H.S.M. Coxeter and Tony Bomford's Colored Hyperbolic Rugs

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Abstract

Tony Bomford made six hooked rugs based on hyperbolic geometry, having received inspiration from the Canadian mathematician H.S.M. Coxeter. All rugs except one exhibit color symmetry to some degree. We will analyze the colorings of the other five rugs and suggest a new related rug pattern.

1. Introduction

A.G. (Tony) Bomford (1927–2003) was an Australian surveyor with wide-ranging interests. One of those interests was making mathematically based hooked rugs, of which Figures 1 and 2 are examples. My paper

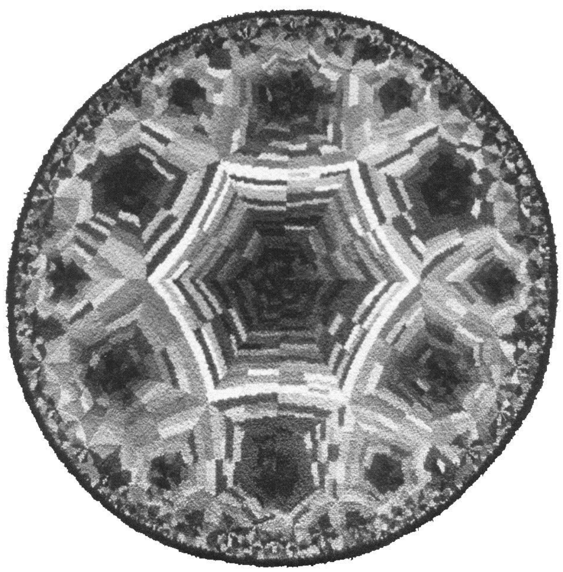

Figure 1: Tony Bomford's Rug #13: *Hyperbolic Lagoon.*

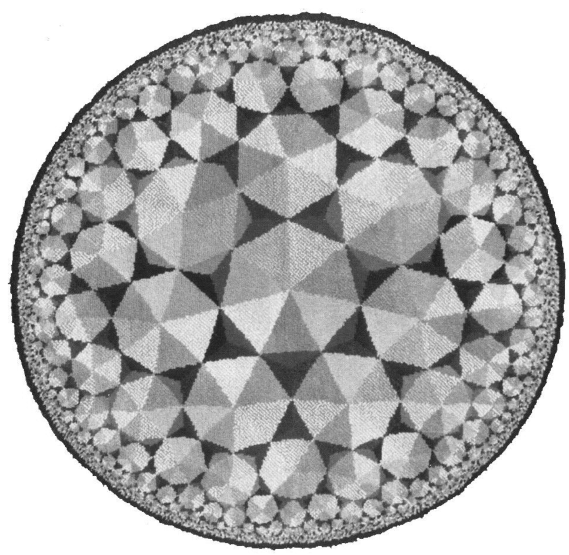

Figure 2: Tony Bomford's Rug #16: *Triangles and Heptagons.*

in Bridges 2004 contains a brief biographical sketch of Tony and some of the mathematics upon which his hyperbolic rugs were based [Dunham1].

The Canadian mathematician H.S.M. (Donald) Coxeter provided Bomford with two key inspirations. The first, in 1981, was a reproduction of M. C. Escher's hyperbolic pattern *Circle Limit IV,* which Tony saw in Coxeter's chapter "Angels and Devils" in *The Mathematical Gardner* [Coxeter3]. This inspired Tony to make his first hyperbolic rug, Rug #12 which he called *Hyperbolic Spiderweb,* (Figure 3). Tony went on to create five more hyperbolic hooked rugs between 1982 and 1989. He did not quite complete his last hyperbolic rug, Rug #18.

A second inspiration followed when Coxeter sent Tony a reprint of his article "Regular compound tessellations of the hyperbolic plane," [Coxeter2]; Figure 10 of that article provided Tony with information on how to color Rug #16 (Figure 2 above) and Rug #17. More details of the extensive Coxeter-Bomford interaction can be found in Doris Schattschneider's article in this volume [Schattschneider2], and in [Schattschneider1]. Tony Bomford's "Rug Book" contains a few sentences about each of his rugs and has a lengthy analysis of various tessellations embedded in Rugs #16 and #17 [Bomford1].

We start with a short review of essential geometrical concepts needed to analyze Tony's rugs. Tony created 18 rugs altogether; Rugs #12 through #18 are hyperbolic, except for Rug #14 which is a Euclidean design. We first analyze Rug #12 for color symmetries, then skip over Rug #13 (Figure 1 above), which, while quite colorful has no apparent color symmetry. Next, we discuss Rugs #15 and #18 which are related and have more color symmetry than Rug #12. Then, we analyze Rugs #16 and #17, which are also related and have even more color symmetry. Finally we show an original Coxeter-inspired design for a possible Bomford-like rug.

2. Hyperbolic Geometry, Tessellations, and Color Symmetry

Coxeter, Escher, and Bomford all used the *Poincaré disk model* of hyperbolic geometry whose points are the interior points of a bounding circle, and whose (hyperbolic) lines are circular arcs orthogonal to the bounding circle (including diameters). These circular arcs are apparent to various degrees in Tony's rugs and are made explicit in Figure 4 below. The hyperbolic measure of an angle is the same as its Euclidean measure in the disk model (we say such a model is *conformal*), but equal hyperbolic distances correspond to ever-smaller Euclidean distances as figures approach the edge of the disk. For example, all the heptagons in Figure 2 are hyperbolically congruent, as are all the hexagons in Figure 4.

There is a *regular tessellation,* $\{p, q\}$, of the hyperbolic plane by regular p-sided polygons meeting q at a vertex provided $(p - 2)(q - 2) > 4$. Figure 4 shows the regular tessellation $\{6, 4\}$, upon which Tony's Rugs #12 and #13 are based. Each regular tessellation $\{p, q\}$ gives rise to a semi-regular (or uniform) tiling ($p.q.p.q$) in which two regular p-sided polygons and two regular q-sided polygons are placed alternately around each vertex. One can construct ($p.q.p.q$) from $\{p, q\}$ by connecting midpoints of successive sides of each of the regular p-sided polygons. It can be seen in Figure 2 that Tony's Rug #16 is based on the tiling (7.3.7.3), as is #17. Tony's Rugs #15 and #18 are based on the tiling ($5.4.5.4$) (see Figures 5 and 6).

A *symmetry* of a pattern is transformation that maps the pattern onto itself. For example, disregarding color, reflection in the vertical diameter of the rug in Figure 2 preserves that pattern, as does rotation by $\frac{360}{7}$ degrees about the center. The only symmetries we will need to consider in our analysis are (hyperbolic) reflections and rotations. A reflection across a hyperbolic line in the disk model can be obtained by inversion in the orthogonal circular arc representing that line (of course reflection across a diameter is just a Euclidean reflection). As in Euclidean geometry, successive reflections across two intersecting lines results in a rotation about the intersection point by twice the angle between the lines.

A *color symmetry* of a pattern is a symmetry of the uncolored pattern that maps all parts of the pattern having the same color onto parts of a single color (possibly the same color) — that is, the symmetry permutes the colors. For example, in Figure 2 the reflection in the vertical diameter swaps pairs of tan colors of opposite triangles in the central heptagon and fixes the dark tan of the bottom triangle. The rotation by $\frac{360}{7}^\circ$ about the center of the rug permutes those seven shades of tan in a cycle. If every symmetry of a pattern is a color symmetry, that pattern is said to have *perfect color symmetry.* Tony's Rugs #16 and #17 exhibit perfect color symmetry (although, as we shall see, some interpretation needs to be made). For more information on color symmetry, see [Grünbaum & Shephard1], [Schwarzenberger1], and [Senechal1].

3. The Color Symmetries of Rug #12

Figure 3 shows Rug #12, *Hyperbolic Spiderweb*, Tony Bomford's first hyperbolic rug, inspired by M. C. Escher's hyperbolic pattern *Circle Limit IV*, which in turn was inspired by Figure 7, a hyperbolic tessellation of triangles in Coxeter's article "Crystal Symmetry and Its Generalizations" [Coxeter1]. (See [Schattschneider2] for details on Escher's inspiration.) Rug #12, *Circle Limit IV*, and Coxeter's figure are all based on the regular tessellation $\{6, 4\}$, in which four hexagons meet at every vertex (see Figure 4).

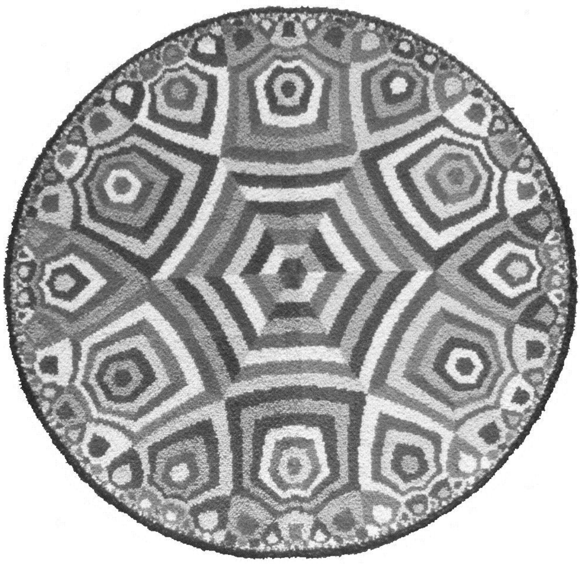

Figure 3: Tony Bomford's Rug #12: *Hyperbolic Spiderweb*

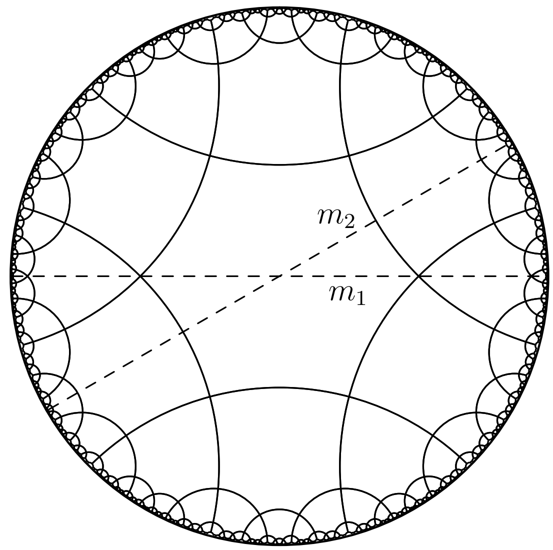

Figure 4: The regular tessellation $\{6, 4\}$ (solid lines), together with a diameter m_1 and side perpendicular bisector m_2 (dashed lines).

Tony used two sets of colors for Rug #12: three shades of tan (the lightest one almost white) and four shades of "reds" (the lightest almost orange and the darkest almost brown). There are eight hexagonal rings of color that make up the central hexagon, alternating between reds and tans, starting with reds at the center. The six edges of a tan ring contain all three shades of tans, progressing from lightest to darkest in clockwise order. The six edges in a red ring have two shades of red that alternate: the central ring (actually a small hexagon), and fifth ring (third red ring) use the two mid-level reds; the third and seventh rings use the orangish and brownish reds. The reds in two red rings of the same color are offset by 60 degrees, so that each of the 60-45-45 triangles comprising the central hexagon contains an edge of each of the four reds. For convenience, we number reds and tans from lightest to darkest: red1, red2, red3, red4, and tan1, tan2, tan3.

There are two kinds of reflections of the central hexagon onto itself: three reflections across the diameters of the hexagon (such as m_1 in Figure 4; these diameters connect opposite vertices), and three reflections across perpendicular bisectors of the sides (such as m_2 in Figure 4). The first kind of reflection interchanges red1 and red4, and red2 and red3, that is, each one produces the permutation (red1 red4)(red2 red3) of reds. But these reflections are not color symmetries of the whole pattern since they 'mix up' the tans. For example the horizontal reflection interchanges tan1 and tan3 in the eighth (outer) ring, but interchanges tan2 and tan3 in the sixth ring. Reflections across perpendicular bisectors also mix up the tans, so they are not color symmetries either.

On the other hand, a 60° counterclockwise rotation about the center of the circle *is* a color symmetry since it induces the following color permutation: (red1 red4)(red2 red3)(tan1 tan3 tan2). This rotation is produced by successive reflections across the horizontal diameter m_1 and the perpendicular side bisector m_2. So even though neither reflection alone is a color symmetry, the composition of the two reflections is a color symmetry. Also, this rotation is a color symmetry of the entire rug, not just the central hexagon.

There is one other kind of limited color symmetry that applies only to the six hexagons that share an edge with the central hexagon. These hexagons are made up of seven colored rings: four red rings alternating with three tan rings. When limited to such a hexagon, reflection across a diameter or rotation by 60° interchanges the two lightest and two darkest reds and fixes the tans: it produces the permutation (red1 red2)(red3 red4)(tan1)(tan2)(tan3) (a 1-cycle indicates a fixed color). Reflections across perpendicular side bisectors of those hexagons do not change any colors. All the other smaller (to our Euclidean eyes) hexagons are composed of rings of constant color, and so only have trivial color symmetry.

4. The Color Symmetry of Rugs #15 and #18

Bomford's next hyperbolic rug to display color symmetry is Rug #15, shown in Figure 5. Figure 6 shows the underlying tiling $(5.4.5.4)$ upon which it is based. Tony used five shades of green to color the triangular regions that make up the pentagons and five shades of tan to color the regions that make up the squares. For discussion purposes, we number the greens and the tans from 1 to 5, from lightest to darkest.

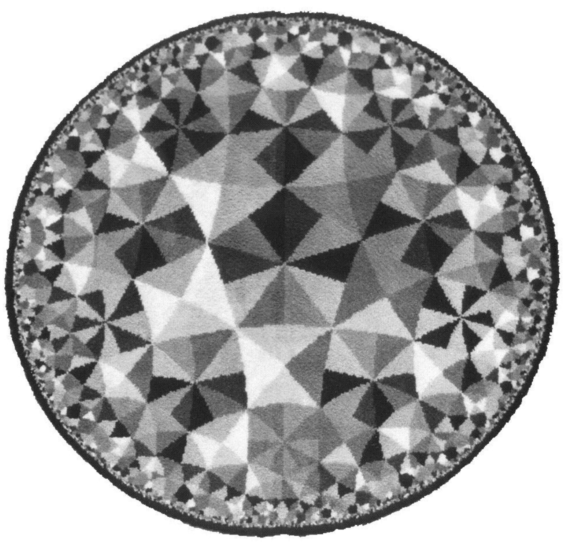

Figure 5: Bomford's rug #15 *Squares and Pentagons.*

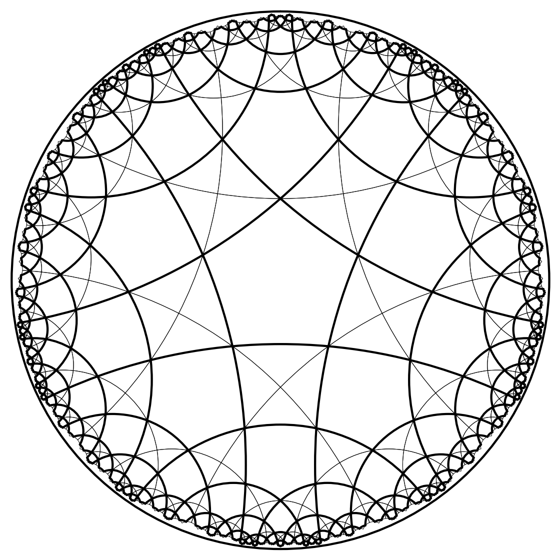

Figure 6: The tiling (5.4.5.4) (bold lines) superimposed on the tessellation $\{5, 4\}$ (thin lines).

Reflection across the vertical diameter of the circle produces the color permutations (green1)(green2 green3)(green4 green5) and (tan1 tan5)(tan2)(tan3 tan4), where we emphasize that green1 and tan2 are sent to themselves by listing them as 1-cycles. The reflection about the diameter that makes an angle of 36° with the vertical diameter induces the color permutations (green1 green2)(green3 green5)(green4) and (tan1 tan4)(tan2 tan5)(tan3). If we perform these reflections in succession, we obtain a 72° counterclockwise rotation, which gives the color permutations (green1 green2 green5 green4 green3) and (tan1 tan2 tan5 tan4 tan3); the cyclic permutations for shades of greens and tans are the same. We note that the color permutation structure is the same for each of the five reflections across a diameter of the bounding circle that goes through a vertex of the central pentagon.

Each of the vertices of the central pentagon touches a vertex of one of a set of five other "smaller" pentagons (which are hyperbolically the same size as the central pentagon). A $180°$ rotation about each of these common vertices interchanges the central pentagon and the "smaller" pentagon, inducing the identity permutation on each of the green shades — that is, each shade of green goes to itself. (This is a local color symmetry, not a color symmetry of the whole pattern.)

There are also pairs of squares that share a common vertex in the $(5.4.5.4)$ tiling, but the $180°$ rotation about that vertex mixes up the shades of tan — the two isosceles triangles of the same shade of tan go to triangles of different shades. Thus such a rotation is not a color symmetry of paired squares.

There is another set of five "even smaller" pentagons, each one sharing two of its vertices with vertices of two of the "smaller" pentagons. Each of these "even smaller" pentagons is separated from the central pentagon by a square — the "even smaller" pentagon and the central pentagon are on opposite sides of the square. The right triangles of an "even smaller" pentagon are alternately colored with the two shades of green of the two right triangles opposite it in the central pentagon. Because of their small size, the pentagons closer to the outer edge of the rug are subdivided into 5 isosceles triangles, and their colorings do not have color symmetry.

Rug #18 was Tony's last; it was not quite completed before his death in 2003. It is also based on the tiling $(5.4.5.4)$ and he chose its colors to bring out the $(5.4.5.4)$ pattern more clearly.

5. Rugs #16 and #17

Rugs #16 (Figure 2) and #17 (Figure 7) are the most symmetrically-colored of Tony's hyperbolic rugs. Both rugs use the (7.3.7.3) tiling (Figure 8), in which heptagons and equilateral triangles alternate about each vertex. The equilateral triangles and the heptagons are divided into three and seven isosceles central triangles respectively. Both rugs use the same color scheme, but differently-colored wool. In each rug,

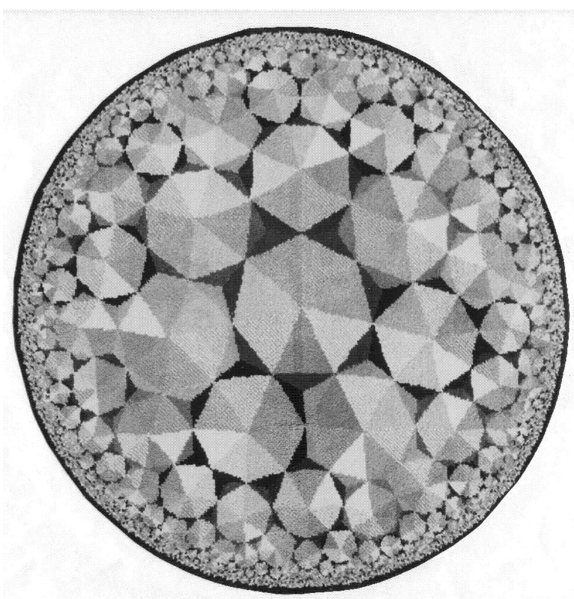

Figure 7: Bomford's Rug #17

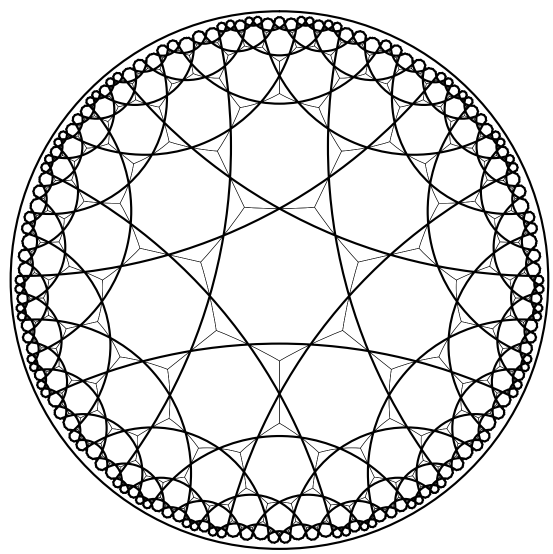

Figure 8: The tiling (7.3.7.3) (bold lines) superimposed on the tessellation $\{7, 3\}$ (thin lines).

eight shades of tan are used to color the triangles that make up the heptagons. The eight shades are actually created from four shades of wool — the four "pure" tans plus four "tweeds" created by alternating pairs of different tans from knot to knot. The tans used in the two rugs appear to be slightly different. Rug #16 uses eight shades of red to color the isosceles triangles within the equilateral triangles from very dark red to almost yellow. Rug #17 uses eight shades of blues and greens for the same purpose and is somewhat more effective since there are more striking differences in the colors. For convenience we use the term "bright colors" for the reds of Rug #16, and for the greens and blues of Rug #17.

In both rugs each shade of tan is paired with a bright color. We can see that each heptagon of the (7.3.7.3) tiling is surrounded by seven isosceles triangles of a single bright color (forming a slightly larger heptagon). Similarly, each equilateral triangle of the (7.3.7.3) tiling is surrounded by three isosceles triangles of a single shade of tan, forming a larger equilateral triangle with a center of bright colors. The bright color surrounding a heptagon is associated with the shade of tan that is missing from that heptagon (which contains only seven of the eight tans). For example the dark blue that surrounds the central heptagon is associated with the lightest tan in Rug #17 above. This relationship is shown in Figure 9 below. Notice that in this figure, the larger light tan equilateral triangles have a "twist" to their arrangement, thus the patterns in Rugs #16 and #17 are chiral — they have a "handedness" to them.

Coxeter's article, "Regular compound tessellations of the hyperbolic plane" contains the (7.3.7.3) tiling in its Figure 10, with the heptagons numbered from 1 to 8, and the triangles labeled with pairs of numbers [Coxeter2]. Here, the two numbers in each triangle are those that are not in the adjacent heptagons or in the heptagons opposite each of its vertices (see Figure 10 below.) Tony used this numbering scheme as a template to arrange the colors in his Rug #17. Coxeter also supplied the permutations of the numbering scheme corresponding to a counterclockwise rotation of $\frac{360}{7}^\circ$ about the center and a 180° rotation about the top vertex of the central heptagon. These permutations are $(1\ 2\ 3\ 4\ 5\ 6\ 7)(8)$ and $(1\ 8)(2\ 7)(3\ 4)(5\ 6)$, respectively. But Tony probably did not make use of these permutations, since he could read the numbers directly from Coxeter's figure. Also, the number pattern in Figure 10 is easily extended as follows. The perpendicular bisector of a heptagon edge goes through the heptagon vertex opposite it. The first heptagons meeting the perpendicular bisector on either side of the original heptagon are assigned the same number.

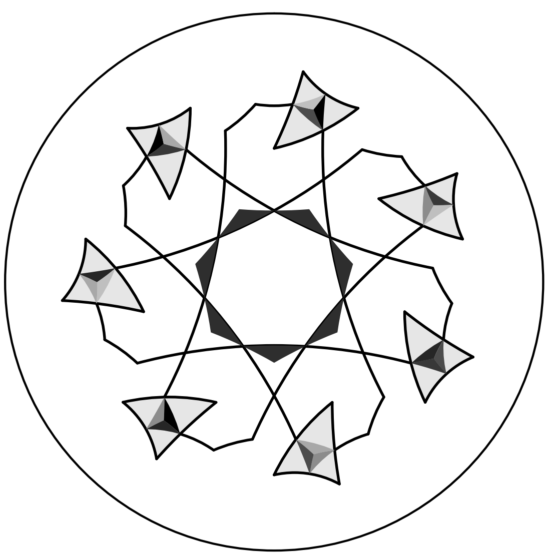

Figure 9: Some of the "larger" equilateral triangles whose lightest tan corresponds to the blue around the central heptagon.

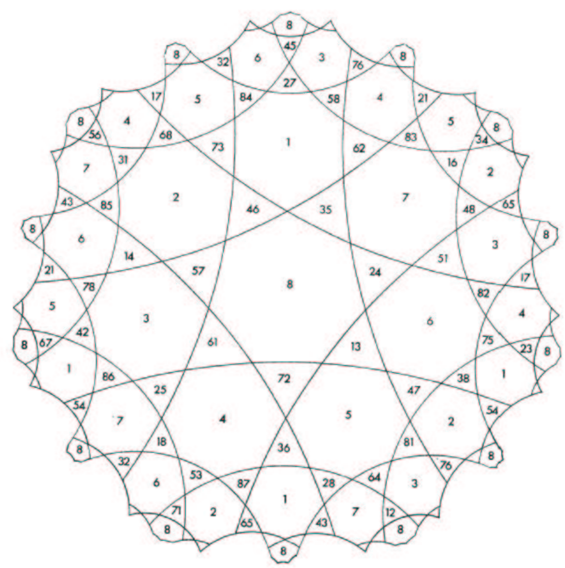

Figure 10: Figure 10 of Coxeter's paper [Coxeter2].

If we consider each pair (bright color, tan) to be a single "color", then Rugs #16 and #17 each exhibit perfect color symmetry — every symmetry of the pattern is a color symmetry. In fact the group of color permutations of this pattern is isomorphic to the simple group $PSL(2, 7)$ of order 168.

Unfortunately Rug #17 was lost on an airplane flight in 1993 enroute to an exhibition in the western United States. Although it seems unlikely that it will be found, I offer $1000 for information leading to its return to the Bomford family.

6. A New Coxeter-Bomford Rug Pattern

Although Tony was in possession of a copy of Coxeter's article "Regular compound tessellations of the hyperbolic plane" at the time he started Rug #15 in March, 1985, he does not appear to have made use of Coxeter's Figure 8 (see Figure 11 below) in that article to color his Rugs #15 and #18. Coxeter's Figure 8 shows a $(5.4.5.4)$ tessellation with the pentagons numbered from 1 to 6 and the squares assigned the two numbers missing from its adjacent pentagons. For example, the square labeled with the number pair 2,4 is surrounded by pentagons numbered 6, 1, 3, and 5. These numbers provide instructions for a perfect coloring of the tessellation $(5.4.5.4)$ that requires six colors. In fact, in coloring patterns based on the $\{5, 4\}$ tessellation, I have found that I always needed six colors to achieve a perfect coloring.

Figure 12 shows my original colored rug pattern based on Coxeter's Figure 8 and the coloring scheme of Tony's Rugs #16 and #17. Here, each pentagon is divided into five isosceles central triangles, colored in

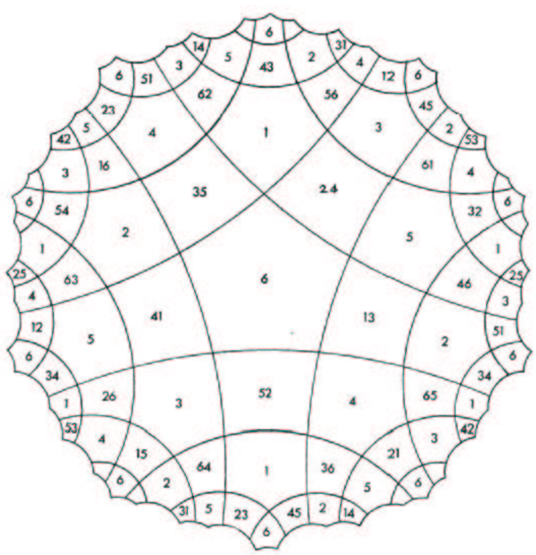

Figure 11: Figure 8 of Coxeter's article [Coxeter2].

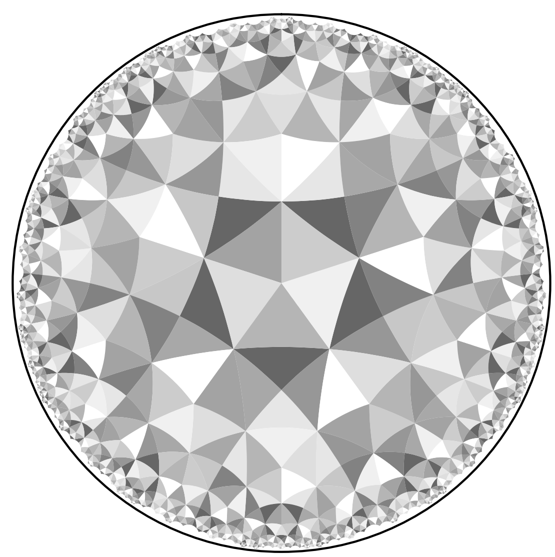

Figure 12: A new rug design with perfect color symmetry.

shades of tan, and each square is divided by its diagonals into four isosceles triangles which are colored in shades of blue and green — the "bright colors". The pentagons and squares correspond to the heptagons and equilateral triangles, respectively, in Rugs #16 and #17. There are six shades of tan (including white) and six bright colors.

As in Rugs #16 and #17, I have paired each shade of tan with a bright color. Each pentagon of the (5.4.5.4) tiling is surrounded by five isosceles triangles of the same bright color (forming a slightly larger pentagon). However, unlike the equilateral triangles of Rugs #16 and #17, each square is surrounded by isosceles triangles of four different shades of tan. A bright pentagon color is associated with the shade of tan that is missing from its pentagon. For example, the dark green that surrounds the central pentagon is associated with the lightest tan (white). One white isosceles triangle touches each of the five vertices of the central pentagon. Like Rugs #16 and #17, the pattern of Figure 12 is chiral since the arrangement of white isosceles triangles has a “twist” to it.

In this new pattern, if we consider each pair (bright color, tan) to be a single “color”, the pattern has perfect color symmetry. As in Figure 10, Coxeter also provided the permutations for his number scheme in Figure 11 above: they are generated by a counterclockwise rotation of 72° about the center and a 180° rotation about the top vertex of the central pentagon. These permutations, which can be taken as instructions for permuting the “colors”, are $(1\ 2\ 3\ 4\ 5)(6)$ and $(1\ 6)(2\ 3)(4\ 5)$ respectively. Thus the group of color permutations of this pattern is isomorphic to the symmetric group S_5 of order 120, as represented in S_6.

Acknowledgments

I would like to thank Richard Bomford for his generous help with information about his father and for supplying me with a copy of his father’s “Rug Book” [Bomford1]. I would also like to thank Doris Schattschneider for her generous help, and with information about the Bomford-Coxeter relationship in particular.

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