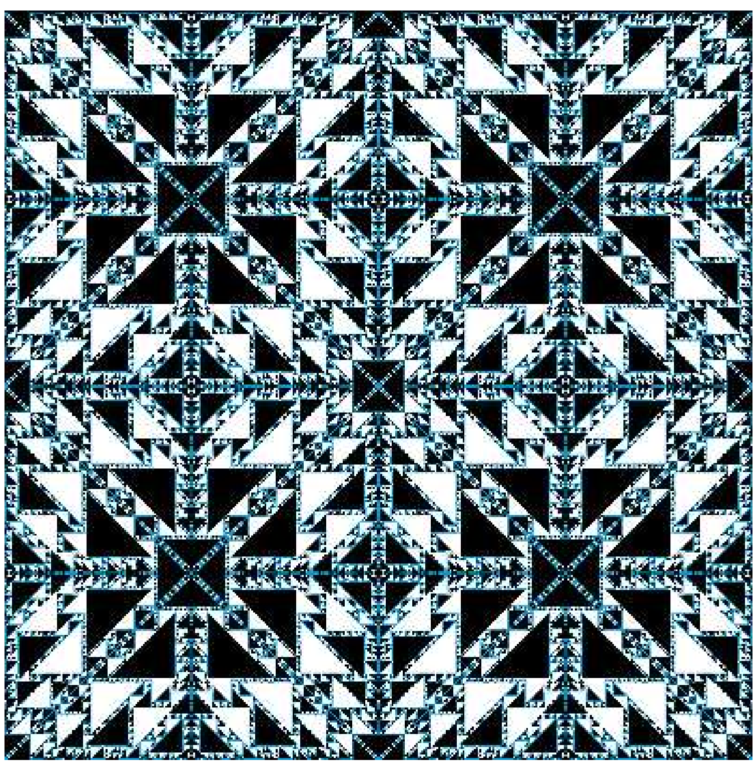

Figure 8: *A pattern of black and white triangles with global $p4mm$ symmetry.*

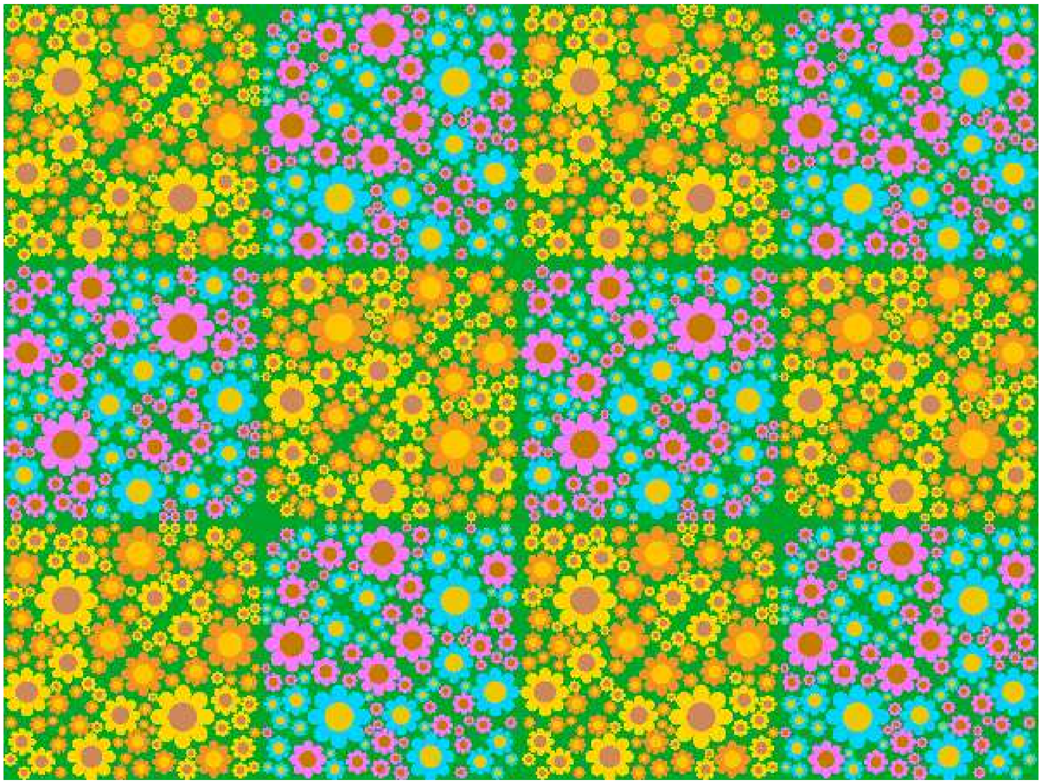

Figure 9: *A flower pattern with color symmetry and global $p4mm$ symmetry.*

Patterns with Symmetry Group $\mathrm{p6mm}$

Figure 1 shows a pattern with $p6mm$ symmetry in which some circles are centered on the mirror boundaries of the $30 - 60 - 90$ triangular fundamental region. Figures 10 and 12 both show patterns with $p6mm$ symmetry in which the motifs avoid the mirror lines of the fundamental region. Figure 10 shows a pattern of triangle motifs whose edges are aligned with the edges of the fundamental region and Figure 12 shows a pattern of arrow motifs that are similarly aligned. Figure 11 shows a pattern of yellow and orange flowers on a green background, with some of the flowers on the mirror lines.

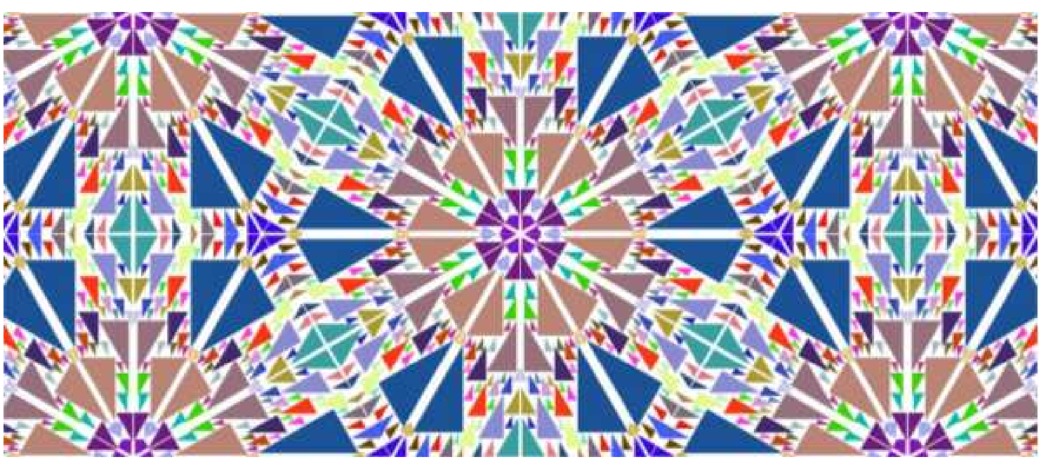

Figure 10: *A pattern of triangles with global $p6mm$ symmetry.*

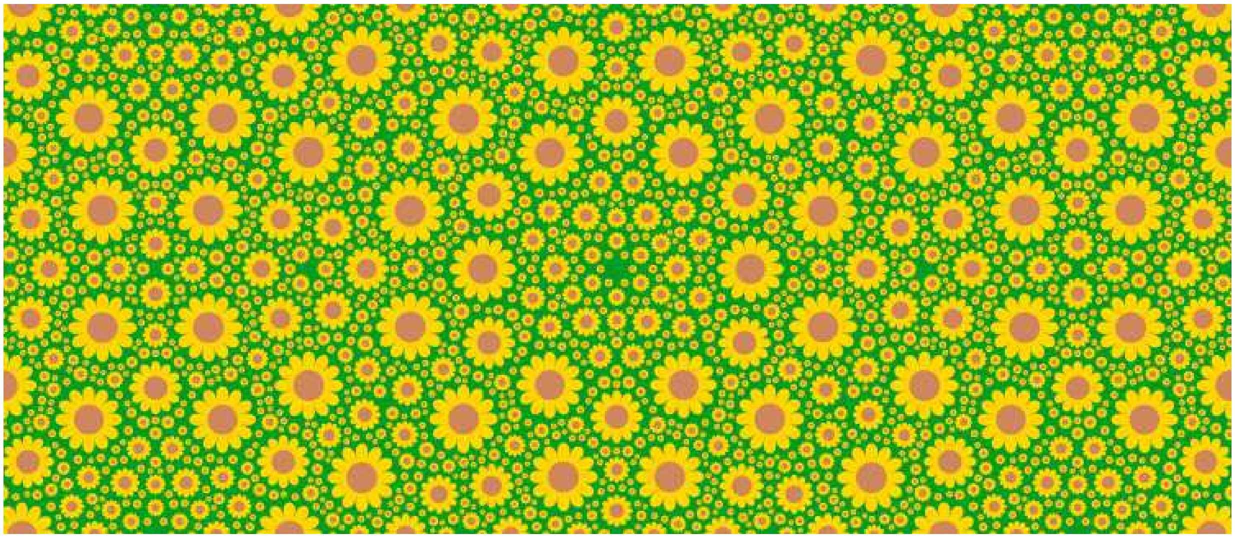

Figure 11: *A flower pattern with* $p6mm$ *symmetry with some flowers on mirror lines.*

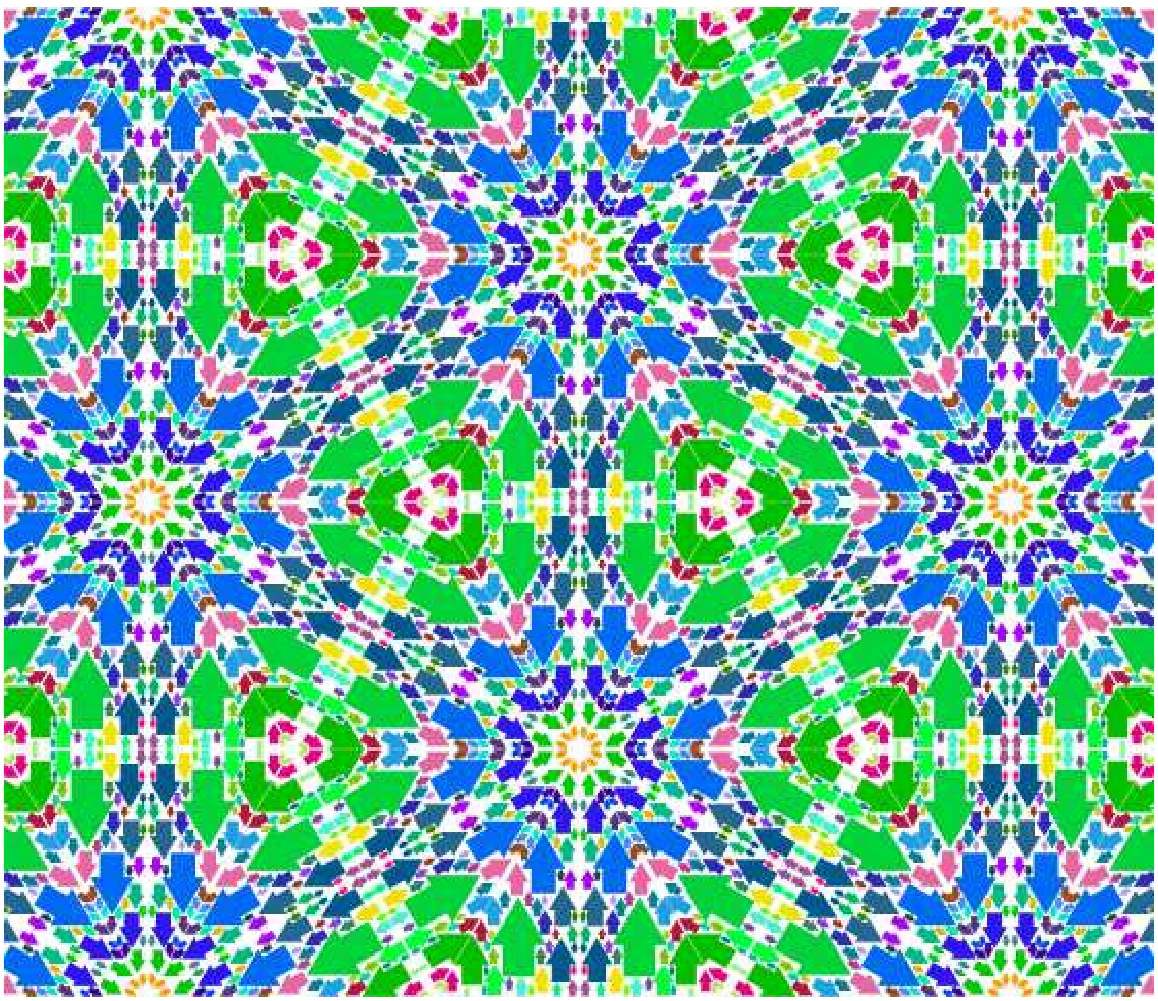

Figure 12: *An arrow pattern with* $p6mm$ *symmetry.*

Summary and Future Work

We have presented a method for creating patterns that have global wallpaper pattern symmetries but are locally fractal in nature. Our goal was to make pleasing patterns with global symmetry but local randomness.

However, we have only implemented our methods for a few of the 17 plane crystallographic groups. In the future we would like to devise methods that would locally fractal patterns with all of the 17 groups. This would involve handling fractal patterns with rotational and glide reflection symmetries. Another challenge would be to create patterns with more interesting color symmetry.

It would also be interesting to create corresponding spherical or hyperbolic patterns that are locally random, but have global symmetries.

References

[1] J. Conway, H. Burgiel, C. Goodman-Strauss, *The Symmetries of Things,* A.K. Peters, Ltd., Wellesley, MA, 2008. ISBN 1-56881-220-5. Wikipedia site for orbifold notation: http://en.wikipedia.org/wiki/Orbifold_notation (accessed Apr. 24, 2015)

[2] Doug Dunham and John Shier, The Art of Random Fractals, in *Bridges Seoul,* (eds. Gary Greenfield, George Hart, and Reza Sarhangi), Seoul, Korea, 2014, pp. 79–86. Also online at: http://archive.bridgesmathart.org/2014/bridges2014-79.html

[3] Christopher Ennis *A Provably "Jam-proof" Algorithm of (Almost) Filling Space*. private communication.

[4] John Shier, Filling Space with Random Fractal Non-Overlapping Simple Shapes *ISAMA 2011 Conference Proceedings,* page 131, June 13–17, 2011.

[5] Doris Schattschneider, The Plane Symmetry Groups: Their Recognition and Notation, *American Mathematical Monthly*, 85, 6, 439-450, July, 1978. Wikipedia site for wallpaper groups: http://en.wikipedia.org/wiki/Wallpaper_group (accessed Apr. 24, 2015)

[6] John Shier and Paul Bourke, An Algorithm for Random Fractal Filling of Space, *Computer Graphics Forum,* Vol. 32, Issue 8, pp. 89-97, December 2013. Also available on Shier's web site: http://www.john-art.com/ (accessed May 3, 2014)

[7] John Shier web site: http://www.john-art.com/stat_geom_linkpage.html (accessed Apr. 24, 2015)