A “Circle Limit III” Calculation

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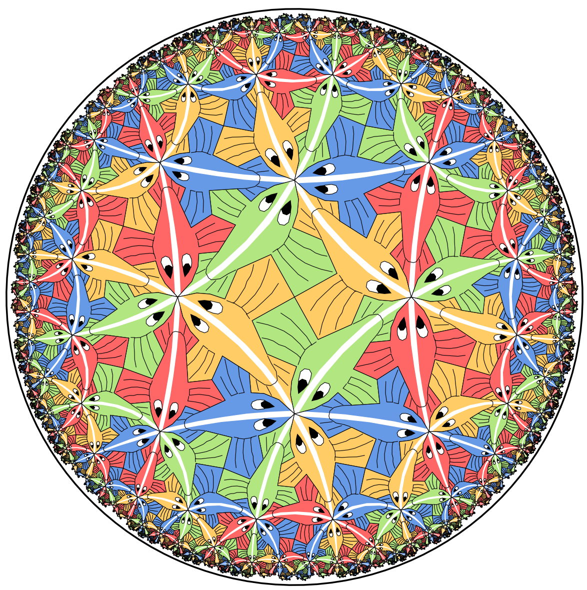


A General Theory?

Mathematical Intelligencer 18, No. 4 (1996):

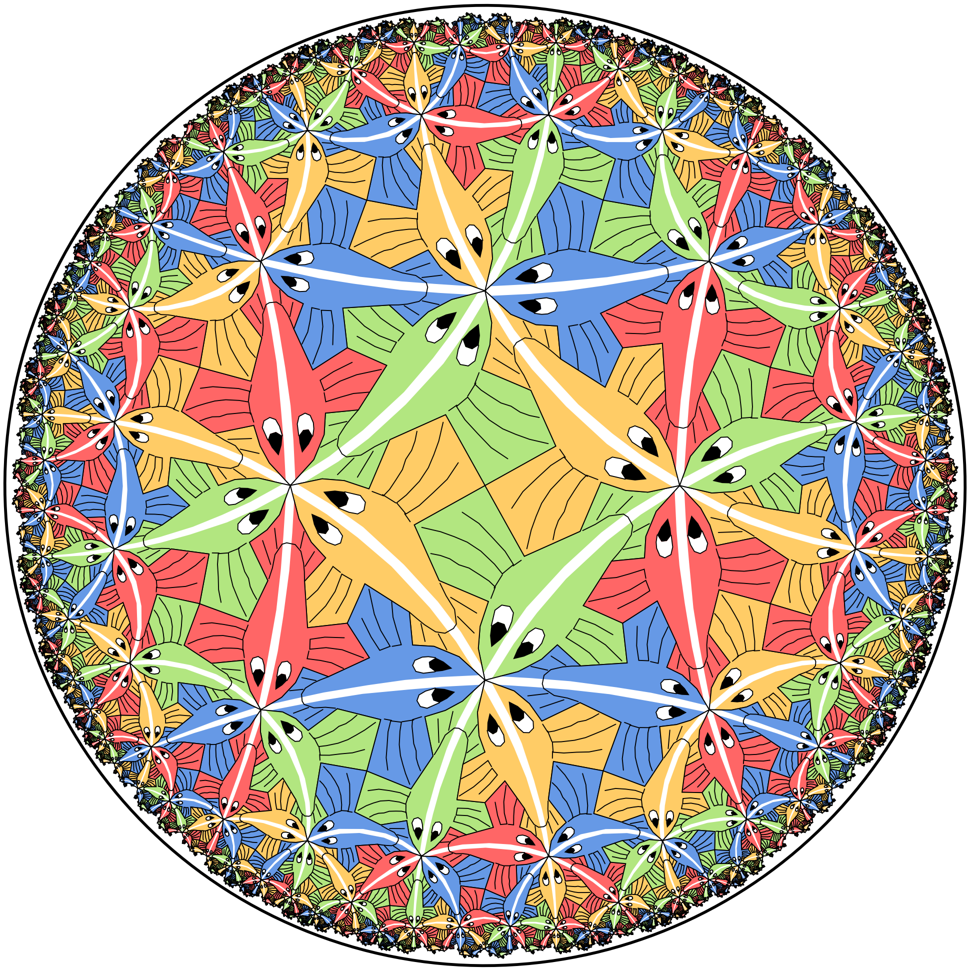
Coxeter's enthusiasm for the gift M.C. Escher gave him, a print of Circle Limit III, is understandable. So is his continuing curiosity. See the articles on pp. 35–46. He has not, however said of what general theory this pattern is a special case. Not as yet.

A General Theory

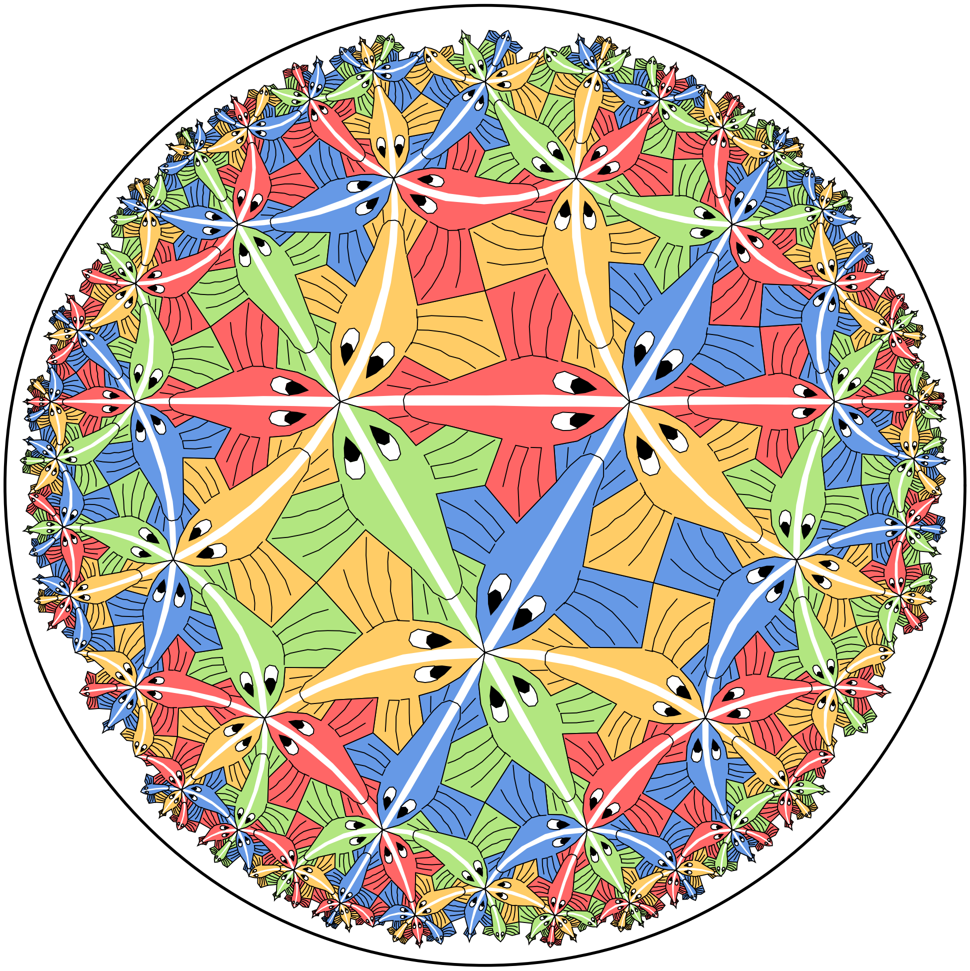
We use the symbolism **(*p*,*q*,*r*)** to denote a pattern of fish in which ***p*** meet at right fin tips, *q* meet at left fin tips, and ***r*** fish meet at their noses. Of course ***p*** and ***q*** must be at least three, and ***r*** must be odd so that the fish swim head-to-tail.

The *Circle Limit III* pattern would be labeled (4,3,3) in this notation.

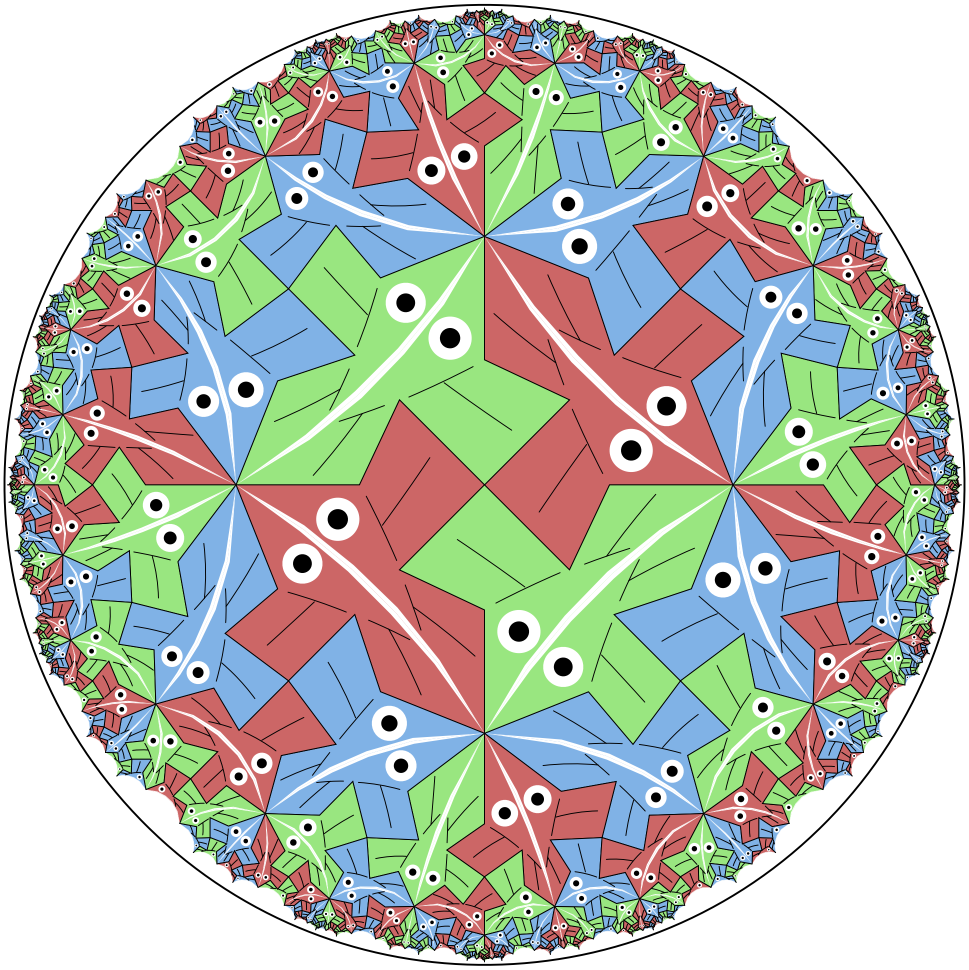
Circle Limit III - a (4,3,3) Pattern



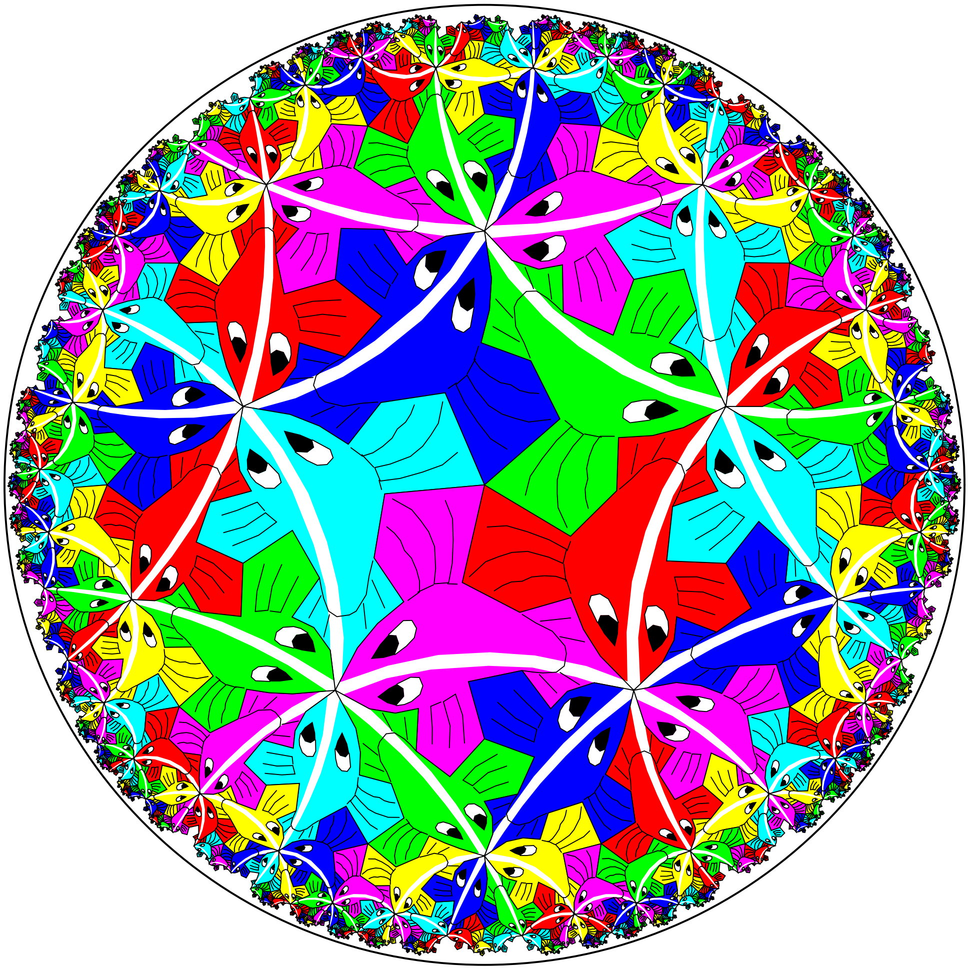
A (3,4,3) Pattern



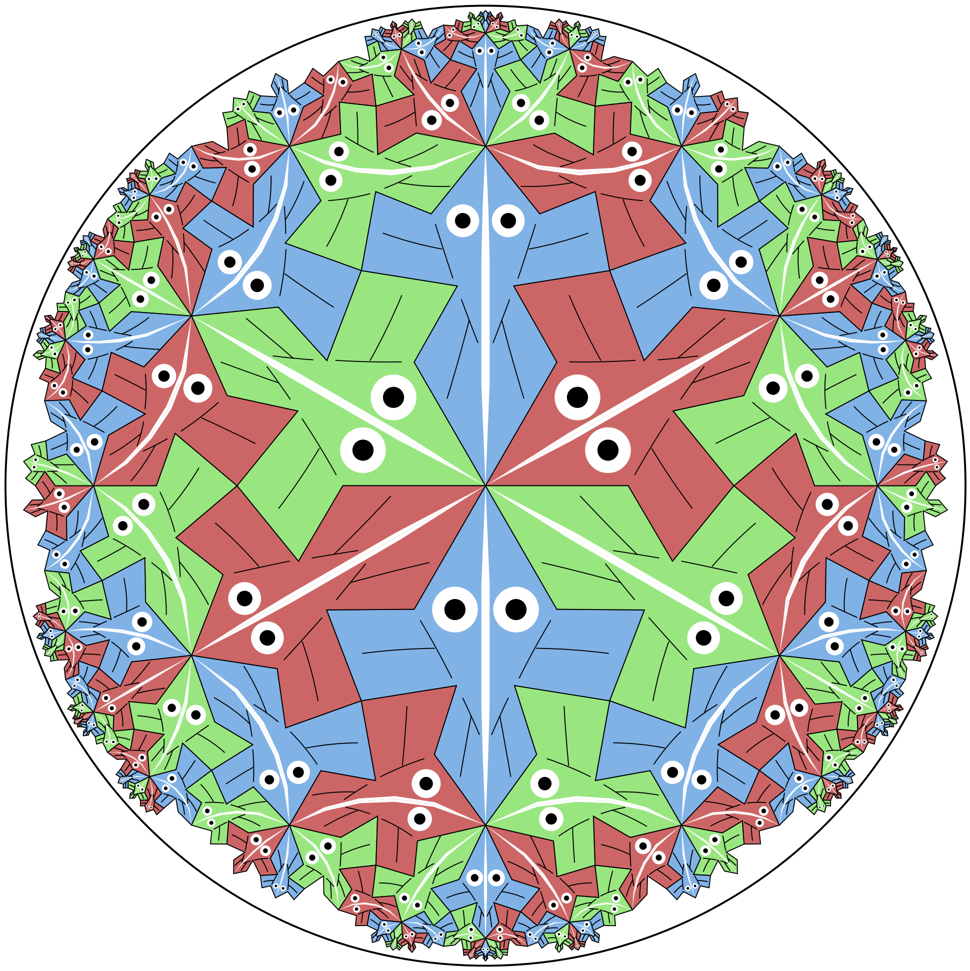
A (4,4,3) Pattern



A (5,3,3) Pattern

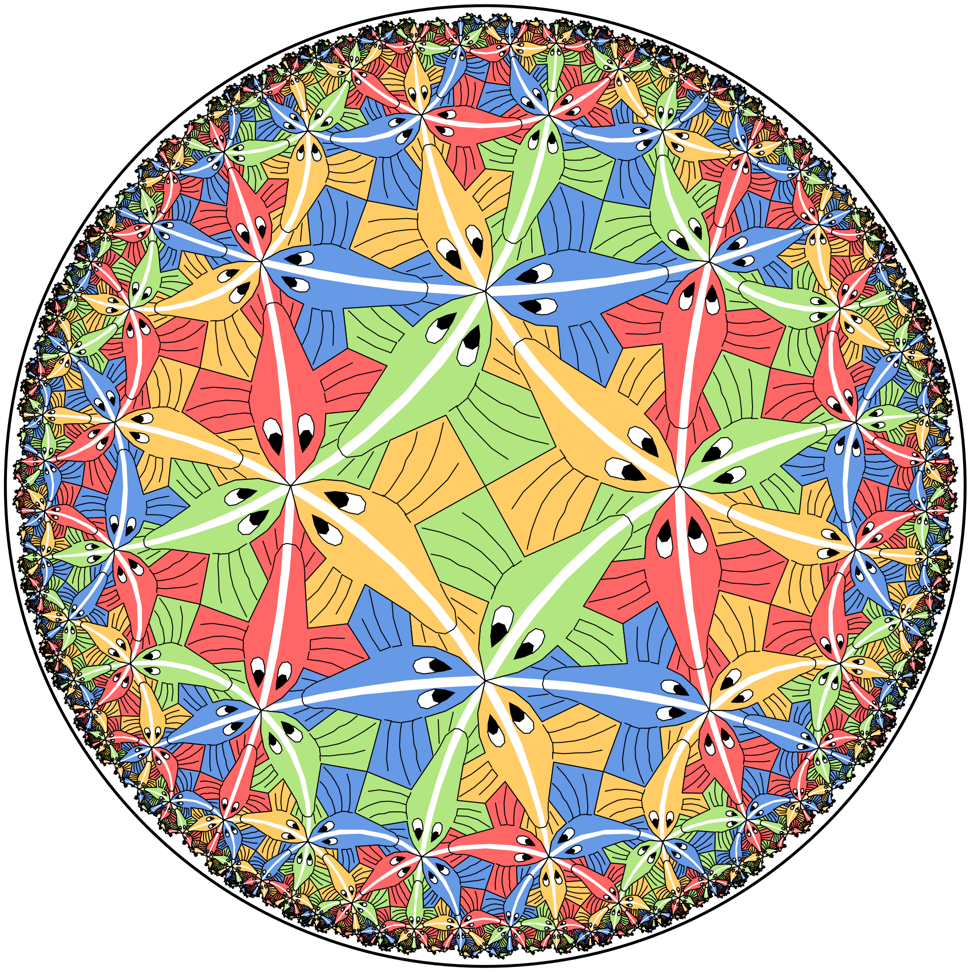


Poincaré Circle Model of Hyperbolic Geometry



- **Points:** points within the **bounding circle**
- **Lines:** circular arcs perpendicular to the bounding circle (including diameters as special cases)

Equidistant Curves



- **Equidistant Curves:** circular arcs *not* perpendicular to the bounding circle (including chords as special cases).

  For each hyperbolic line and a given hyperbolic distance, there are two equidistant curves, one on each side of the line, all of whose points are that distance from the given line.

Weierstrass Model of Hyperbolic Geometry

- **Points:** points on the upper sheet of a hyperboloid of two sheets: $x^2 + y^2 - z^2 = -1$, $z \geq 1$.

- **Lines:** the intersection of a Euclidean plane through the origin with this upper sheet (and so is one branch of a hyperbola).

  A line can be represented by its *pole*, a 3-vector $\begin{bmatrix} \ell_x \\ \ell_y \\ \ell_z \end{bmatrix}$ on the dual hyperboloid $\ell_x^2 + \ell_y^2 - \ell_z^2 = +1$, so that the line is the set of points satisfying $x\ell_x + y\ell_y - z\ell_z = 0$.

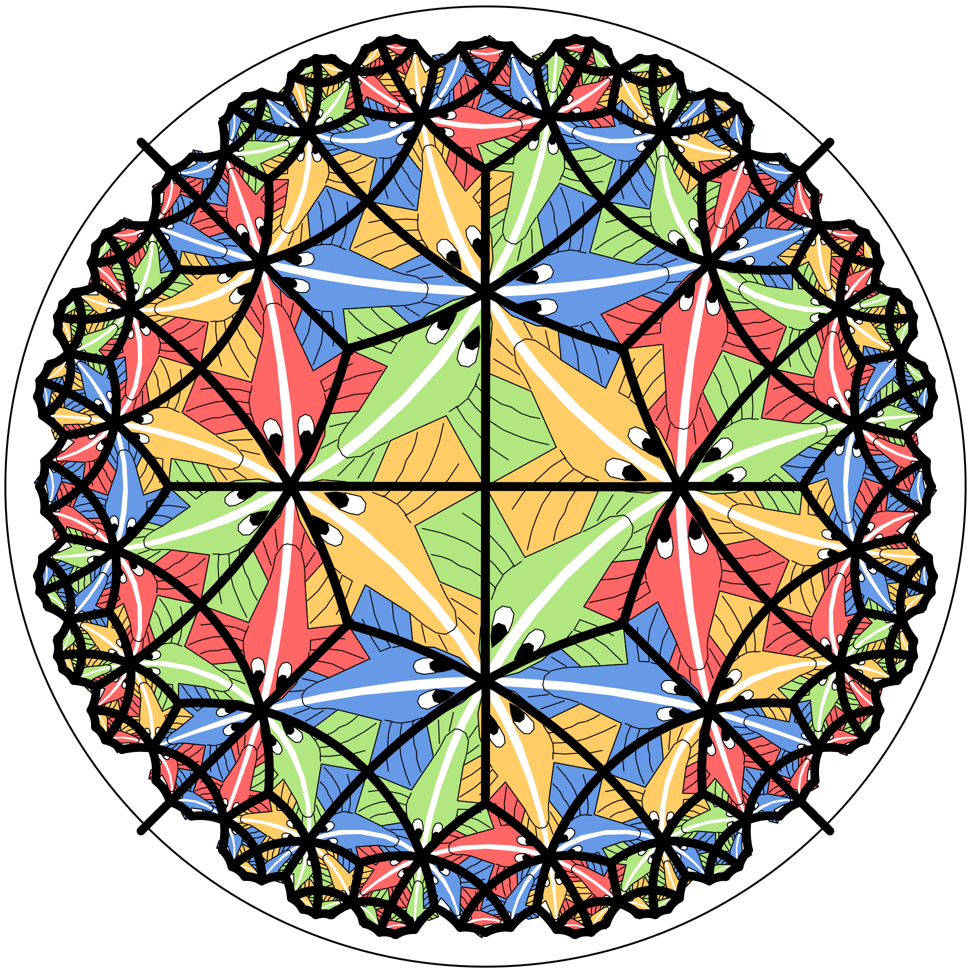
The Relation Between the Models

Stereographic projection from the Weierstrass model onto the Poincaré disk in the xy-plane toward the point $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$,

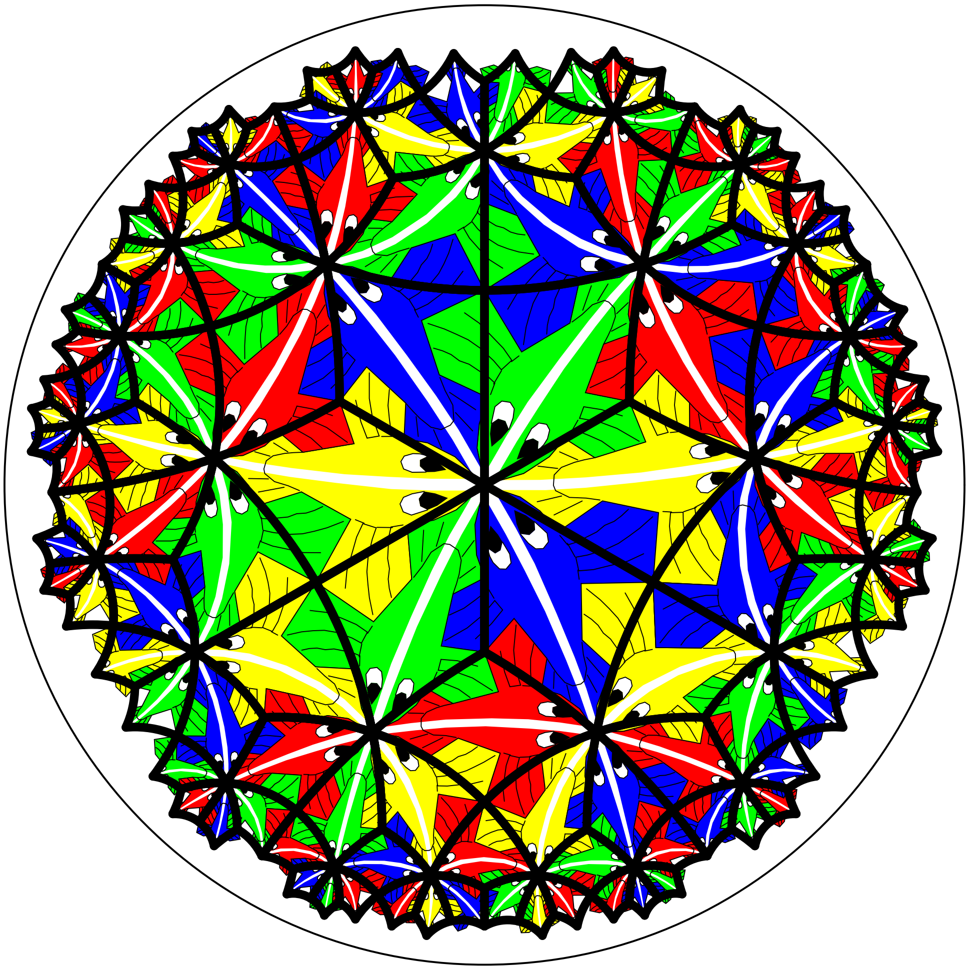
Given by the formula: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x/(1+z) \\ y/(1+z) \\ 0 \end{bmatrix}$.

The Kite Tessellation

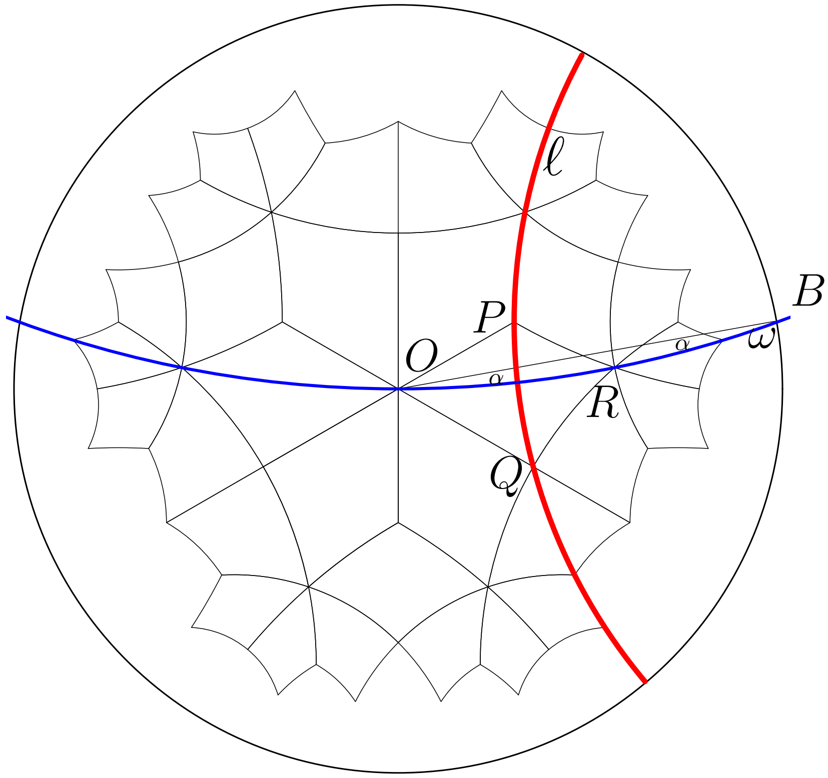
The **fundamental region** for a (p, q, r) pattern can be taken to be a ***kite*** — a quadrilateral with two opposite angles equal. The angles are $2\pi/p$, π/r, $2\pi/q$, and π/r.



A Nose-Centered Kite Tessellation



The Geometry of the Kite Tessellation

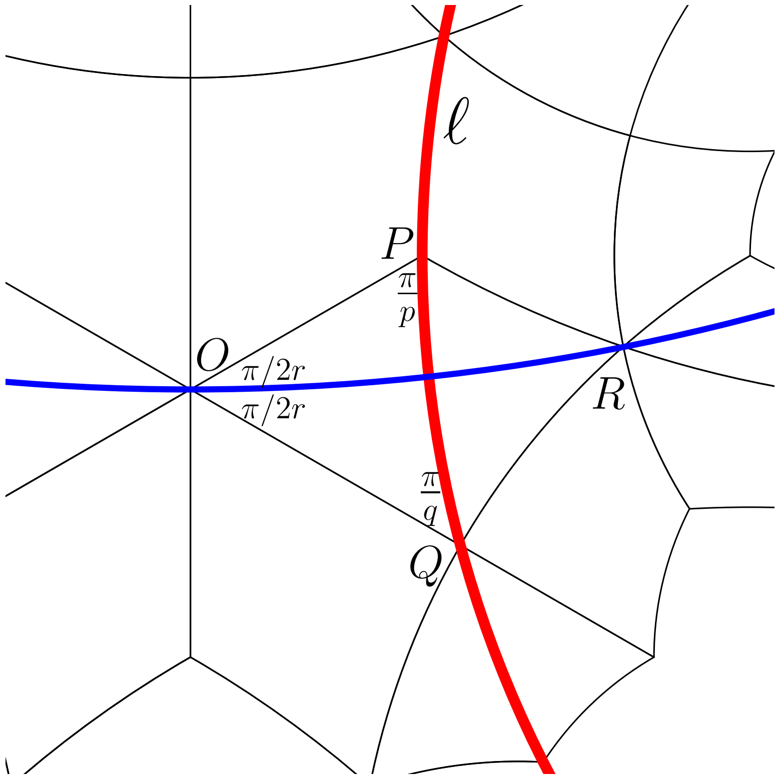


The kite $OPRQ$, its bisecting line, ℓ, the backbone line (equidistant curve) through O and R, and radius OB.

Outline of the Calculation

1. Calculate the Weierstrass coordinates of the points P and Q.
2. Find the coordinates of ℓ from those of P and Q.
3. Use the coordinates of ℓ to compute the matrix of the reflection across ℓ.
4. Reflect O across ℓ to obtain the Weierstrass coordinates of R, and thus the Poincaré coordinates of R.
5. Since the backbone equidistant curve is symmetric about the y-axis, the origin O and R determine that circle, from which it is easy to calculate ω, the angle of intersection of the backbone curve with the bounding circle.

Details of the Central Kite



1. The Weierstrass Coordinates of P and Q

From a standard trigonometric formula for hyperbolic triangles, the hyperbolic cosines of the hyperbolic lengths of the sides OP and OQ of the triangle OPQ are given by:

$$cosh(d_p) = \frac{\cos(\pi/q)\cos(\pi/r) + \cos\pi/p}{\sin(\pi/q)\sin(\pi/r)}$$

and

$$cosh(d_q) = \frac{\cos(\pi/p)\cos(\pi/r) + \cos\pi/q}{\sin(\pi/p)\sin(\pi/r)}$$

From these equations, we obtain the Weierstrass coordinates of P and Q:

$$P = \begin{bmatrix} \cos(\pi/2r)sinh(d_q) \\ \sin(\pi/2r)sinh(d_q) \\ cosh(d_q) \end{bmatrix} \quad Q = \begin{bmatrix} \cos(\pi/2r)sinh(d_p) \\ -\sin(\pi/2r)sinh(d_p) \\ cosh(d_p) \end{bmatrix}$$

2. The Coordinates of ℓ

The coordinates of the pole of ℓ are given by

$$\ell = \begin{bmatrix} \ell_x \\ \ell_y \\ \ell_z \end{bmatrix} = \frac{P \times Q}{|P \times Q|}$$

Where the hyperbolic cross-product $P \times Q$ is given by:

$$P \times Q = \begin{bmatrix} P_y Q_z - PzQy \\ P_z Q_x - PxQz \\ -P_x Q_y + PyQx \end{bmatrix}$$

(and where the norm of a pole vector V is given by: $|V| = \sqrt(V_x^2 + V_y^2 - V_z^2))$

3. The Reflection Matrix - A Simple Case

The pole corresponding to the hyperbolic line perpendicular to the x-axis and through the point $\begin{bmatrix} \sinh d \\ 0 \\ \cosh d \end{bmatrix}$ is given by $\begin{bmatrix} \cosh d \\ 0 \\ \sinh d \end{bmatrix}$.

The matrix *Ref* representing reflection of Weierstrass points across that line is given by:

$$Ref = \begin{bmatrix} -\cosh 2d & 0 & \sinh 2d \\ 0 & 1 & 0 \\ -\sinh 2d & 0 & \cosh 2d \end{bmatrix}$$

where d is the the hyperbolic distance from the line (or point) to the origin.

3. The Reflection Matrix - The General Case

In general, reflection across a line whose nearest point to the origin is rotated by angle θ from the x-axis is given by: $Rot(\theta) Ref\, Rot(-\theta)$ where, as usual,

$$Rot(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From ℓ we identify $\sinh d$ as ℓ_z, and $\cosh d$ as $\sqrt{(}\ell_x^2 + \ell_y^2)$, which we denote ρ. Then $\cos\theta = \frac{\ell_x}{\rho}$ and $\sin\theta = \frac{\ell_y}{\rho}$.

Further, $\sinh 2d = 2\sinh d \cosh d = 2\rho\ell_z$ and $\cosh 2d = \cosh^2 d + \sinh^2 d = \rho^2 + \ell_z^2$.

Thus ***Ref***$_\ell$, the matrix for reflection across ℓ is given by:

$$Ref_{\,\ell} = \begin{bmatrix} \frac{\ell_x}{\rho} & -\frac{\ell_y}{\rho} & 0 \\ \frac{\ell_y}{\rho} & \frac{\ell_x}{\rho} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -(\rho^2 + \ell_z^2) & 0 & 2\rho\ell_z \\ 0 & 1 & 0 \\ -2\rho\ell_z & 0 & (\rho^2 + \ell_z^2) \end{bmatrix} \begin{bmatrix} \frac{\ell_x}{\rho} & \frac{\ell_y}{\rho} & 0 \\ -\frac{\ell_y}{\rho} & \frac{\ell_x}{\rho} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. The Coordinates of R

We use Ref_ℓ to reflect the origin to R since the kite $OPRQ$ is symmetric across ℓ:

$$R = \mathit{Ref}_\ell \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\ell_x\ell_z \\ 2\ell_y\ell_z \\ \rho^2 + \ell_z^2 \end{bmatrix}$$

Then we project Weierstrass point R to the Poincaré model:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2\ell_x\ell_z}{1+\rho^2+\ell_z^2} \\ \frac{2\ell_y\ell_z}{1+\rho^2+\ell_z^2} \\ 0 \end{bmatrix}$$

5. The Angle ω

The three points $\begin{bmatrix} u \\ v \\ 0 \end{bmatrix}, \begin{bmatrix} -u \\ v \\ 0 \end{bmatrix}$, and the origin determine the (equidistant curve) circle centered at $w = (u^2+v^2)/2v$ on the y-axis.

The y-coordinate of the intersection points of this circle, $x^2 + (y - w)^2 = w^2$, with the unit circle to be $y_{int} = 1/2w = v/(u^2 + v^2)$.

In the figure showing the geometry of the kite tessellation, the point B denotes the right-hand intersection point.

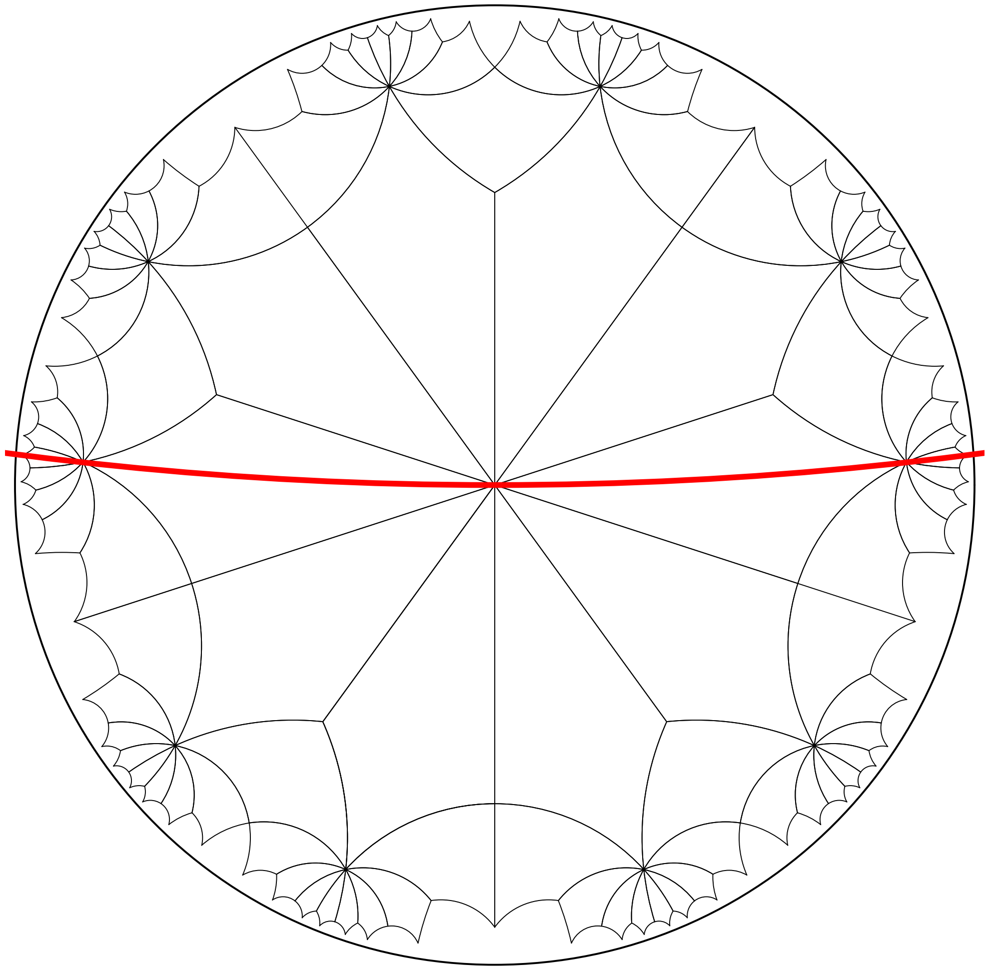
The central angle, α, made by the radius OB with the x-axis is the complement of ω (which can be seen since the equidistant circle is symmetric across the perpendicular bisector of OB).

Thus $y_{int} = \sin\alpha = \cos\omega$, so that

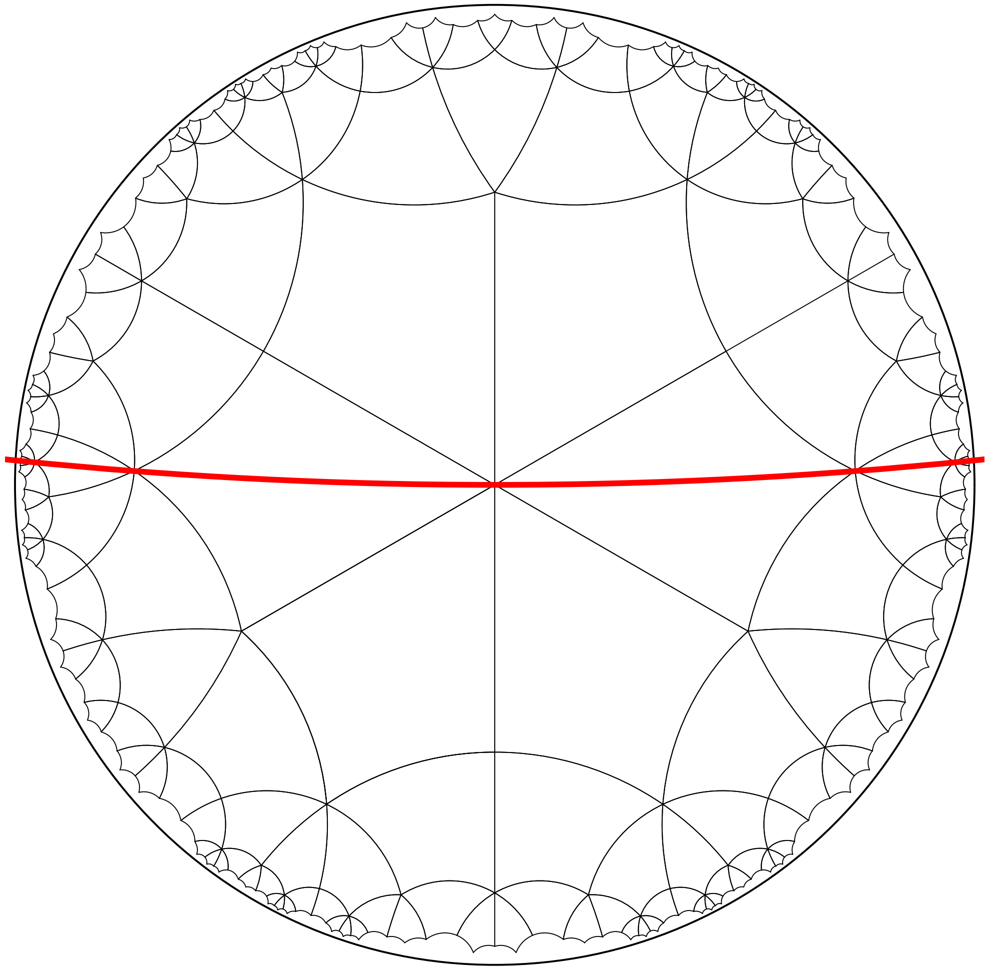
$$\cos\omega = y_{int} = v/(u^2 + v^2)$$

which is the desired result.

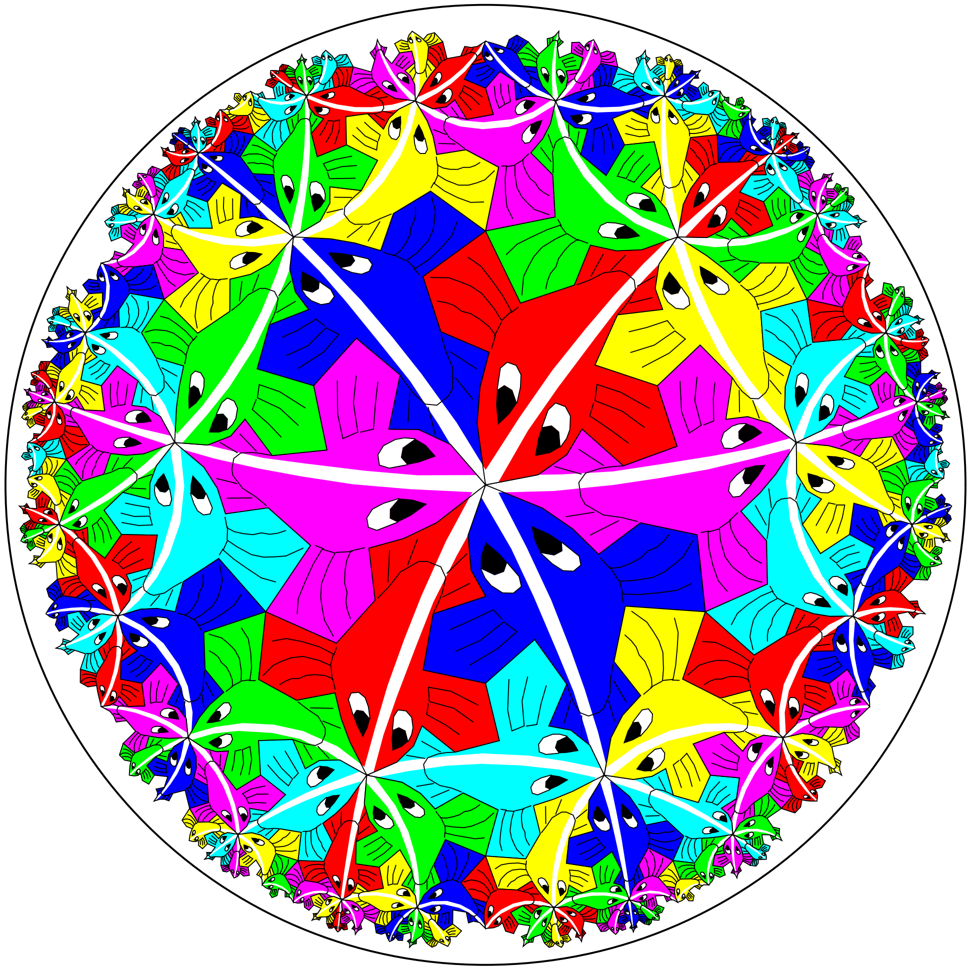
Examples: A (3,4,5) Kite Tessellation



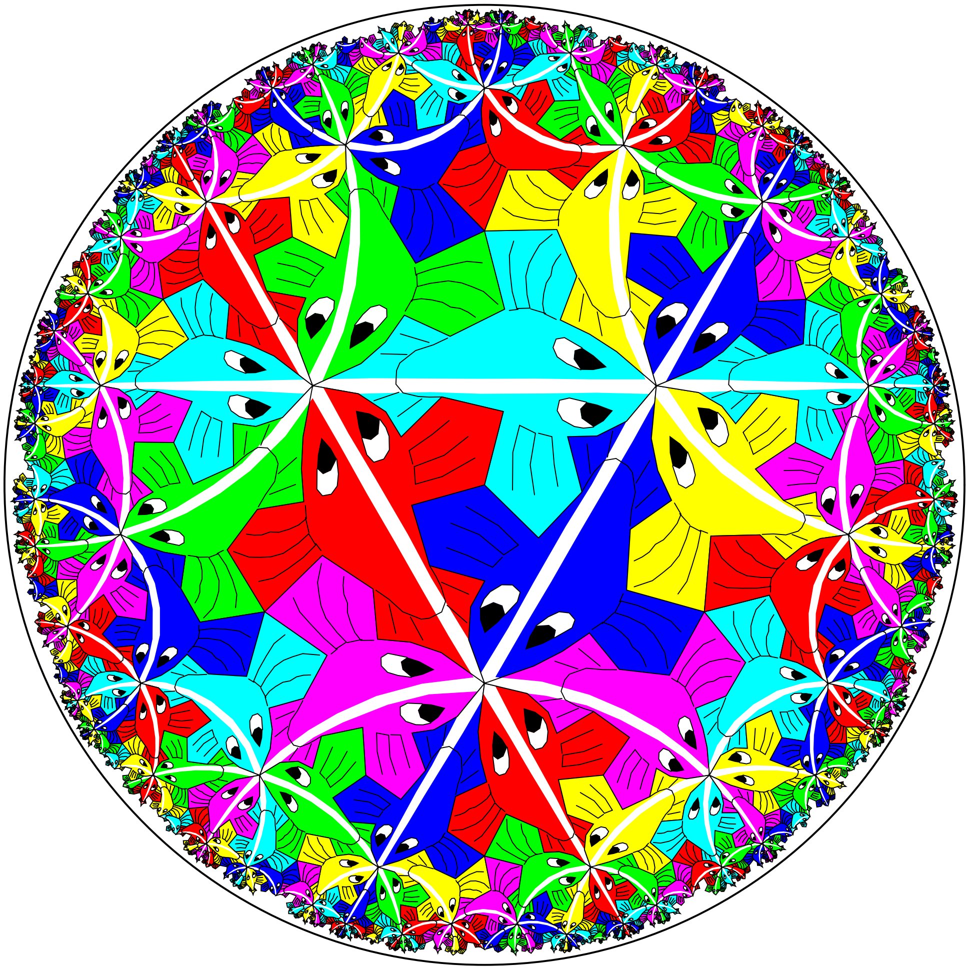
A (4,5,3) Kite Tessellation



A Nose-Centered (5,3,3) Pattern.



A (3,5,3) Pattern.



Future Work

- Find a general formula for ω in terms of *p*, *q*, and *r*. Note: this has been obtained by Luns Tee:

$$\cos(\omega) = \frac{\sin(\pi/2r) * (\cos(\pi/p) - \cos(\pi/q))}{\sqrt{(\cos(\pi/p)^2 + \cos(\pi/q)^2 + \cos(\pi/r)^2 + 2\cos(\pi/p)\cos(\pi/q)\cos(\pi/r) - 1)}}$$

- Write software to automatically convert the motif of a (p,q,r) pattern to a (p',q',r') motif.
- Investigate patterns in which one of *q* or *r* (or both) is infinity. Also, extend the current program to draw such patterns.
- Find an algorithm for computing the minimum number of colors needed for a (p,q,r) pattern as in *Circle Limit III*: all fish along a backbone line are the same color, and adjacent fish are different colors (the "map-coloring principle").