The Family of “Circle Limit III” Escher Patterns

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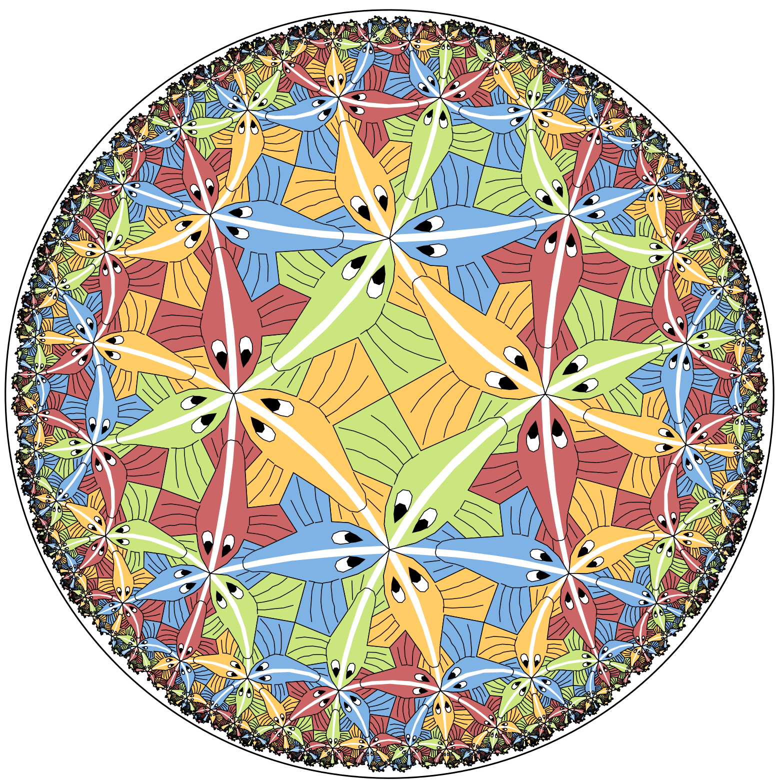
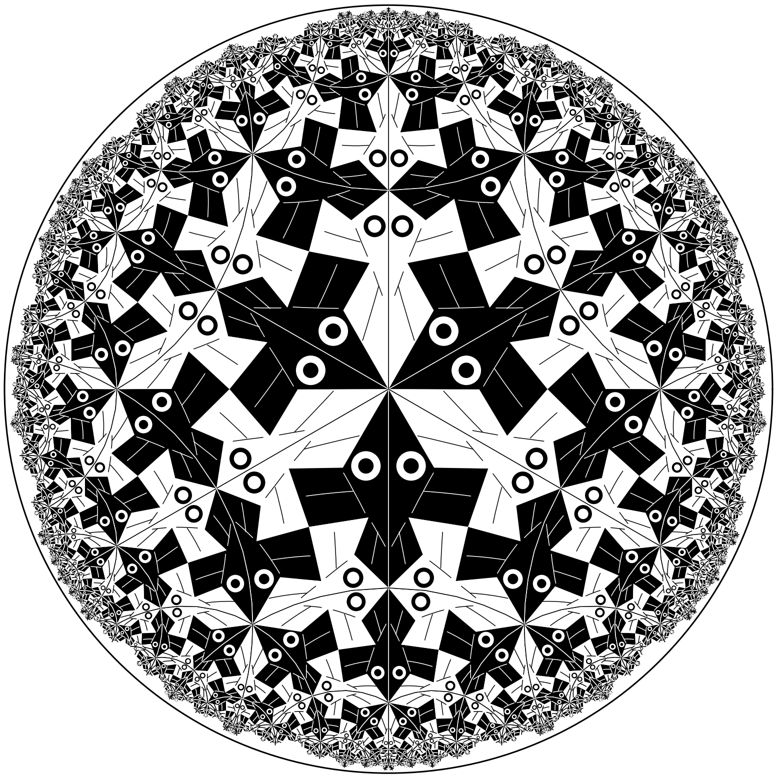
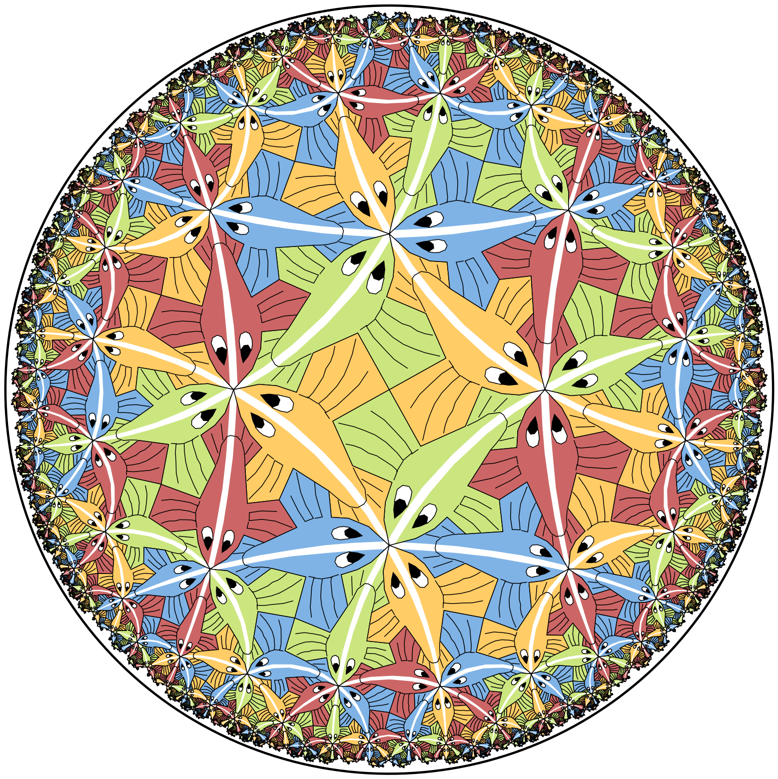
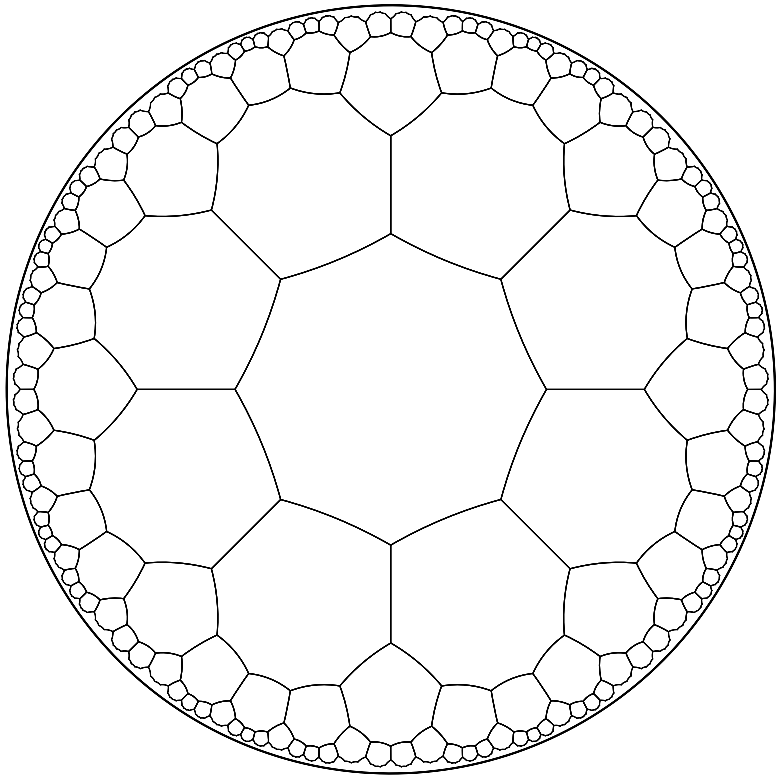




A rendition of Escher's *Circle Limit I*.



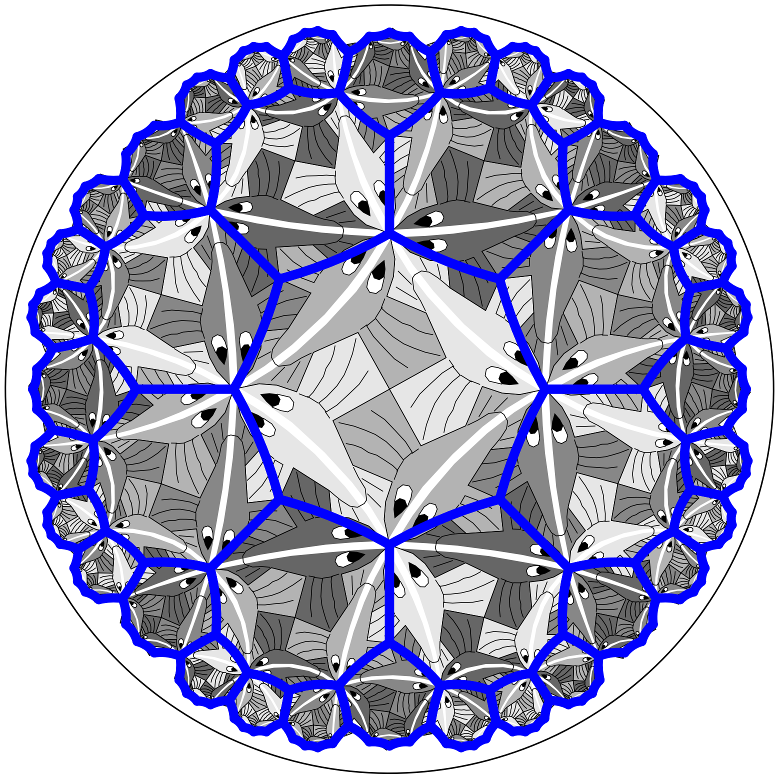
A rendition of Escher's *Circle Limit III*.



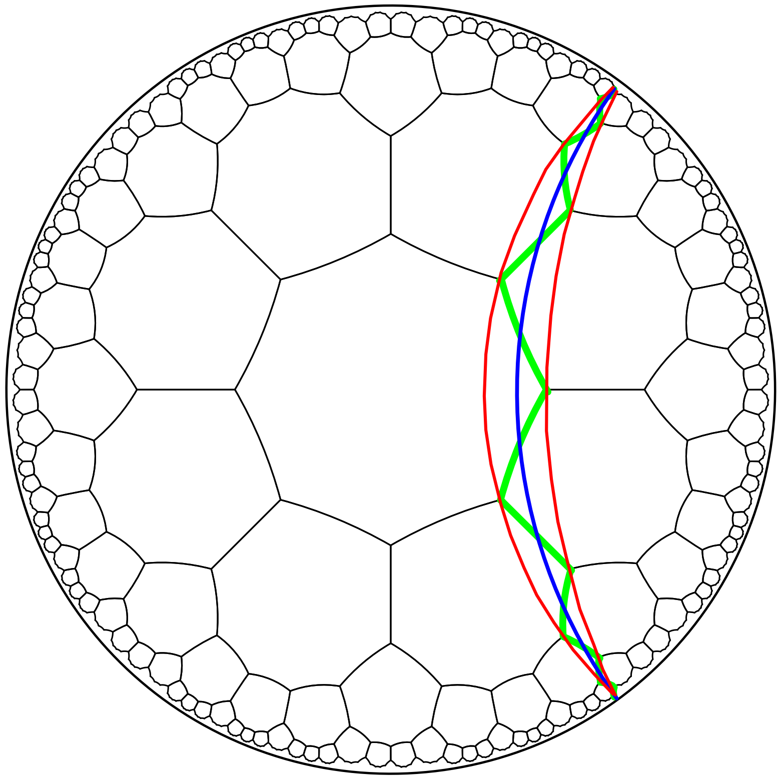
The tessellation $\{8,3\}$.

In general $\{m,n\}$ denotes the ***regular tessellation*** by regular m-sided polygons meeting n at a vertex.

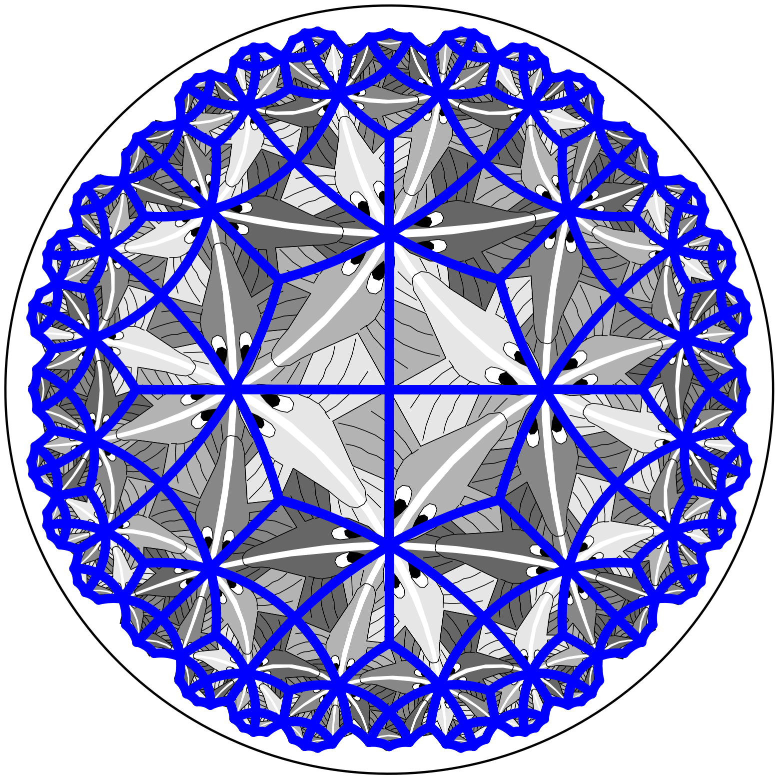
The tessellation is hyperbolic if $(m-2)(n-2) > 4$.



The tessellation $\{8{,}3\}$ underlying *Circle Limit III*.

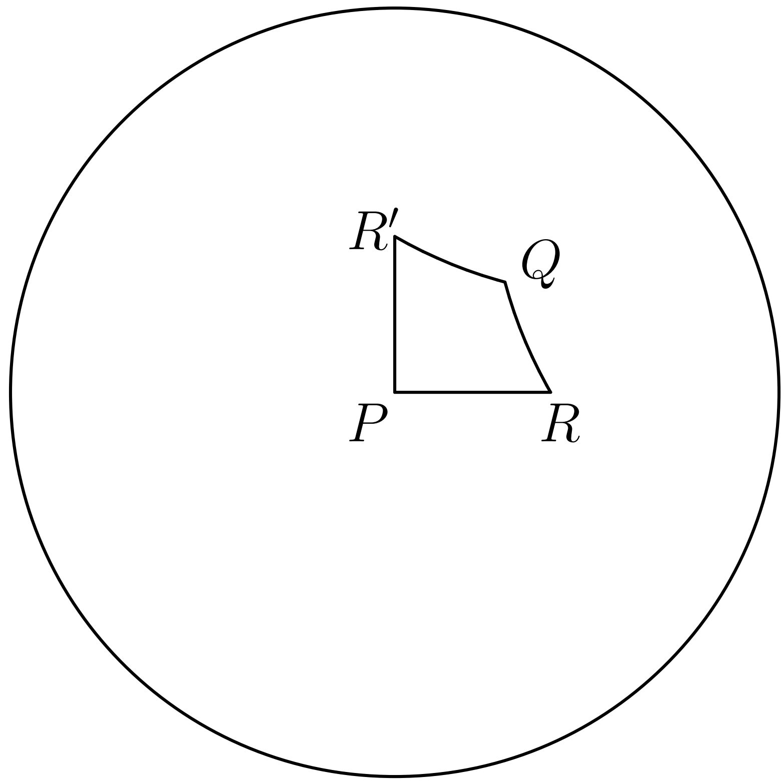


A Petrie polygon (green), a hyperbolic line through the midpoints of its edge (blue), and two equidistant curves (red).

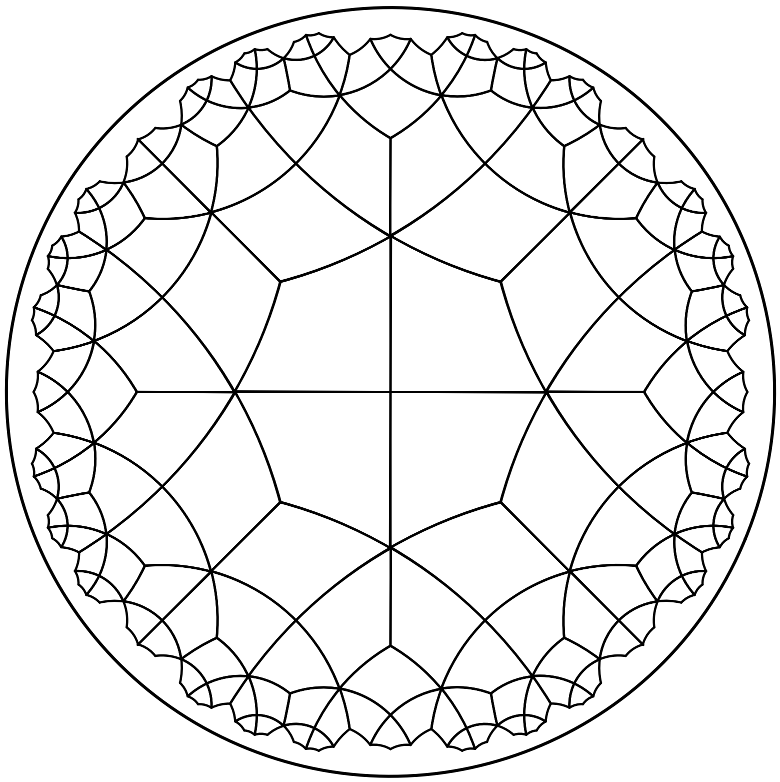


A kite tessellation superimposed on the *Circle Limit III* pattern.

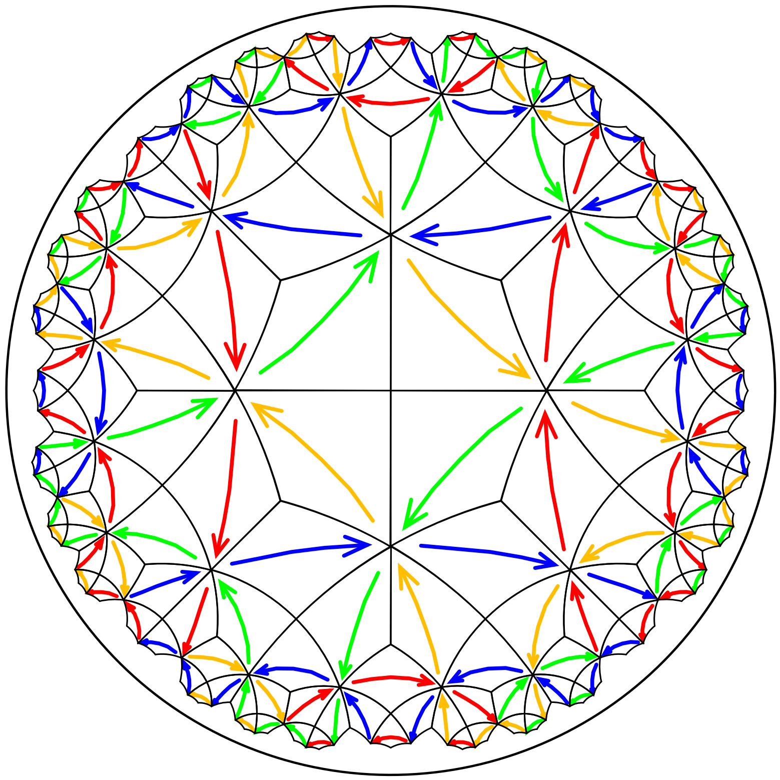
A *kite* is a quadrilateral $PRQR'$ with two pairs of congruent edges PR, PR', and QR, QR' (so $\angle PRQ \cong \angle PR'Q$).



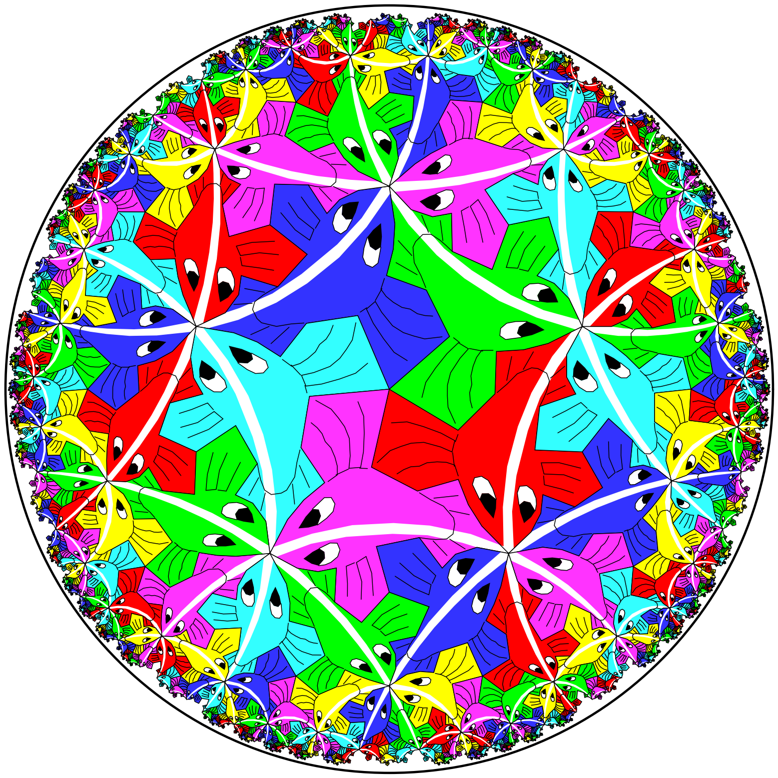
A kite is the fundamental region for a tessellation if the angles at P, Q, and R are $2\pi/p$, $2\pi/q$, and π/r respectively, for integers $p, q, r \geq 3$. Such a *kite tessellation* is hyperbolic if $2\pi/p + \pi/r + 2\pi/q + \pi/r < 2\pi$, i.e. if $1/p + 1/q + 1/r < 1$.



Now r must be odd to make the fish swim head-to-tail, giving a kite tessellation that is the basis for a *Circle Limit III* pattern — which we denote (p, q, r).



A $(5,3,3)$ *Circle Limit III* pattern.



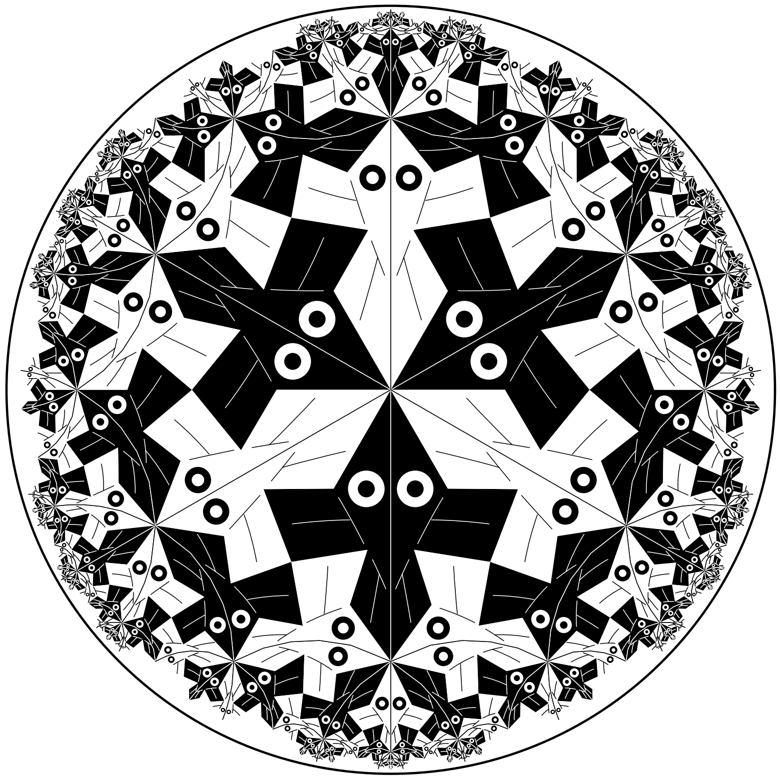
If $p \neq q$, the backbone lines form equidistant curves, which make an angle $\omega < 90$ degrees with the bounding circle.

For *Cirlce Limit III*, Coxeter determined that $cos(\omega) = \sqrt{\frac{3\sqrt{2}-4}{8}}$, or $\omega \approx 79.97°$.

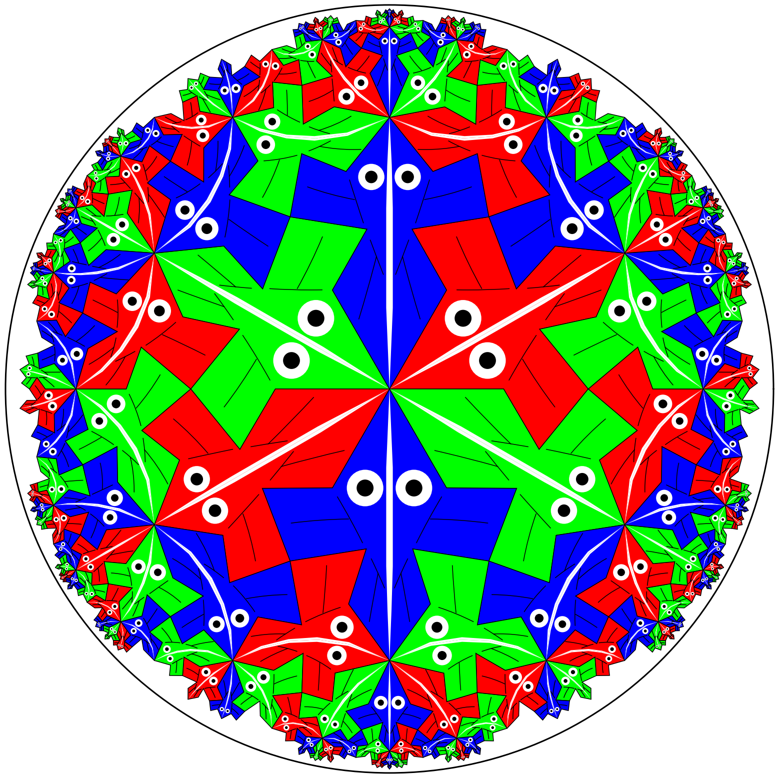
For the previous pattern, ($5, 3, 3$), $cos(\omega) = \sqrt{\frac{3\sqrt{5}-5}{40}}$, or $\omega \approx 78.07°$.

If $p = q$, of course $\omega = 90°$.

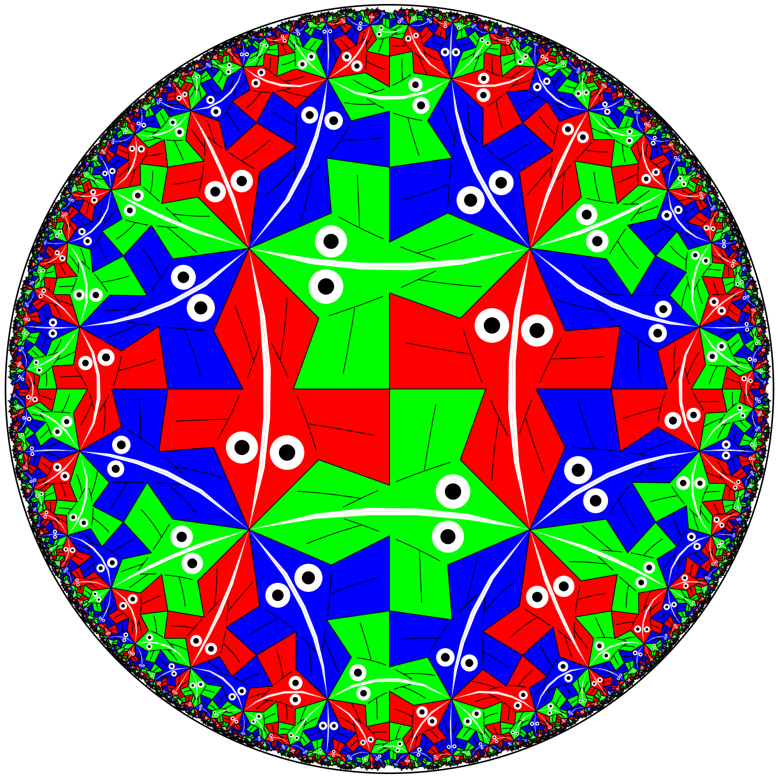
A *Circle Limit I* pattern that solves the "traffic flow" problem.



A recoloring of the previous pattern that gives a uniform color along lines of fish.



A recentering of the previous pattern giving a ($4, 4, 3$) *Circle Limit III* pattern.



Future Work

Calculate ω for arbitrary (p, q, r).

Automatically transform the *Circle Limit III* fish motif to any (p, q, r).

Automatically compute the minimal coloring of a (p, q, r) pattern that satisfies the *Circle Limit III* conditions.