1.2 Sets

Intuitively, a “set” is a collection of things, called its *elements*, or *members*.

To say that $x$ is an element of $S$, we write

$$x \in S.$$ 

Other ways of saying this: “$x$ belongs to $S$”, “$S$ contains $x$”, ”$x$ is in $S$”, . . .

Notice that we don’t actually define the term “set”. Instead, we take sets as primitives, whose essential property is membership.

If $x$ doesn’t belong to $S$, we write

$$x \notin S.$$ 

If $x$ and $y$ belong to $S$, we sometimes write

$$x, y \in S.$$ 

The set with no elements is called the *empty set*, and is often written

$$\emptyset.$$ 

A set with exactly one element is called a *singleton*. 
Equality of sets, notation for sets

So what a given set is simply a matter of what belongs to it. Consequently, two sets are equal (identical) if they contain the same elements.

There is some standard notation for specifying sets. (That is, for saying what are the elements of the set.)

For example,

\[ \{a, b, c\} \]

is the set whose elements are \(a\), \(b\) and \(c\).

Notice that

\[ \{a, b\} = \{b, a\} . \]

These are just two different representations of the same set.

In case you are wondering, the expression \(\{a, a, b\}\) for instance, would just be an unusual way of writing the set \(\{a, b\}\).
Sets may have sets as elements.

For example, the set

$$\{\emptyset\}$$

has one element, and the set

$$\{\emptyset, \{\emptyset\}\}$$

has two elements.

A set with finitely many elements is, of course, called *finite*.
You can guess what it means to say that a set is *infinite*.

Often we use ellipses to indicate elements of a set. For example,

$$\{0, 1, 2, \ldots, 9\}$$

can be understood to represent the set consisting of the natural numbers 0 through 9.

This convention is especially useful for representing infinite sets, such as

$$\{\ldots, -3, 0, 3, 6, \ldots\}.$$
We will write $\mathcal{N}$ to denote the set of natural numbers, and $\mathbb{Z}$ to denote the set of integers.

We can also specify sets using “set constructor” notation. For example,

$$\{ n \mid n \in \mathcal{N} \text{ and } n \text{ is even} \}$$

is the set of even natural numbers.

$$\{ n \mid n \in \mathbb{Z} \text{ and } 3 \mid n \}$$

is another way of representing the integers that are multiples of 3. (This expression looks like it might be ambiguous, but it isn’t — there’s only one way to parse it that makes sense.)

What is the following set?

$$\{ 2n \mid n \in \mathbb{Z} \}$$

How is it different from this one?

$$\{ \{ 2n \} \mid n \in \mathbb{Z} \}$$

What is this set?

$$\{ \{ n \mid n \in \mathcal{N} \text{ and } n < i \} \mid i \in \mathbb{Z} \}$$
Subset

For sets \(A, B\), we say \(A\) is a *subset* of \(B\), written

\[ A \subset B, \]

if every element of \(A\) belongs to \(B\).

Notice that, for any set \(S\),

\[ S \subset S \]

and

\[ \emptyset \subset S. \]

If \(A \subset B\) and \(A \neq B\), we say \(A\) is a *proper* subset of \(B\).

For example, \(\mathbb{N}\) is a proper subset of \(\mathbb{Z}\).

\[ \emptyset \in \emptyset? \]
\[ \emptyset \subset \emptyset? \]
\[ \emptyset \in \{\emptyset\}? \]
\[ \emptyset \subset \{\emptyset\}? \]
\[ \{\emptyset\} \subset \{\emptyset\}? \]
Claim: If $A \subset B$ and $B \subset C$, then $A \subset C$.

[Try to prove this.]
The power set of a set $S$, denoted by

$$\text{power}(S),$$

is the set of all subsets of $S$.

That is, $\text{power}(S) = \{ A \mid A \subset S \}$.

$$\text{power}(\{0\}) =$$
$$\text{power}(\emptyset) =$$
$$\text{power}(\text{power}(\emptyset)) =$$
$$\text{power}(\mathbb{N}) =$$

The cardinality, or size, of a finite set is the number of elements in it. For a finite set $S$, we denote its cardinality by

$$|S|.$$
Claim: \( A \subseteq B \) iff \( \text{power}(A) \subseteq \text{power}(B) \).

[Try proof in two parts. If you are having trouble seeing how to proceed in the RL part, you might take a look at its contrapositive. (Either way works nicely; good to try both.)]
Notice that $A = B$ iff $A \subset B$ and $B \subset A$.

(Do you see why?)

We can use that observation, plus the previous claim, to prove the following. . .

Claim: $A = B$ iff $\text{power}(A) = \text{power}(B)$.

Proof:

\[
A = B \iff A \subset B \text{ and } B \subset A
\]
\[
\iff \text{power}(A) \subset \text{power}(B)
\]
\[
\text{and } \text{power}(B) \subset \text{power}(A) \quad \text{(previous claim)}
\]
\[
\iff \text{power}(A) = \text{power}(B)
\]
Set union

The *union* of sets $A$ and $B$, written $A \cup B$, is defined by

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}.$$ 

For example, $\{0\} \cup \text{power}(\emptyset) =$

For sets $A_1, \ldots, A_n$,

$$\bigcup_{i=1}^{n} A_i = \{ x \mid \text{for some } i \in \{1, \ldots, n\}, x \in A_i \}.$$ 

For example, if $A_i = \{i\}$ for all $i \in \mathcal{N}$, then

$$\bigcup_{i=1}^{n} A_i =$$

What if $n = 1$?

What if $n = 0$?
For sets $A_0, A_1, A_2, \ldots$,

$$\bigcup_{i=0}^{\infty} A_i = \{ x \mid \text{for some } i \in \mathbb{N}, x \in A_i \}.$$  

For example, if $A_i = \{i, -i\}$ for all $i \in \mathbb{N}$, then

$$\bigcup_{i=0}^{\infty} A_i =$$

Similarly, if $I$ is a set of indices, and, for every $i \in I$, $A_i$ is a set, then

$$\bigcup_{i \in I} A_i = \{ x \mid \text{for some } i \in I, x \in A_i \}.$$  

Notice that, for any sets $A_0, A_1, A_2, \ldots$,

$$\bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i=0}^{\infty} A_i.$$  

Finally, if $S$ is a set of sets, then

$$\bigcup_{A \in S} A = \{ x \mid \text{for some } A \in S, x \in A \}.$$
Set intersection

The intersection of sets $A$ and $B$, written $A \cap B$, is defined as follows.

$$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$$

For example, $\text{power}(\emptyset) \cap \text{power}(\{0\}) = \emptyset$.

If $A \cap B = \emptyset$, we say that $A$ and $B$ are disjoint.

Notation for intersection is extended as it was for union...

$$\bigcap_{i=1}^{n} A_i = \emptyset$$
$$\bigcup_{i=0}^{\infty} A_i = \emptyset$$
$$\bigcap_{i \in I} A_i = \emptyset$$
$$\bigcap_{A \in S} A = \emptyset$$
Set difference

For sets $A, B$,

$$A - B = \{ x \mid x \in A \text{ and } x \notin B \} .$$

For example, take $A_1 = \{ n \mid n \in \mathbb{Z} \text{ and } n \text{ is odd} \}$ and $A_2 = \{ 2^n \mid n \in \mathcal{N} \}$.

$$A_1 - A_2 =$$

$$A_2 - A_1 =$$
Set complement

We can also take the complement of a set $A$, written $A'$, (wrt a given “universe of discourse” $U$):

$$A' = U - A.$$  

Of course for a use of this definition to make sense, one must know the identity of the universal set $U$. (Sometimes it is explicitly specified, but often it must be inferred from context.)

For example, take $U = \{0, 1\}$.

What is $\{0\}'$?

What is $\{0, 1\}'$?

If $A = \emptyset$, what is $A'$?

Now let $U$ be $\mathbb{N}$. Then

$$\{0, 1\}' = \{n + 2 \mid n \in \mathbb{N}\}.$$  

If $A = \{2n + 1 \mid n \in \mathbb{N}\}$, what is $A'$?
Claim: For every $x \in U$, $x \in A$ iff $x \notin A'$.

We can say this another way: For every $x \in U$,

- if $x \in A$ then $x \notin A'$, and
- if $x \notin A$ then $x \in A'$.

Another alternative is to replace the second if-then with its contrapositive, yielding: For every $x \in U$,

- if $x \in A$ then $x \notin A'$, and
- if $x \notin A$ then $x \in A'$.

With the claim stated in this form, it is easy to imagine a proof with the following structure.

Take any $x \in U$.

(i) Assume $x \in A$. Derive $x \notin A'$.
(ii) Assume $x \notin A$. Derive $x \in A'$.

The proof is easy to complete according to this plan...
Claim: For every $x \in U$, $x \in A$ iff $x \notin A'$.

Proof. First observe that

$$A' = U - A = \{x \mid x \in U \text{ and } x \notin A\} \tag{1}$$

Now take any $x \in U$.

(Left-to-right): Assume $x \in A$. By (1) we can conclude in this case that $x \notin A'$.

(Right-to-left): Assume $x \notin A$. By (1) we can conclude in this case that $x \in A'$.
\[ A \cap B = \{ x \mid x \in A \text{ and } x \in B \} \]

Claim: For every subset \( A \) of \( U \), \( A \cap A' = \emptyset \).

Proof. Take any \( x \). We need to show that \( x \notin A \cap A' \).

Consider two cases.

Case 1: \( x \in A \). By the previous result, it follows that \( x \notin A' \). And since

\[ A \cap A' = \{ x \mid x \in A \text{ and } x \in A' \} , \]

we can conclude in this case that \( x \notin A \cap A' \).

Case 2: \( x \notin A \). Since

\[ A \cap A' = \{ x \mid x \in A \text{ and } x \in A' \} , \]

we can conclude in this case that \( x \notin A \cap A' \).

Or, better, it is enough to observe that since no \( x \) belongs to both \( A \) and \( A' \) (by the prior result), we can conclude that no \( x \) belongs to \( A \cap A' \).
Counting finite sets

We'll assume we're working only with finite sets here...

The “union rule”:

\[ |A \cup B| = |A| + |B| - |A \cap B|. \]

That's easy. Let's extend it to three sets...

For this, we'll want a simple fact about intersection and union:

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \]

[There are many such facts listed in the textbook. You needn't remember them all, but it will help to be familiar with them — and to know how to check similar claims.]

OK, now we apply the union rule for two sets to get a similar rule for three sets:

\[
\begin{align*}
|A \cup (B \cup C)| &= |A| + |B \cup C| - |A \cap (B \cup C)| \\
&= |A| + (|B| + |C| - |B \cap C|) - |A \cap (B \cup C)| & \text{(union rule)} \\
&= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| & \text{(distribution)} \\
&= |A| + |B| + |C| - |B \cap C| \\
&\quad - (|A \cap B| + |A \cap C| - |A \cap B \cap C|) & \text{(union rule)} \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \\
&\quad + |A \cap B \cap C|
\end{align*}
\]
Russell’s paradox (1901)

Naive set theory is a little dangerous...

\[ X = \{ S \mid S \text{ is a set and } S \notin S \} . \]

It may seem to you that a set can never be an element of itself. In which case, this seems to be just an odd way to say that \( X \) is the set of all sets.

But we have a problem...

Consider two cases.

Case 1: \( X \in X \). Then \( X \) is a set that belongs to itself, so by the above equation \( X \notin X \).

Case 2: \( X \notin X \). Then \( X \) is a set that doesn’t belong to itself, so by the above equation \( X \in X \).