Comparing sizes of sets

Sets $A$ and $B$ are the same size if there is a bijection from $A$ to $B$.

(That was a definition!)

For finite sets $A, B$, it is not difficult to verify that

there is a bijection from $A$ to $B$ iff $|A| = |B|$.

Let’s do it. . .

Take arbitrary finite sets $A$ and $B$.

**LR**: Assume $f : A \rightarrow B$ is bijective.
Then $f$ is injective.
So, by the pigeonhole principle, $|A| \leq |B|$.
Also $f^{-1} : B \rightarrow A$ is injective. [Do you follow this step?]
So, again by the pigeonhole principle, $|B| \leq |A|$.
We can conclude that $|A| = |B|$.

**RL**: Assume that $|A| = |B|$. Since $A$ is finite, there is a bijection $f : A \rightarrow \{1, \ldots, |A|\}$. And since $B$ is also finite, there is a similar bijection $g : B \rightarrow \{1, \ldots, |B|\}$. Moreover, since $|A| = |B|$, the codomains of bijections $f$ and $g$ are the same. It follows that $g^{-1} \circ f$ is a bijection from $A$ to $B$. 
Let’s not write $|S|$ when $S$ is an infinite set

The textbook proposes the notation

$$|A| = |B|$$

to say that $A$ and $B$ are the same size. But this is bad notation when $A$ or $B$ is infinite!

Why? Because $|A|$ is not defined for an infinite set $A$.

What do I mean? Well, it won’t do, for instance, to say that $|A| = \infty$ whenever $A$ is infinite. Why? Because then any two infinite sets would be the same size!

But aren’t infinite sets all “the same size”?

(Namely, infinite.)

Well no, but that’s a long story...
Countable sets

A set is *countable* if it is finite or is the same size as $\mathbb{N}$.

So countable sets can be either finite or infinite.

The obvious question is: Are there any sets that are *not* countable?

Short answer: Yes.

Familiar example of an uncountable set: The set of real numbers.

Or the open interval of real numbers between 0 and 1.

Or $\text{power}(\mathbb{N})$.

Or $\text{power}(\{0, 1\}^*)$.

Soon we will learn a method — called “diagonalization” — for proving that one infinite set is larger than another.

But first let’s get a firmer understanding of countability...
Countability as “enumerability”

Pick a natural number $k$. If you start listing the natural numbers in their “standard” order — that is, enumerating them — you will reach $k$ in a finite number of steps (namely, $k + 1$ steps).

Intuitively, this is an argument that $\mathbb{N}$ is “enumerable”, or countable.

Similarly, take any set $S$ for which there is a bijection

$$f : S \rightarrow \mathbb{N}.$$ 

Each element $x$ of $S$ corresponds to a natural number $f(x)$.

If you start listing the elements of $S$ in the order given by $f$ ($f^{-1}(0), f^{-1}(1), f^{-1}(2) \ldots$), you will reach $x$ (for any given $x$) in a finite number of steps (namely $f(x) + 1$ steps).

This is the correct understanding of enumerability.

So how about this (fallacious) argument? To see that the real numbers are countable, do the following. Take an arbitrary real number $x$. Now start listing real numbers, one by one, and after some number of steps, include $x$ as the next element in the list. Since you wrote $x$ within a finite number of steps, the set of real numbers is countable.
Every subset of $\mathbb{N}$ is countable

**Claim:** Every subset of the natural numbers is countable.

**Proof:** Take any subset $S$ of $\mathbb{N}$. [So what is our goal now?]

If $S$ is finite, we’re done. So assume $S$ is infinite.

Notice that for each $x \in S$, $\{ n \mid n \in S, n < x \}$ is finite.

Take $f : S \to \mathbb{N}$ as follows. For all $x \in S$,

$$f(x) = | \{ n \mid n \in S, n < x \} | .$$

To see that $f$ is injective, consider any two distinct elements $x, y$ of $S$. Wlog assume that $x < y$. Then $f(x) < f(y)$. To see this, notice that

$$y \notin \{ n \mid n \in S, n < x \}$$

while

$$\{ n \mid n \in S, n < x \} \subset \{ n \mid n \in S, n < y \} .$$

To see that $f$ is surjective, take any $k \in \mathbb{N}$. Let $A$ be the set consisting of the $k + 1$ smallest elements of $S$. (That is, $|A| = k + 1$ and every element of $A$ is less than every element of $S - A$.) Let $x$ be the largest element of $A$, and notice that $f(x) = k$. 

To show that $A$ is countable, it is sufficient to show that there is an injection from $A$ to $\mathcal{N}$.

Indeed, if $f : A \rightarrow \mathcal{N}$ is injective, then there is a bijection $g : A \rightarrow \text{range}(f)$ such that, for all $x \in A$, $g(x) = f(x)$.

[How hard is it to see that $g$ is bijective?] The existence of such a $g$ shows that $A$ is the same size as $\text{range}(f)$, which is a subset of $\mathcal{N}$, and so, a countable set.

Or, equivalently, it suffices to show that there is a surjection from $\mathcal{N}$ to $A$.

Why? (Because this implies that there is an injection from $A$ to $\mathcal{N}$.)
Claim: Every subset of a countable set is countable.

Proof: Let $A$ be a subset of countable set $S$. Since $S$ is countable, there is an injection $f : S \rightarrow \mathbb{N}$. Take $g : A \rightarrow \mathbb{N}$ s.t. for all $x \in A,$

$$g(x) = f(x).$$

Then $g$ is an injection from $A \rightarrow \mathbb{N}$, which shows that $A$ is countable.
Proving countability (more generally)

To show that $A$ is countable, it is sufficient to show that there is an injection from $A$ to some countable set.

To see this, assume that $B$ is countable, and that

$$f : A \rightarrow B$$

is injective. Since $B$ is countable, there is an injection

$$g : B \rightarrow \mathcal{N}.$$ 

It follows that $(g \circ f)$ is an injection from $A$ to $\mathcal{N}$, from which we can conclude that $A$ is countable.

And from this it follows that we can also prove $A$ countable by showing that there is a surjection from some countable set to $A$.

[Why?]
The image of a countable set is countable

**Claim**: The image of a countable set (under any function) is countable.

Proof: Let $f$ be a function from $A$ to $B$.
Assume that $S$ is a countable subset of $A$.

[So what is our goal now?]
Take $g : S \rightarrow f(S)$ s.t. for all $x \in S$,

$$g(x) = f(x).$$

Notice that $g$ is a surjection from a countable set, namely $S$, to $f(S)$.
From this we can conclude that $f(S)$ is countable.
\( \mathbb{N} \times \mathbb{N} \) is countable

**Claim:** \( \mathbb{N} \times \mathbb{N} \) is countable.

Proof idea:

\[
\begin{align*}
(0, 0) & \leftrightarrow 0 \\
(0, 1), (1, 0) & \leftrightarrow 1, 2 \\
(0, 2), (1, 1), (2, 0) & \leftrightarrow 3, 4, 5 \\
(0, 3), (1, 2), (2, 1), (3, 0) & \leftrightarrow 6, 7, 8, 9 \\
(0, 4), (1, 3), (2, 2), (3, 1), (4, 0) & \leftrightarrow 10, 11, 12, 13, 14 \\
\vdots \\
(0, n), (1, n - 1), \ldots, (n, 0) & \leftrightarrow \sum_{i=0}^{n} i, \ldots, (\sum_{i=0}^{n} i) + n \\
\vdots
\end{align*}
\]

This bijection is given by Cantor’s pairing function, mentioned in the last set of lecture notes as an example of an injective function.

\[
f(x, y) = \left( \sum_{i=0}^{x+y} i \right) + x + \frac{(x+y)(x+y+1)}{2} + x
\]

\[
= \frac{x^2 + xy + x + y + y^2 + y}{2} + \frac{2x}{2}
\]

\[
= \frac{(x+y)^2 + 3x + y}{2}
\]
Every “countable union” of countable sets is countable

**Claim:** If $S_0, S_1, \ldots$ is a sequence of countable sets, then

$$\bigcup_{n \in \mathbb{N}} S_n$$

is also countable.

**Proof:** For each set $S_i$, let $f_i$ be a surjection from $\mathbb{N}$ to $S_i$. (Such a function $f_i$ exists, since $S_i$ is countable.) Take

$$g : \mathbb{N} \times \mathbb{N} \to \bigcup_{n \in \mathbb{N}} S_n$$

s.t. for all $m, n \in \mathbb{N}$,

$$g(m, n) = f_m(n).$$

Observe that $g$ is surjective. Indeed, take any

$$x \in \bigcup_{n \in \mathbb{N}} S_n.$$

Then, for some $m \in \mathbb{N}$, $x \in S_m$.

And since $f_m : \mathbb{N} \to S_m$ is surjective, there is an $n \in \mathbb{N}$ s.t. $f_m(n) = x$.

And since $g$ is a surjection from the countable set $\mathbb{N} \times \mathbb{N}$, we can conclude that $\bigcup_{n \in \mathbb{N}} S_n$ is countable.
The rational numbers are countable

In the last set of lecture notes, we considered a bijection between the integers and the natural numbers. The existence of such a function shows that the integers are countable.

Now, let’s show that the rational numbers are countable.

We’ll do this by representing them as a countable union of countable sets...

For each positive integer $d$, let

$$S_d = \left\{ \frac{n}{d} \mid n \in \mathbb{Z} \right\}.$$  

Since the integers are countable, so is $S_d$ (for every $d \in \mathbb{Z}^+$).

Let $S_0 = \emptyset$.

Now the set of rational numbers can be written as a countable union of countable sets, as follows.

$$\bigcup_{d \in \mathbb{N}} S_d$$

And this shows that the set of rational numbers is indeed countable.
The set of all strings (over any alphabet) is countable

Recall: An alphabet is a finite set of symbols, and for any alphabet $A$, $A^*$ is the set of all strings over $A$.

If $A$ is empty, then $A^* =$

If $A$ is a singleton, then it is still easy to see that $A^*$ is countable.

Indeed, take $f : A^* \to \mathbb{N}$ s.t. for all $x \in A^*$, $f(x) = |x|$.

If $|A| > 1$, we need a different approach.

As it happens though, the definition of closure is perfect for this:

$$A^* = \bigcup_{n \in \mathbb{N}} A^n$$

Notice that, for all $n \in \mathbb{N}$, $A^n$ is countable (in fact, finite).

Thus, $A^*$ is a countable union of countable sets, and so, countable.

So, is every language (over every alphabet) countable? Why?