New Kinds of Fractal Patterns

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Abstract

We extend our previous work in three different directions. First we show how to create locally fractal patterns that have $p6$ symmetry. Then we modify the basic algorithm to orient motifs according to their positions. Finally we extend the algorithm to create a detailed realistic looking motif, producing an aesthetically pleasing pattern.

Introduction

Previously we have created pleasing patterns using an algorithm that fills a planar region with a series of ever smaller randomly-placed motifs [1]. In this paper we extend those ideas in three ways: first we show how to create locally fractal patterns with global $p6$ symmetry; second we restrict the orientations of the motifs according to their position; third we enhance the algorithm to handle more complex motifs. Figure 1 shows a pattern of butterflies, an example of the third extension.

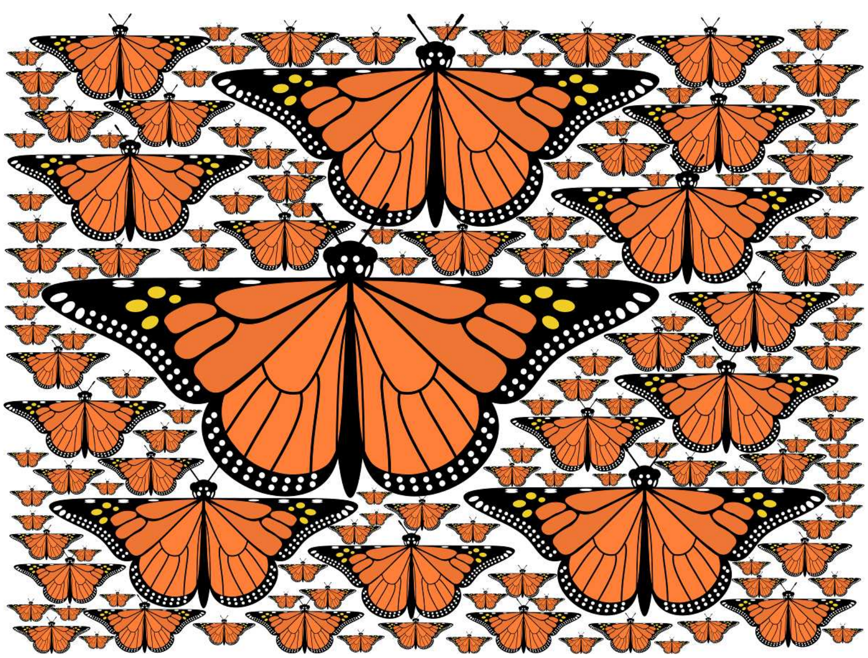

Figure 1: *A pattern of monarch butterflies.*

We begin by reviewing the basic algorithm and the *area rule*, which ensures that algorithm doesn't halt. Then we discuss patterns with 6-fold symmetry, $p6$ patterns in particular. Next we examine orientation restrictions on motifs, followed by a consideration of more complex motifs. Finally, we indicate directions of future work.

The Algorithm, A Review

The goal is to fill a region R with randomly placed, progressively smaller copies of a motif so that in the limit R will be completely filled with non-overlapping motifs. It seems intuitive that the algorithm should place the i-th motif m_i (of area A_i) as follows:

For each $i = 0, 1, 2, \ldots$

  Repeat:

    Randomly choose a point within R to place m_i

  Until (m_i doesn't intersect any of $m_0, m_1, ..., m_{i-1}$)

  Add m_i to the list of successful *placements*

Until some stopping condition is met, such as a maximum value of i or a minimum value of A_i.

The subtlety in this approach is how to specify the A_is so that the algorithm doesn't halt before the stopping condition is met. A natural choice is for the A_is to form a geometric series, but it has been found (by many experiments) that this doesn't work. What *has* been found to work in a wide variety of cases is to have the A_is obey an inverse power law, specifically if A is the area of region R:

$$A_i = \frac{A}{\zeta(c, N)(N + i)^c} \tag{1}$$

where $c > 1$ and $N > 1$ are parameters, and $\zeta(c, N)$ is the Hurwitz zeta function: $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$. We call this the *area rule*. Thus $\lim_{n\to\infty} \sum_{i=0}^{n} A_i = A$, i.e. the process is space-filling in the limit, as desired. Due to the random placement of ever-smaller motifs, the patterns produced by the algorithm have the same appearance at every scale, a common definition "fractal". In fact he fractal dimension D of the placed motifs can be computed to be $D = 2/c$. Avoiding premature halting depends mostly on the choice of c and not very much on N. For "good" shapes of R and motifs (e.g. smooth, convex), the maximum values of c that work can be as high as 1.5.

Patterns with 6-fold Symmetry

It has been known for over a century that there are 17 kinds of repeating plane patterns, often called "wallpaper" patterns. In 1978 Schattschneider wrote a paper clarifying wallpaper group notation [4]. Among the 17 possible kinds of wallpaper patterns, those with 6-fold symmetry may be the most interesting/complicated since in addition to 6-fold rotation points, they also have 3-fold and 2-fold rotation points. There are two kinds of patterns with 6-fold symmetry, those with the "reflection" symmetry group $p6mm$, and those with the "rotation" symmetry group $p6$. We have previously shown examples of patterns with $p6mm$ symmetry [2], and patterns with $p4$ symmetry [3]. Here we consider patterns with $p6$ symmetry.

To create our patterns we use our algorithm to fill a fundamental region for the symmetry group with motifs, giving a fractal pattern. Then we transform that filled region about the plane to create the whole pattern. Similarly to our treatment of $p4$, we modify the basic algorithm to treat motifs that overlap one of the rotation points. In this case, we move the motif so that it is centered on the rotation point if it has that kind of rotational symmetry (otherwise we discard it). Also the area calculation must be adjusted since only part of the motif is within the fundamental region. Specifically, if the motif has n-fold rotational symmetry, we increase its area by a factor of n so that it takes up the same space as it would if it did not overlap the rotation point. There are other possible adjustment options also.

Figure 2 shows a $p6$ pattern of circles with yellow, orange, and red circles on 6-fold, 3-fold, and 2-fold rotation points respectively. Figure 2 also shows an equilateral triangular fundamental region to the right of center and hexagonal "translation unit" that can be used to tile the plane using only translation. Here the Hurwitz parameters are: $c = 1.33$ and $N = 5$. But higher c values would probably work too because the motif is as simple as possible.

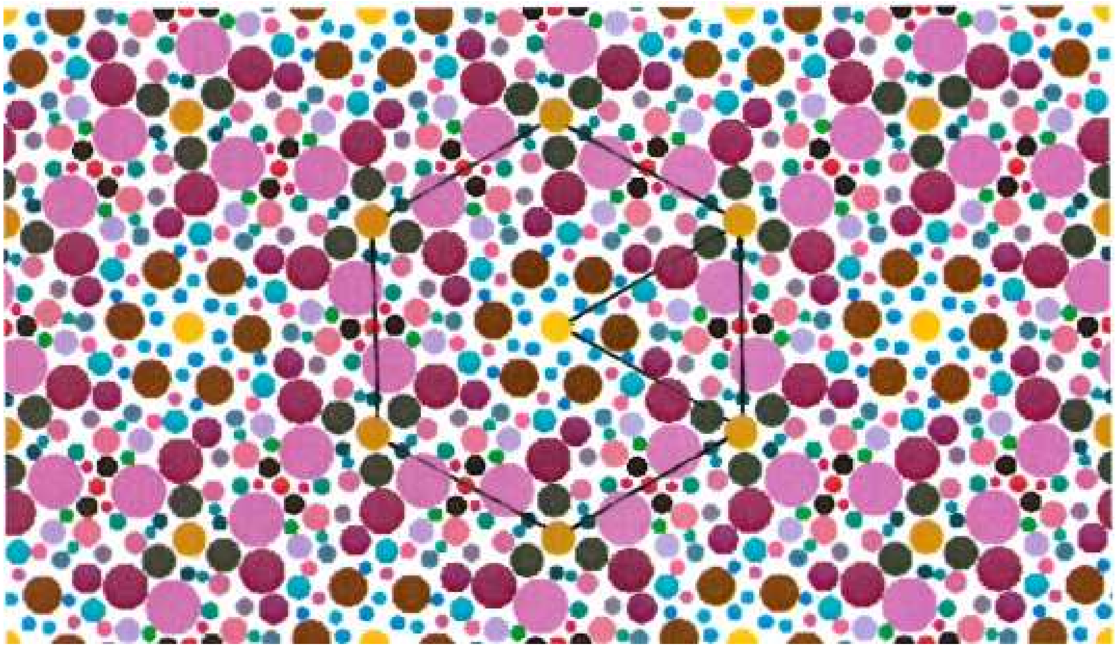

Figure 2: *A* $p6$ *pattern of circles.*

Figure 3 shows a $p6$ pattern of flowers with magenta flowers with yellow centers on the 6-fold rotation points, but no flowers on the 3-fold or 2-fold rotation points. In this case $c = 1.29$ and $N = 3$.

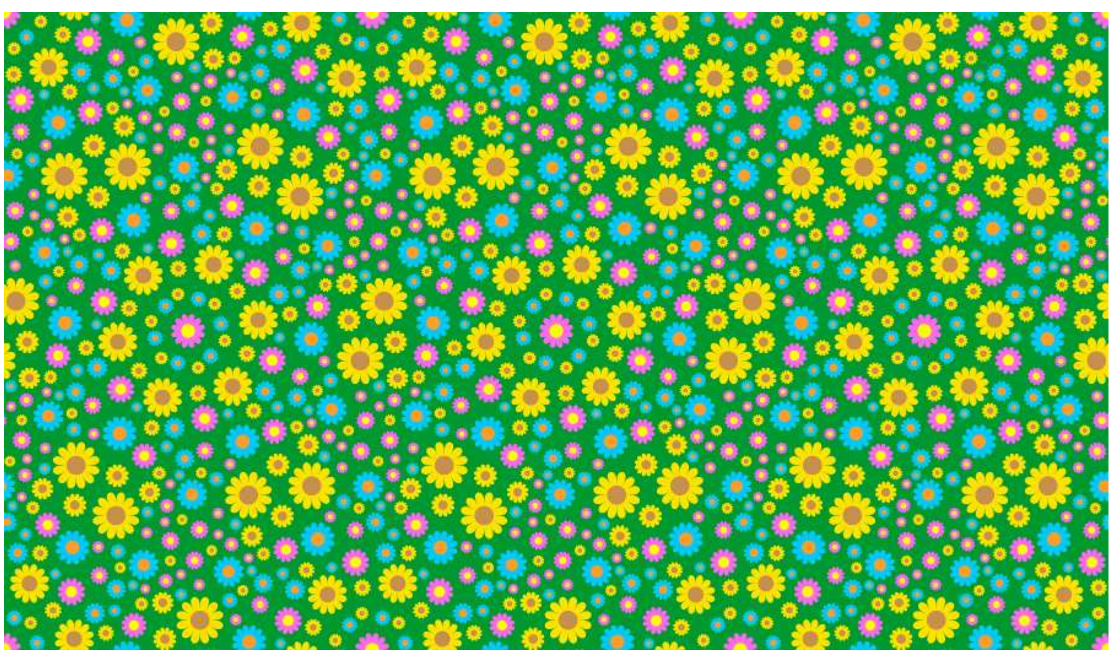

Figure 3: *A* $p6$ *pattern of flowers.*

Patterns Restricted by Motif Orientation

Initially we used our algorithm to create patterns whose motifs all had a fixed orientation as exhibited by the monarch butterflies of Figure 1. But we extended the algorithm to allow for different orientations. Previously we showed patterns whose motifs alternated between two or three orientations [1]. Figure 4 shows a pattern of rhombi in which the orientations (and colors) cycle between three values. We also showed patterns with motifs at random orientations. Figure 5 shows a randomly oriented pattern of arrows with the color determined by orientation.

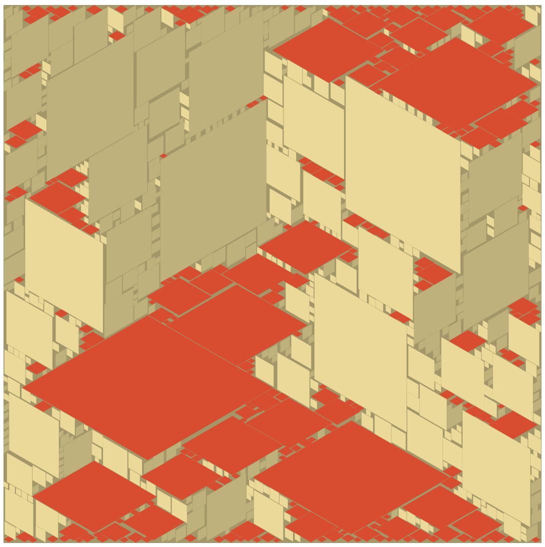

Figure 4: *Rhombi in three orientations.*

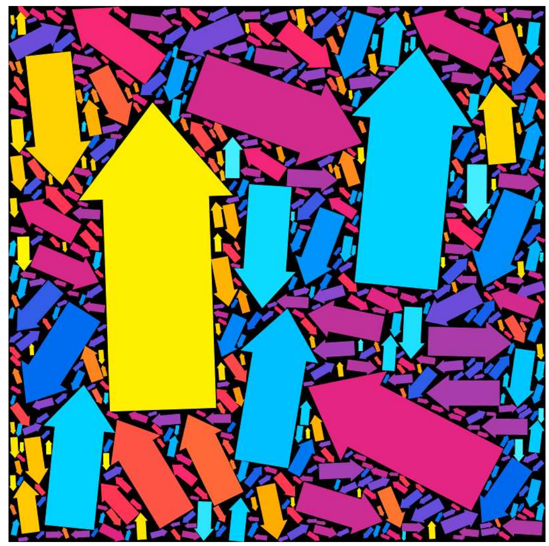

Figure 5: *Randomly oriented arrows.*

In what follows, we modify the algorithm to restrict the orientation of a motif according to its position. It is simplest to consider motifs with a reflection symmetry axis, though the technique is more general than that. We also assume that the region R and each motif has a "center". One modification to the algorithm that we make is to discard any motif whose center coincides with the center of R. This is a very unlikely possibility, given the randomness of the choices of positions of motifs. The significant modification we make is that we orient each motif as follows. The center of R and the center of the motif determine a direction in the plane — the motif is then rotated so that its symmetry axis is aligned with that direction.

Figure 6 shows a pattern of hearts within a circular region. The bilateral symmetry axis of each of the hearts goes through the center of the circle. In this pattern, $c = 1.32$ and $N = 6$. Figure 7 shows a pattern of squares again in a circular region. In this case a diagonal axis of symmetry of each square is used to orient it, i.e. each diagonal symmetry axis goes through the center of the circle. There are some of things to note about this pattern. First the simple shape of the motifs allows for a relatively high value of c, namely 1.50. Second, a moderately high value of N ($N = 12$) ensures that the largest (first placed) motifs are close to each other in size. And finally we note that the orientations of the squares produce a spiral effect since the angle each edge makes with the radial direction is constant (which produces a logarithmic spiral).

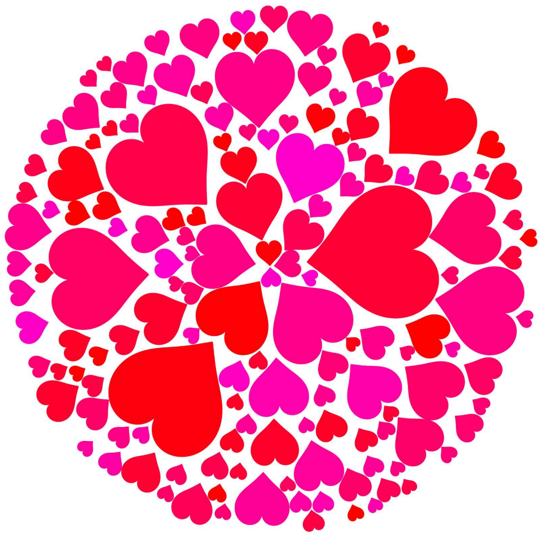

Figure 6: *An oriented heart pattern.*

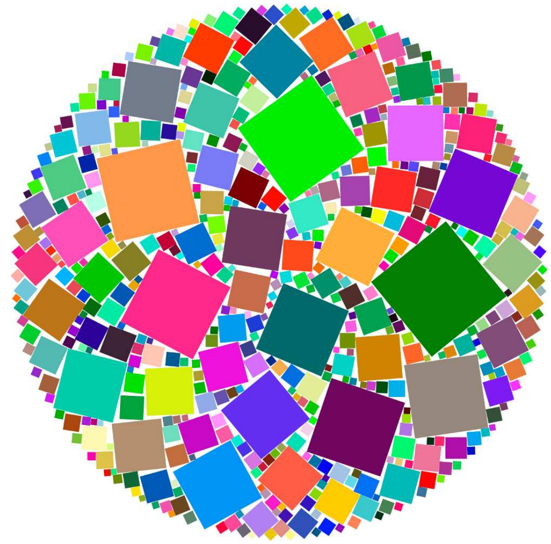

Figure 7: *An oriented pattern of squares.*

In Figures 8 and 9, the squares and rhombi are not oriented "radially" as in Figure 7, but each one is oriented by a smooth function of the coordinates of its position. The smoothness of the orientation function creates a "flowing" pattern which is more evident in Figure 9 than in Figure 8, since the rhombi guide our eyes to the flow direction. Also, in these figures, the center of the region plays no special role, as it does in Figure 7.

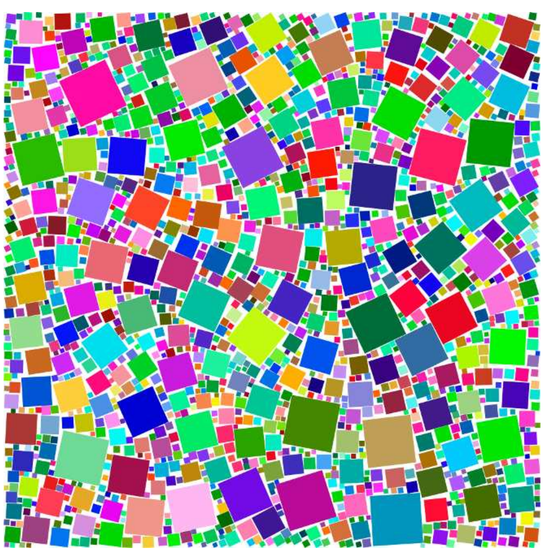

Figure 8: *A flowing pattern of colored squares, oriented by position.*

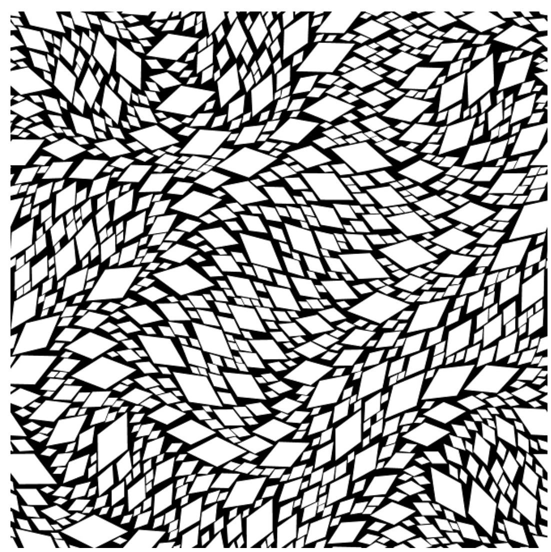

Figure 9: *A pattern of flowing white rhombi, oriented by position.*

A Pattern with a Complex Motif

It is possible to create motifs with detailed interior decorations without changing the basic algorithm. The monarch butterfly of Figure 1 is the most complicated motif we have designed so far. Even with such a motif, the run time was a few seconds and the file size was a few megabytes, so we haven't reached time or memory limitations with our algorithm yet.

In order to create the monarch pattern of Figure 1, the basic algorithm only had to be modified slightly. As one can see by examining Figure 1, the antennae of some butterflies overlap other butterflies. This is because we didn't include the antennae in the overlap test (for simplicity and to speed up the algorithm). Also the bigger butterflies are "on top", so their antennae only overlap smaller butterflies. Since the algorithm keeps a list of all the motifs as they are processed, this overlap behaviour is easily achieved by iterating through the list from smallest to largest butterflies.

The biggest challenge was to mathematically model the monarchs. The outline is described in polar coordinates (with the origin in the center of the butterfly), the radius being given as a slightly complicated Fourier polynomial. The outline is filled with black and all the details are drawn on top of it. The orange scales on the wings are each done "by hand" with Bezier spline curves. The ellipses are simply differentially scaled circles rotated to the proper angle. We note that the butterfly motif is not strictly a monarch of one gender, although it most closely resembles the female.

Finally, we note that the *Fractal Monarchs* pattern of Figure 1 was awarded Best photograph, painting, or print at the 2017 Joint Mathematics Meeting Art Exhibition in Atlanta on January 5, 2017. For this pattern $c = 1.26, N = 1.5$, and 150 butterflies achieved a 72% fill of the surrounding rectangular region.

Summary and Future Work

We have presented methods for creating fractal patterns with $p6$ symmetry, patterns with orientation restrictions, and patterns with a complicated motif.

The methods shown here to create $p6$ patterns extend those previously used to produce fractal wallpaper patterns based on the symmetry groups $p1$, $p2mm$, $p4mm$, and $p6mm$ [2], and $p4$ [3]. It seems evident that the techniques presented in those papers and here for $p6$ could be used to implement algorithms for generating repeating fractal patterns with the remaining wallpaper groups as their symmetry groups.

It would seem that there are more possibilities for specifying orientation functions for motifs in these fractal patterns. And the possibilities for creating patterns with complex motifs also appear to be limitless.

Finally, it would seem possible to modify our basic algorithm to fractally fill the fundamental region of a spherical or hyperbolic symmetry group with copies of motifs. And as we have done for some of the Euclidean wallpaper groups, that filled fundamental region could be transformed around the sphere or hyperbolic plane to create patterns in those respective geometries.

References

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