Our topic this time is asymptotic notation, discussed in Section 5.6 of the textbook. In some details the treatment in the lecture notes diverges from that of the textbook; in all such cases, prefer the treatment in the lecture notes.

An excellent way to study asymptotic notation is to attempt to duplicate the big-$O$ results from the lecture notes, or to prove similar claims about $\Omega$ and $\Theta$. Exercises 14 – 20 (pages 391, 392) are good examples. Exercises 1 – 9 (pages 389, 390) may be easier (skip Exercises 10 – 13).

1  Prove that $f(n) \in \Theta(g(n))$ iff $f(n) \in \Omega(g(n))$ and $f(n) \in O(g(n))$.

LR: Assume that $f(n) \in \Theta(g(n))$. So there are positive $c, d, n_0$ s.t.

$$c|g(n)| \leq |f(n)| \leq d|g(n)| \quad \text{for all } n \geq n_0.$$ 

This implies the desired result. That is, we have $f(n) \in \Omega(g(n))$ because

$$c|g(n)| \leq |f(n)| \quad \text{for all } n \geq n_0,$$

and we have $f(n) \in O(g(n))$ because

$$|f(n)| \leq d|g(n)| \quad \text{for all } n \geq n_0.$$

RL: Assume $f(n) \in \Omega(g(n))$ and $f(n) \in O(g(n))$. So there are positive $c, d, n'_0, n''_0$ s.t.

$$c|g(n)| \leq |f(n)| \quad \text{for all } n \geq n'_0,$$

and

$$|f(n)| \leq d|g(n)| \quad \text{for all } n \geq n''_0.$$

Take $n_0 = n'_0 + n''_0$ ( or max($n'_0, n''_0$) ). It follows that

$$c|g(n)| \leq |f(n)| \leq d|g(n)| \quad \text{for all } n \geq n_0.$$

Thus, $f(n) \in \Theta(g(n))$. 

1
2 Prove that if \( f(n) = \Omega(g(n)) \) and \( g(n) = \Omega(h(n)) \), then \( f(n) = \Omega(h(n)) \).

Assume \( f(n) = \Omega(g(n)) \) and \( g(n) = \Omega(h(n)) \).

It follows that there are positive \( c', n'_0, c'', n''_0 \) s.t.
\[
c'|g(n)| \leq |f(n)| \quad \text{for all } n \geq n'_0
\]
and
\[
c''|h(n)| \leq |g(n)| \quad \text{for all } n \geq n''_0.
\]
Take \( c = c'c'' \) and \( n_0 = n'_0 + n''_0 \) (or \( \max(n'_0, n''_0) \)). Notice that
\[
c|h(n)| = c'c''|h(n)| \leq c'|g(n)| \leq |f(n)| \quad \text{for all } n \geq n_0.
\]
Thus, \( f(n) = \Omega(h(n)) \).

3 Prove that \( f(n) = O(g(n)) \) iff \( g(n) = \Omega(f(n)) \).

LR: Assume that \( f(n) = O(g(n)) \). So there are positive \( c, n_0 \) s.t.
\[
|f(n)| \leq c|g(n)| \quad \text{for all } n \geq n_0.
\]
Since \( c \) is positive, it follows that
\[
\frac{1}{c} \cdot |f(n)| \leq |g(n)| \quad \text{for all } n \geq n_0.
\]
And since \( \frac{1}{c} \) is also positive, we have shown that \( g(n) = \Omega(f(n)) \).
Proof in the other direction is similar.
4 Prove that $f(n) = o(g(n))$ iff $g(n) = \omega(f(n))$.

LR: Assume that $f(n) = o(g(n))$. [Need to show $g(n) = \omega(f(n))$.] Let $c$ be an arbitrary positive constant. We need to show that there is a positive $n_0$ s.t.

$$c|f(n)| < |g(n)| \quad \text{for all } n \geq n_0.$$ 

From the assumption that $f(n) = o(g(n))$ and the fact that $\frac{1}{c}$ is positive, we can conclude that there is a positive $n_0$ s.t.

$$|f(n)| < \frac{1}{c}|g(n)| \quad \text{for all } n \geq n_0,$$

and the desired result then follows easily (multiply both sides by $c$).

Proof in the other direction is similar.

5 Prove that if $f(n) \in o(g(n))$, then $f(n) \notin \Omega(g(n))$.

Suppose that $f(n) \in o(g(n))$ and $f(n) \in \Omega(g(n))$. [We complete the proof by obtaining a contradiction.]

Since $f(n) \in \Omega(g(n))$, there are positive $c, n_0$ s.t.

$$c|g(n)| \leq |f(n)| \quad \text{for all } n \geq n_0.$$ 

It then follows from the assumption that $f(n) \in o(g(n))$ that there is an $m_0$ s.t.

$$|f(n)| < c|g(n)| \quad \text{for all } n \geq m_0.$$ 

Consequently, for all $n \geq m_0 + n_0$ (or $\max(n_0', n_0'')$),

$$c|g(n)| \leq |f(n)| < c|g(n)|$$

which is impossible. Contradiction.