Bridges 2012
Towson University, Towson, Maryland, USA

Patterned Triply Periodic Polyhedra

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Outline

- Some previously designed patterned (closed) polyhedra
- Triply periodic polyhedra
- Inspiration for this work
- Hyperbolic geometry and regular tessellations
- Relation between periodic polyhedra and regular tessellations
- Patterns on the $\{4, 6|4\}$ polyhedron
- Patterns on the $\{6, 4|4\}$ polyhedron
- A Pattern of fish on the $\{6, 6|3\}$ polyhedron
- A Pattern of fish on a $\{3, 8\}$ polyhedron
- Future research

Previously Designed Patterned Polyhedra

- M.C. Escher (1898–1972) created at least 3 such polyhedra.
- In 1977 Doris Schattschneider and Wallace Walker placed Escher patterns on each of the Platonic solids and the cuboctahedron.
- Schattschneider and Walker also put Escher patterns on rotating rings of tetrahedra, which they called "kaleidocycles".
- In 1985 H.S.M. Coxeter showed how to place 18 Escher butterflies on a torus.

Triply Periodic Polyhedra

- A *triply periodic polyhedron* is a (non-closed) polyhedron that repeats in three different directions in Euclidean 3-space.
- We will consider the special case of *uniform* triply periodic polyhedra which have the same vertex figure at each vertex — i.e. there is a symmetry of the polyhedron that takes any vertex to any other vertex..
- We will mostly discuss a speciallization of uniform triply periodic polyhedra: *regular* triply periodic polyhedra which are "flag-transitive" — there is a symmetry of the polyhedron that takes any vertex, edge containing that vertex, and face containing that edge to any other such (vertex, edge, face) combination.
- In 1926 John Petrie and H.S.M. Coxeter proved that there are exactly three regular triply periodic polyhedra, which Coxeter denoted $\{4, 6|4\}$, $\{6, 4|4\}$, and $\{6, 6|3\}$, where $\{p, q|r\}$ denotes a polyhedron made up of p-sided regular polygons meeting q at a vertex, and with regular r-sided holes.

Inspirations for this Work

- Two papers by Steve Luecking at ISAMA 2011:
  - Building a Sherk Surface from Paper Tiles
  - Sculpture From a Space Filling Saddle Pentahedron
- Bead sculptures that approximate three triply periodic minimal surfaces (TPMS) by Chern Chuang, Bih-Yaw Jin, and Wei-Chi Wei at the 2012 Joint Mathematics Meeting Art Exhibit.

  As we will see, some TPMS's are related to triply periodic polyhedra.

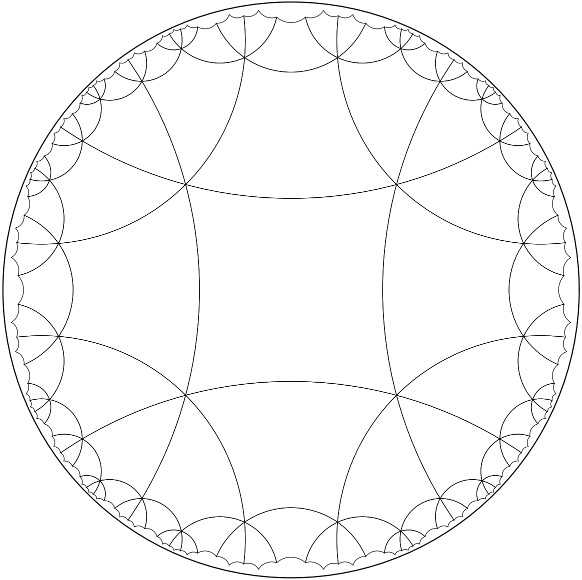
Hyperbolic Geometry and Regular Tessellations

- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.
- Thus we must use *models* of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.
- One such model is the *Poincaré disk model*. The hyperbolic points in this model are represented by interior point of a Euclidean circle — the *bounding circle*. The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (with diameters as special cases).
- This model is appealing to artests since (1) angles have their Euclidean measure (i.e. it is conformal), so that motifs of a repeating pattern retain their approximate shape as they get smaller toward the edge of the bounding circle, and (2) it can display an entire pattern in a finite area.

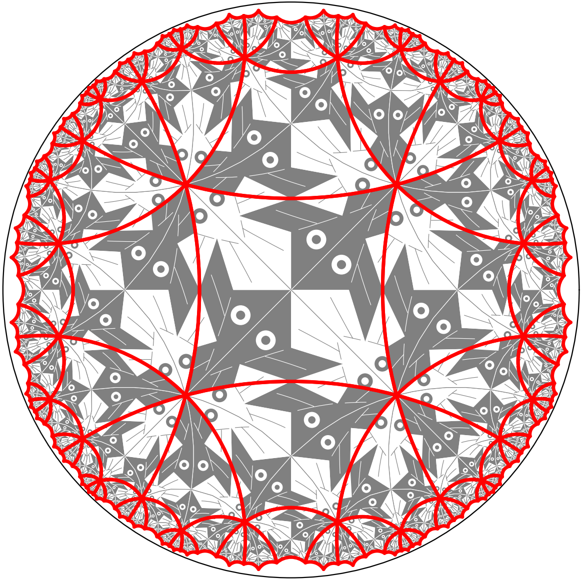
Repeating Patterns and Regular Tessellations

- A *repeating pattern* in any of the 3 "classical geometries" (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or *motif*.
- The *regular tessellation*, $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{6, 4\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

The Regular Tessellation $\{4, 6\}$



The tessellation $\{4, 6\}$ superimposed on the pattern of angular fish of the title slide pattern



Relation between periodic polyhedra and regular tessellations — a 2-Step Process

- (1) Some triply periodic polyhedra approximate TPMS's.

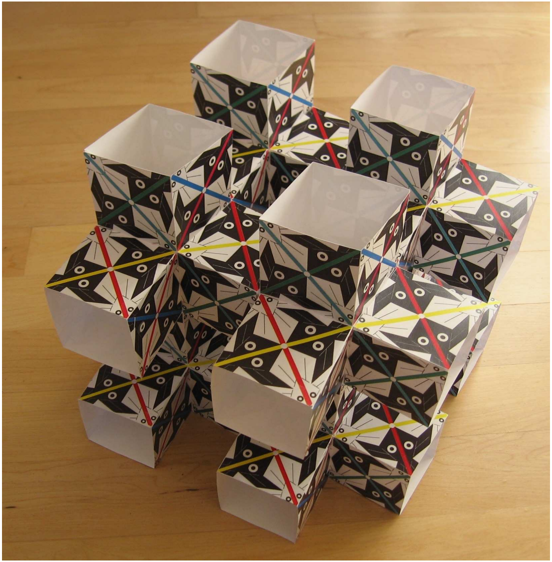
  As a bonus, some triply periodic polyhedra contain embedded Euclidean lines which are also lines embedded in the corresponding TPMS.

- (2) As a minimal surface, a TPMS has negative curvature (except for isolated points of zero curvature), and so its universal covering surface also has negative curvature and thus has the same large-scale geometry as the hyperbolic plane.

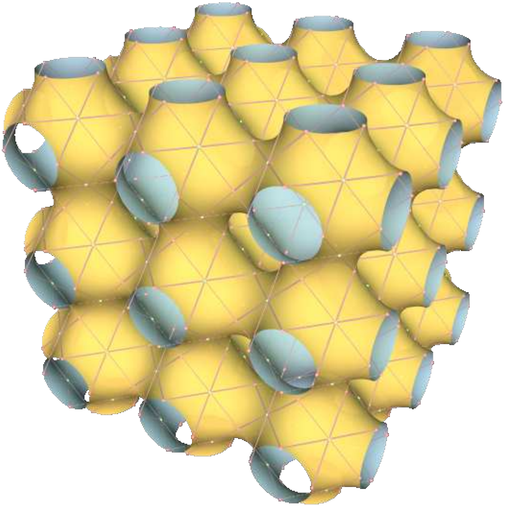
  So the polygons of the triply periodic polyhedron can be transferred to the polygons of a corresponding regular tessellation of the hyperbolic plane.

- We show this relationship in the next slides.

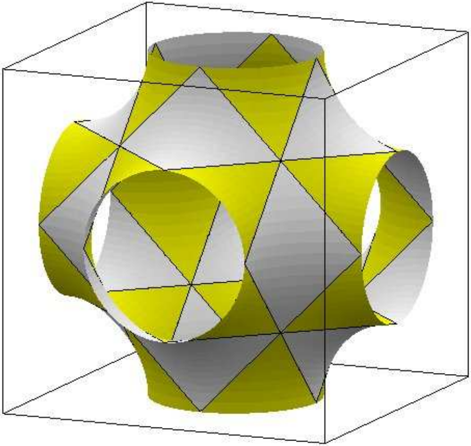
The triply periodic polyhedron of the Title Slide — showing colored embedded lines



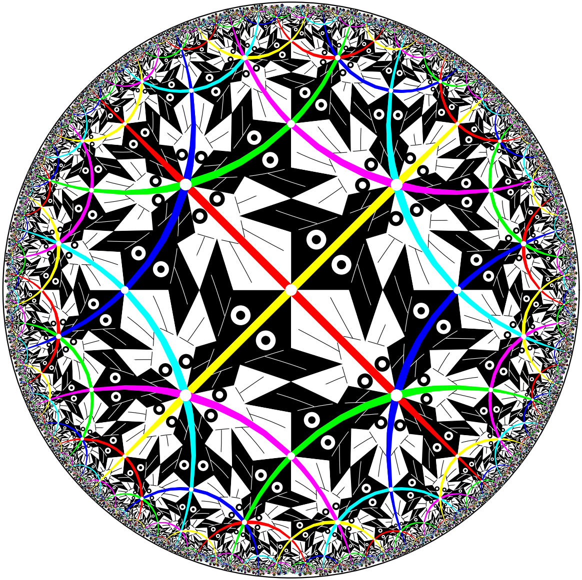
Schwarz's P-surface — approximated by the previous triply periodic polyhedron, and showing corresponding embedded lines



A close-up of Schwarz's P-surface showing corresponding embedded lines and "skew rhombi"



The pattern of the Title Slide "unfolded" onto a repeating pattern of the hyperbolic plane — showing the embedded lines as hyperbolic lines, which bound the "skew rhombi".



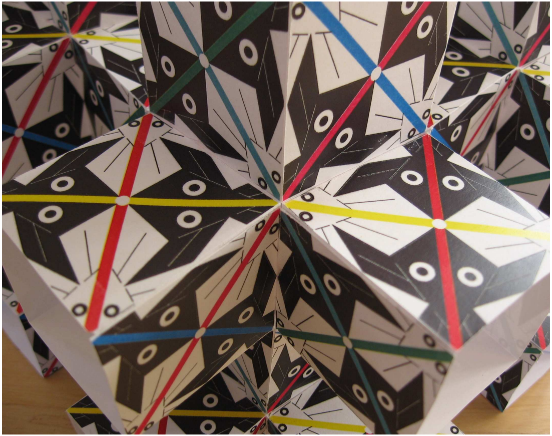
Patterns on the $\{4, 6|4\}$ Polyhedron

We show two patterns on the $\{4, 6|4\}$ polyhedron:

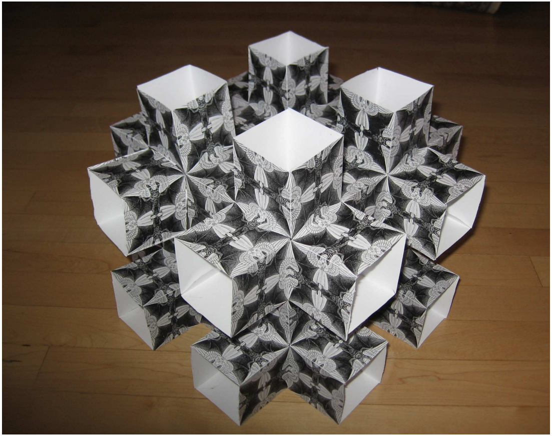
- The pattern of the Title Slide, which we have seen.

  Here we show a close-up of one of the vertices.
- A pattern of angels and devils, inspired by Escher. We show both the patterned polyhedron and the corresponding pattern in the hyperbolic plane

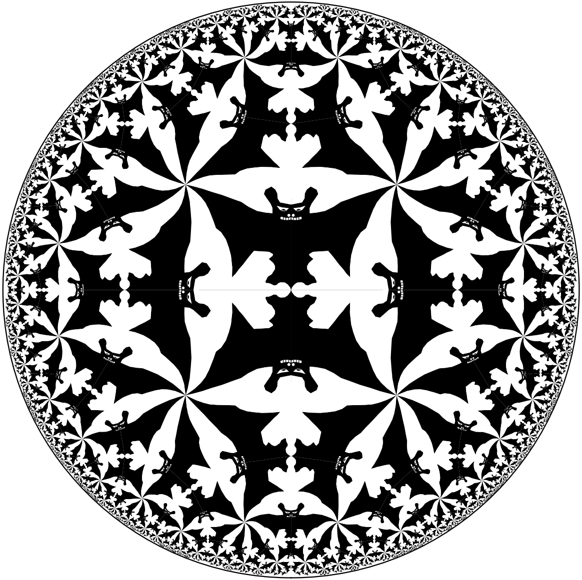
A close-up of a vertex of the Title Slide polyhedron



Angels and Devils on the $\{4, 6|4\}$ polyhedron

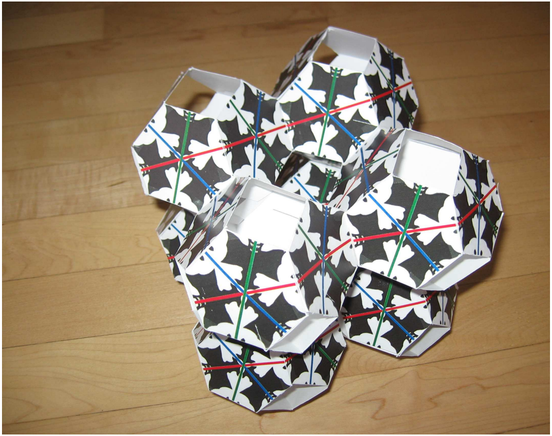


The corresponding Angels and Devils pattern in the hyperbolic plane

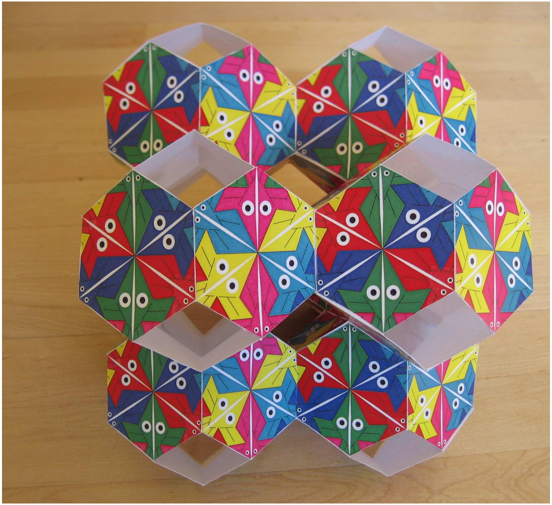


Patterns on the $\{6, 4|4\}$ Polyhedron

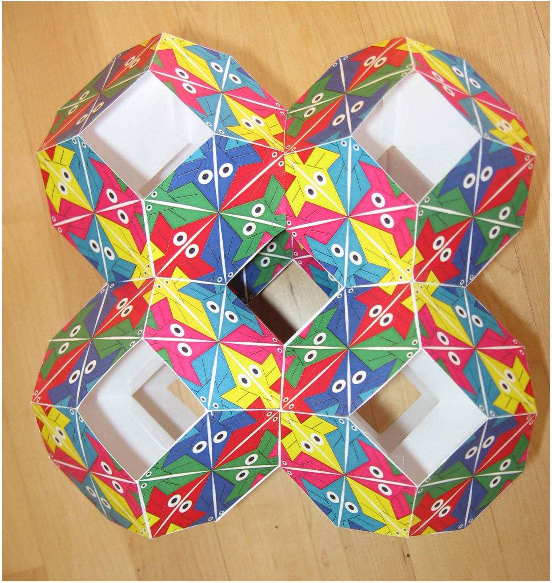
A pattern of angels and devils on the $\{6, 4|4\}$ polyhedron



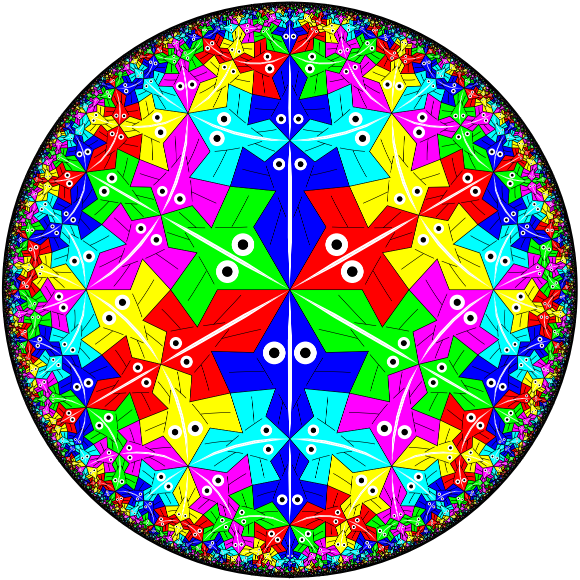
A Pattern of Fish on the $\{6, 4|4\}$ Polyhedron



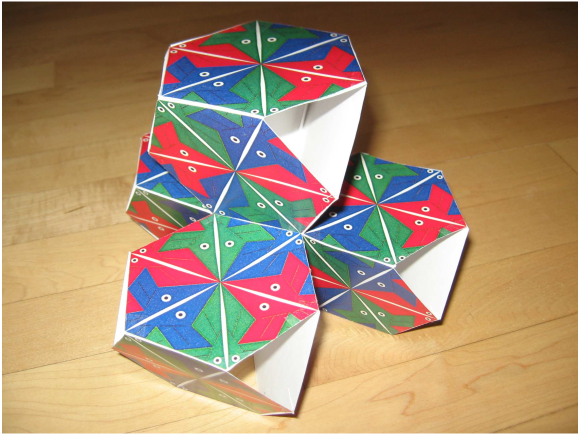
A top view of the fish on the $\{6,4|4\}$ polyhedron — showing fish along embedded lines



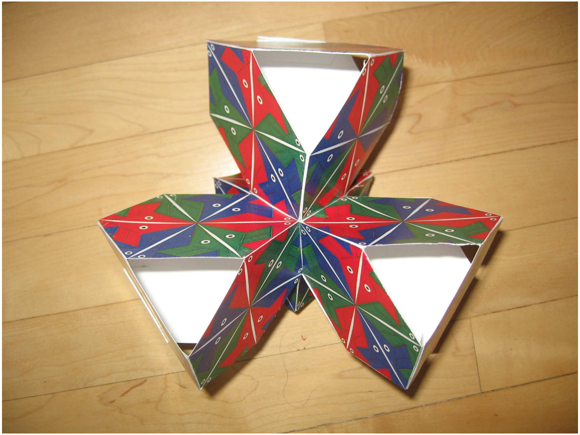
The corresponding hyperbolic pattern of fish — a version of Escher's Circle Limit I pattern with 6-color symmetry



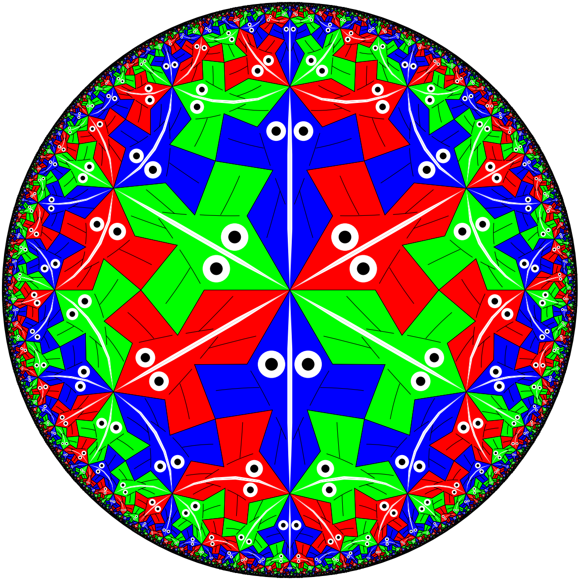
A Pattern of Fish on the $\{6, 6|3\}$ Polyhedron



A top view of the fish on the $\{6, 6|3\}$ polyhedron — showing a vertex



The corresponding hyperbolic pattern of fish — based on the $\{6,6\}$ tessellation

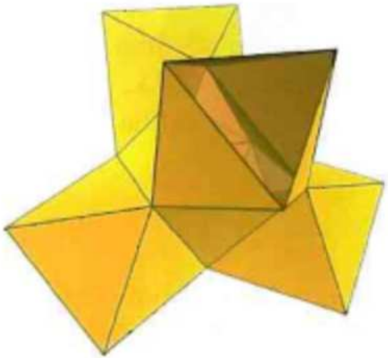


Patterns of Fish on a $\{3, 8\}$ Polyhedron

Using a uniform triply periodic $\{3, 8\}$ polyhedron, we show a pattern of fish inspired by Escher's hyperbolic print *Circle Limit III*, which is based on the regular $\{3, 8\}$ tessellation. This polyhedron is related to Schwarz's D-surface, a TRMS with the topology of a thickened diamond lattice, which has embedded lines. The red, green, and yellow fish swim along those lines (the blue fish swim in loops around the "waists"). We show:

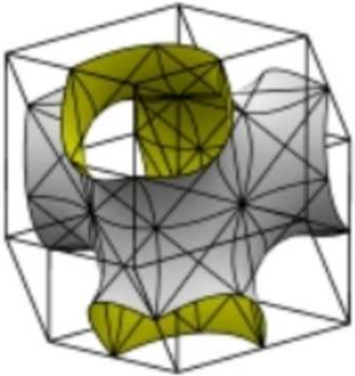
- A piece of the triply periodic polyhedron.
- A corresponding piece of the patterned polyhedron.
- A piece of Schwarz's D-surface showing embedded lines.
- Escher's *Circle Limit III* with the equilateral triangle tessellation superimposed.
- A large piece of the patterned polyhedron.
- A top view of the large piece.

A piece of the triply periodic polyhedron

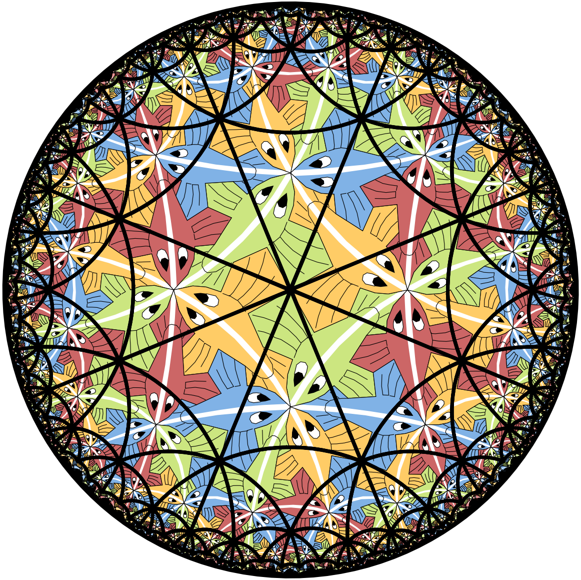


A corresponding piece of the patterned polyhedron

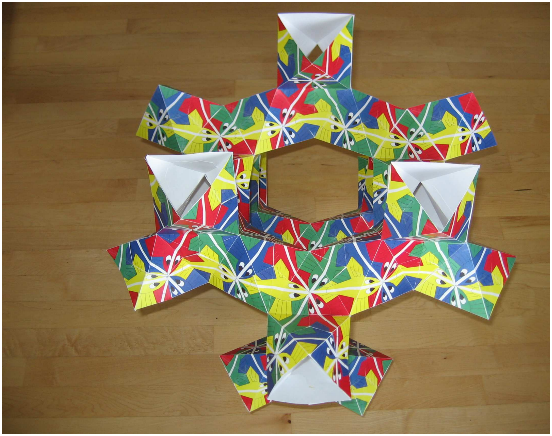
A piece of Schwarz's D-surface showing embedded lines



Escher's Circle Limit III with the equilateral triangle tessellation superimposed



A large piece of the patterned polyhedron



A top view of the large piece

Future Work

- Put other patterns on the regular triply periodic polyhedra, exploiting their embedded lines.
- Place patterns on non-regular, uniform triply periodic polyhedra.
- Put patterns on non-uniform triply periodic polyhedra — especially those that more closely approximate triply periodic minimal surfaces.
- Draw patterns on TPMS's — the gyroid, for example.

Thank You!

Reza and all the other Bridges organizers

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