3.1 Inductively defined sets

Take

\[ A = \{3, 5, 7, \ldots \} . \]

Although we can’t be entirely certain, presumably this means that

\[ A = \{2n + 3 \mid n \in \mathbb{N}\} . \]

Another way to describe \( A \) is to say:

\[ 3 \in A \text{ and, for all } n \in \mathbb{N}, \text{ if } n \in A, \text{ then } n + 2 \in A . \]

If we want to understand this last description of \( A \) as a definition, it has three (!) parts, as follows:

- There is an “initial” element of \( A \), namely 3.
- You construct additional elements of \( A \) by adding 2 to any element of \( A \).
- Nothing else belongs to \( A \).

We call this an “inductive definition” of \( A \).
General form of inductive definition of a set

An inductive definition of a set $S$ has the following form:

- **Basis**: Specify one or more “initial” elements of $S$.
- **Induction**: Give one or more rules for constructing “new” elements of $S$ from “old” elements of $S$.
- **Closure**: Understand that $S$ consists of exactly the elements that can be obtained by starting with the initial elements of $S$ and applying the rules for constructing new elements of $S$.

Typically the closure condition is assumed (that is, left *unstated*), since it is standard.

There is another, more mathematically elegant way to understand the closure condition: *$S$ is the least set satisfying both the basis and induction conditions.*

Another way to understand this: *$S$ is the intersection of all sets that satisfy both the basis and induction conditions.*
Example: Let $S$ be defined as follows:

- **Basis:** $0 \in S$.
- **Induction:** If $n \in S$, then $n + 1 \in S$.

Then $S = \mathbb{N}$.

And we can check this. How? We can verify that $\mathbb{N}$ is the *least* set that satisfies the basis and induction conditions in the definition of the set $S$. [Actually, we'll check that $\mathbb{N}$ is minimal, which for reasons we won't fully explain (yet?), guarantees that it is least.]

1. $\mathbb{N}$ satisfies the basis condition.
2. $\mathbb{N}$ satisfies the induction condition.
3. No proper subset of $\mathbb{N}$ satisfies both conditions. Let's check this...

Take any proper subset $X$ of $\mathbb{N}$. [We need to show that $X$ doesn't satisfy both conditions. Why?] There is a *least* natural number $n$ missing from $X$. (That's a powerful claim.) Consider two cases. [Why are these cases exhaustive?]

Case 1: $n = 0$. Then $X$ doesn't satisfy the basis condition.

Case 2: $n = k + 1$ for some $k \in X$. Then $X$ doesn't satisfy the induction condition. (Why?)
Another example: Let $S$ be defined as follows:

- **Basis**: $0 \in S$.
- **Induction**: If $n \in S$, then $2n + 1 \in S$.

So what is $S$?

Notice: $2^0 - 1 = 0$, and for all $x \in \mathbb{N}$,

$$2^{x+1} - 1 = 2(2^x - 1) + 1.$$
An inductive definition of $A^*$ and other languages

If $A$ is an alphabet (a finite set), the set of all strings over $A$, $A^*$, can be defined as follows:

- **Basis:** $\Lambda \in A^*$.
- **Induction:** If $s \in A^*$ and $x \in A$, then $xs \in A^*$.

[What happens if we replace $xs$ above by $sx$?]

Let $L$ be the language over $\{0, 1\}$ defined as follows:

- **Basis:** $\Lambda \in L$.
- **Induction:** If $s \in L$, then $0s1 \in L$.

Let $L$ be the language over $\{0, 1\}$ defined as follows:

- **Basis:**
  1. $\Lambda \in L$.
  2. If $x \in \{0, 1\}$, then $x \in L$.
- **Induction:** If $s \in L$ and $x \in \{0, 1\}$, then $xsx \in L$. 
The set of binary trees over \( A \)

Define the set \( B \) of binary trees over set \( A \) as follows:

- **Basis**: \( \langle \rangle \in B \).
- **Induction**: If \( L, R \in B \) and \( x \in A \), then \( \langle L, x, R \rangle \in B \).

**Note**: This is the list representation of binary trees, given as a shorthand for the longer notation “tree\((L, x, R)\)” which is also presented in the text.

Define the set \( Twins \) over set \( A \) as follows:

- **Basis**: \( \langle \rangle \in Twins \).
- **Induction**: If \( x \in A \) and \( T \in Twins \), then \( \langle T, x, T \rangle \in Twins \).
For any nonempty binary tree $T = \langle L, x, R \rangle$, let
\[ \text{left}(T) = L, \quad \text{root}(T) = x, \quad \text{right}(T) = R. \]

Define the set $\text{Opps}$ over set $\{0, 1\}$ as follows:

- **Basis**: If $x \in \{0, 1\}$, then $\langle \langle \rangle, x, \langle \rangle \rangle \in \text{Opps}$.

- **Induction**: If $x, y \in \{0, 1\}$, $T \in \text{Opps}$, and $y \neq \text{root}(T)$,
  then $\langle T, x, \langle \text{right}(T), y, \text{left}(T) \rangle \rangle \in \text{Opps}$. 
Define the set $F$ of subsets of $\mathcal{N}$ as follows:

- **Basis:** $\emptyset \in F$.
- **Induction:** If $n \in \mathcal{N}$ and $S \in F$, then $S \cup \{n\} \in F$.

What is $F$?

Can we prove this? (By showing that it is a minimal set satisfying the basis and induction conditions.)