Escher Patterns on Triply Periodic Polyhedra

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Outline

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Previously Designed Patterned Polyhedra

- M.C. Escher (1898–1972) created at least 3 such polyhedra.
- In 1977 Doris Schattschneider and Wallace Walker placed Escher patterns on each of the Platonic solids and the cuboctahedron.
- Schattschneider and Walker also put Escher patterns on rotating rings of tetrahedra, which they called “kaleidocycles”.
- In 1985 H.S.M. Coxeter showed how to place 18 Escher butterflies on a torus.
Triply Periodic Polyhedra

- A *triply periodic polyhedron* is a (non-closed) polyhedron that repeats in three different directions in Euclidean 3-space.

- We will consider the special case of *uniform* triply periodic polyhedra which have the same vertex figure at each vertex — i.e., there is a symmetry of the polyhedron that takes any vertex to any other vertex..

- We will further specialize to what we call *semiregular triply periodic polyhedra* — polyhedra composed of copies of a single $p$-sided regular polygon. We denote such polyhedra $\{p, q\}$ if $q$ polygons meet at each vertex.

- We note that $p$ and $q$ do not uniquely specify such polyhedra — unlike that situation for regular tessellations.
Inspirations for this Work

- Two papers by Steve Luecking at ISAMA 2011:
  - Building a Sherk Surface from Paper Tiles
  - Sculpture From a Space Filling Saddle Pentahedron

- Bead sculptures that approximate three triply periodic minimal surfaces (TPMS) by Chern Chuang, Bih-Yaw Jin, and Wei-Chi Wei at the 2012 Joint Mathematics Meeting Art Exhibit.

As we will see, some TPMS's are related to triply periodic polyhedra.
Hyperbolic Geometry and Regular Tessellations

- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.

- Thus we must use models of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.

- One such model is the \textit{Poincaré disk model}. The hyperbolic points in this model are represented by interior point of a Euclidean circle — the \textit{bounding circle}. The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (with diameters as special cases).

- This model is appealing to artists since (1) angles have their Euclidean measure (i.e. it is conformal), so that motifs of a repeating pattern retain their approximate shape as they get smaller toward the edge of the bounding circle, and (2) it can display an entire pattern in a finite area.
Repeating Patterns and Regular Tessellations

- A repeating pattern in any of the 3 “classical geometries” (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or motif.

- The regular tessellation, \( \{p, q\} \), is an important kind of repeating pattern composed of regular \( p \)-sided polygons meeting \( q \) at a vertex.

- If \((p - 2)(q - 2) < 4\), \( \{p, q\} \) is a spherical tessellation (assuming \( p > 2 \) and \( q > 2 \) to avoid special cases).

- If \((p - 2)(q - 2) = 4\), \( \{p, q\} \) is a Euclidean tessellation.

- If \((p - 2)(q - 2) > 4\), \( \{p, q\} \) is a hyperbolic tessellation. The next slide shows the \( \{4, 5\} \) tessellation.

- Escher based his 4 “Circle Limit” patterns, and many of his spherical and Euclidean patterns on regular tessellations.
The Regular Tessellation \( \{4, 5\} \)
The tessellation \{4, 5\} superimposed on a pattern of angels and devils
Relation between periodic polyhedra and regular tessellations — a 2-Step Process

- (1) Some triply periodic polyhedra approximate TPMS’s.
- (2) As a minimal surface, a TPMS has negative curvature (except for isolated points of zero curvature), and so its universal covering surface also has negative curvature and thus has the same large-scale geometry as the hyperbolic plane. So the polygons of the triply periodic polyhedron can be transferred to the polygons of a corresponding regular tessellation of the hyperbolic plane.
- We show this relationship when we analyze the patterned polyhedra below.
- Just as the regular hyperbolic tessellations are “universal covering tessellations” of the polyhedra, we can consider patterns based on those tessellations as universal covering patterns of the patterned polyhedra.
Escher’s “Angels and Devils” Patterns

- Escher only realized one pattern, his “Angles and Devils” pattern in each of the three classical geometries:
  - His Euclidean Regular Division Drawing # 45 (1941), which is based on the \( \{4, 4\} \) tessellation.
  - “Heaven and Hell” on a Maple Sphere (1942), based on the \( \{4, 3\} \) spherical tessellation.
  - His hyperbolic pattern “Circle Limit IV” (1960), based on the \( \{6, 4\} \)
Regular Division Drawing # 45
“Heaven and Hell” carved sphere, 1942. Maple, stained in two colors, diameter 235 mm.
“Circle Limit IV”
Filling a “Gap” — a \(\{4, 5\}\) Pattern

- We have seen the progression from Escher’s carved spherical pattern based on the \(\{4, 3\}\) tessellation, to Notebook Pattern 45, based on the \(\{4, 4\}\) tessellation, and *Circle Limit IV* based on the \(\{6, 4\}\) tessellation.

- But Escher did not create an “angels and devils” pattern based on the \(\{4, 5\}\) tessellation.

- We fill that “gap” by displaying an angels and devils pattern on a \(\{4, 5\}\) polyhedron.
A \{4, 5\} Polyhedron
A “Construction Unit” for the \{4, 5\} Polyhedron
The $\{4,5\}$ Polyhedron with Angels and Devils
The Hyperbolic “Universal Covering Pattern”
The “Complement” of the \( \{4, 5\} \) Polyhedron

- The complement of the \( \{4, 5\} \) polyhedron is perhaps easier to understand.
- It is composed of truncated octahedral “hubs” having their hexagonal faces connected by regular hexagonal prisms as “struts”.
- The next slide shows one of the hubs with the struts that are connected to it.
A “Hub” and 8 “struts” of the \( \{4, 5\} \) Complement Polyhedron
A Piece of the TPMS IWP Surface Corresponding to the Hub
A View of the Original \(\{4, 5\}\) Polyhedron
Looking down a Hexagonal Hole
A Pattern of Butterflies on a \( \{3, 8\} \) Polyhedron

The butterfly pattern on the triply periodic \( \{3, 8\} \) polyhedron of the title slide was inspired by Escher’s Regular Division Drawing # 70. This polyhedron is related to Schwarz’s D-surface, a TPMS with the topology of a thickened diamond lattice. We show:

- Escher’s Regular Division Drawing # 70.
- A hyperbolic pattern of butterflies based on the \( \{3, 8\} \) tessellation — the “universal covering pattern” of the patterned polyhedron.
- A construction unit of a \( \{3, 8\} \) polyhedron consisting of a regular octahedral “hub” and four octahedral “struts” placed on alternate faces of the hubs.
- Part of Schwarz’s D-surface corresponding to the construction unit.
- Another view of the patterned \( \{3, 8\} \) polyhedron of the title slide down one of its “tunnels”.
- A close-up of a vertex of the patterned polyhedron.
Escher’s Regular Division Drawing # 70
A pattern of butterflies based on the \( \{3, 8\} \) tessellation — the “universal covering pattern” for the polyhedron.
A “construction unit” of the triply periodic polyhedron
A corresponding piece of Schwarz’s D-surface
A view down one of the “tunnels” of the $\{3, 8\}$ polyhedron.
A close-up of a vertex of the patterned polyhedron.
Butterflies on Another \( \{3, 8\} \) Polyhedron

We show the pattern of butterflies on a different the triply periodic \( \{3, 8\} \) polyhedron. This butterfly pattern was also inspired by Escher’s Regular Division Drawing # 70. Thus the hyperbolic “covering pattern” is the same as for the previous polyhedron. This polyhedron has the same topology as Schwarz’s P-surface, a TPMS with the topology of a thickened version of the 3-D coordinate lattice. We show:

- Schwarz’s P-surface.
- The \( \{3, 8\} \) polyhedron, which is made up of snub cubes arranged in a cubic lattice, attached by their (missing) square faces, and alternating between left-handed and right-handed versions.
- A close-up of the patterned polyhedron.
- A close-up of a vertex of the patterned polyhedron.
Schwarz’s P-surface
Another Patterned $\{3, 8\}$ Polyhedron
A Close-up of the \( \{3, 8\} \) Polyhedron
A close-up of a vertex of the patterned polyhedron.
Future Work

- Put other patterns on the triply periodic polyhedra shown above.
- Place patterns on other triply periodic polyhedra, with various regularity conditions.
- Place a butterfly pattern on a \( \{3, 7\} \) polyhedron.
- Draw patterns on TPMS’s — the gyroid, for example.
Thank You!

Reza and all the other Bridges organizers

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