PATTERNS ON TRIPLY PERIODIC POLYHEDRA

Douglas DUNHAM
University of Minnesota Duluth, USA

ABSTRACT:
This paper discusses patterns on *triply periodic polyhedra*, infinite polyhedra that repeat in three independent directions in Euclidean 3-space. We further require that all the vertices be congruent by a symmetry of the polyhedron, i.e. that they be uniform, and also that each of the faces is a single regular polygon. We believe that we are the first to apply patterns to such polyhedra. The patterns we use are inspired by the Dutch artist M.C. Escher. The patterns are preserved, up to color symmetry, by the symmetries of the polyhedra.

Keywords: periodic polyhedra, Escher designs, hyperbolic geometry

1. INTRODUCTION

A number of artists, including M.C. Escher, have placed patterns on convex polyhedra. In contrast, we show Escher-inspired patterns on *triply periodic polyhedra*, infinite polyhedra that repeat in three independent directions in Euclidean 3-space. Doris Schattschneider's book *M.C. Escher: Visions of Symmetry* is a good reference for Escher's works [3]. We require that those polyhedra (1) be composed of copies of a regular polygon, and (2) that they be *uniform*: all vertices are congruent by a symmetry of the polyhedron. Figure 1 shows a pattern of angular fish on the simplest triply periodic polyhedron, which is based on the cubic lattice.

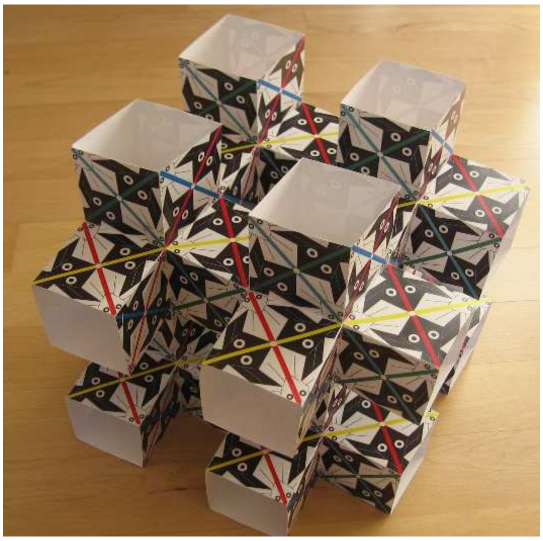

Figure 1: A pattern of angular fish on the $\{4,6 \mid 4\}$ polyhedron.

The triply periodic polyhedra we discuss are often called hyperbolic since the sum of the angles of the polygons at each vertex is greater than 360 degrees (if the angle sum were 360 degrees, the "polyhedron" would be flat; and if it were less than 360 degrees, the polyhedron would be finite).

In 1926 H.S.M. Coxeter and John Petrie discovered a regular class of triply periodic polyhedra which they called *infinite skew polyhedra* and have symmetry groups that are flag-transitive, and thus are natural analogs of the Platonic Solids. They designated those polyhedra by the extended Schläfli symbol $\{p,q \mid n\}$, indicating that there are q p-gons around each vertex and n-gonal holes. Coxeter and Petrie showed there are three possibilities: $\{4,6 \mid 4\}$ (shown in Figure 1), $\{6,4 \mid 4\}$, and $\{6,6 \mid 3\}$ [1].

In the following section we dicuss the relationship between triply periodic polyhedra and regular hyperbolic tessellations. Then we show patterns on each of the infinite skew polyhedra. Next we show a patterns on less regular polyhedra. Finally we draw conclusions and indicate directions of future work.

2. PERIODIC POLYHEDRA AND REGULAR TESSELLATIONS

We use the Schläfli symbol $\{p,q\}$ to denote the regular tessellation formed by regular p-sided polygons or *p-gons* with q of them meeting at each vertex. If $(p-2)(q-2) > 4$, $\{p,q\}$ is a tessellation of the hyperbolic plane (otherwise it is Euclidean or spherical). Figure 2 shows the tessellation $\{4,6\}$ in the Poincaré disk model of hyperbolic geometry. Escher based all four of his hyperbolic "Circle Limit" patterns, and many of his Euclidean patterns on regular tessellations. Figure 3 shows that tessellation superimposed on an Escher-like pattern of angular fish similar to those of Figure 1.

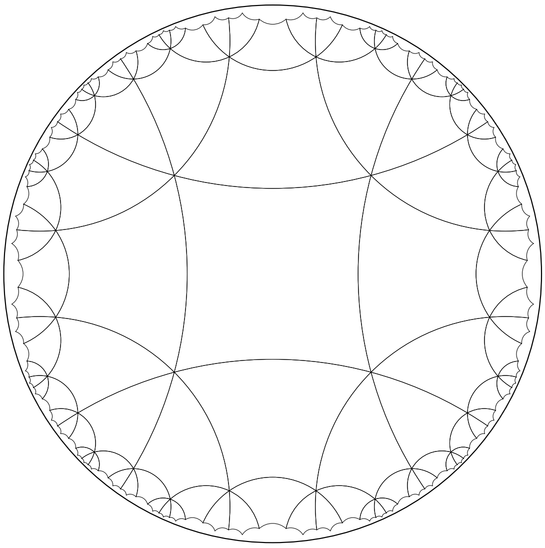

Figure 2: The regular tessellation $\{4,6\}$.

As mentioned, each of the triply periodic polyhedra we consider have a p-gon for each of its faces, with q p-gons around each vertex (since the polyhedron is uniform). Thus we can also use the Schläfli symbol $\{p,q\}$ to refer to such polyhedra, however, unlike regular tessellations, different polyhdera can have the same $\{p,q\}$, as we will see below. We have already introduced the extended extended Schläfli symbol $\{p,q \mid n\}$ used by Coxeter and Petrie to (uniquely) specify their more regular infinite skew polyhedra.

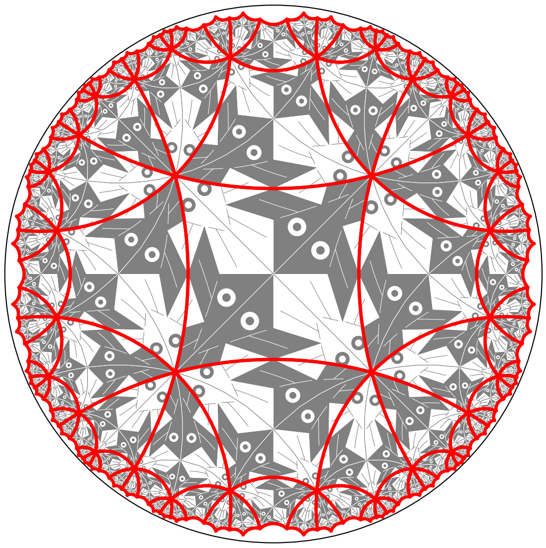

Figure 3: The regular tessellation $\{4,6\}$ superimposed on a pattern of angular fish.

For the three infinite skew polyhedra and the other triply periodic polyhedra discussed below, there is an intermediate "connecting surface" between the polyhedron $\{p,q\}$ (or $\{p,q \mid n\}$) and the corresponding regular tessellations $\{p,q\}$. First, these periodic polyhedra are approximations to triply periodic minimal surfaces (TPMS). Alan Schoen has done extensive investigations into TPMS [2]. Figure 4 shows the Schwarz P-surface, which has the same topology as the polyhedron in Figure 1. In fact the embedded Euclidean lines of the P-surface are just the backbone lines of the fish in Figure 1.

Second, each smooth surface has a *universal covering surface*: a simply connected surface with a covering map onto the original surface, which is a sphere, the Euclidean plane, or the hyperbolic plane. Since each TPMS has negative curvature (except for possible isolated points), its universal covering surface does too, and thus has the same large-scale geometry as the hyperbolic plane. In the same vein, we might call a hyperbolic pattern based on the tessellation $\{p,q\}$ the "universal covering pattern" for the related pattern on the polyhedron $\{p,q\}$. Thus Figure 5 shows the universal covering pattern for the

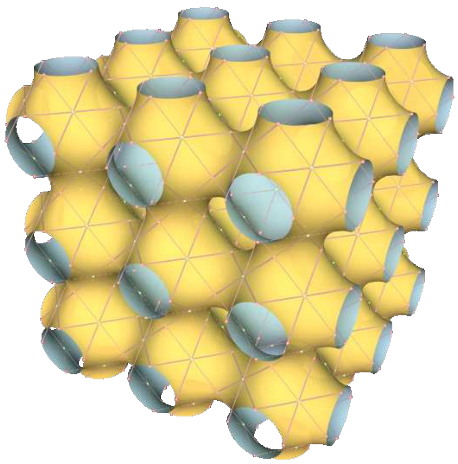

Figure 4: The Schwarz P-surface, showing embedded lines.

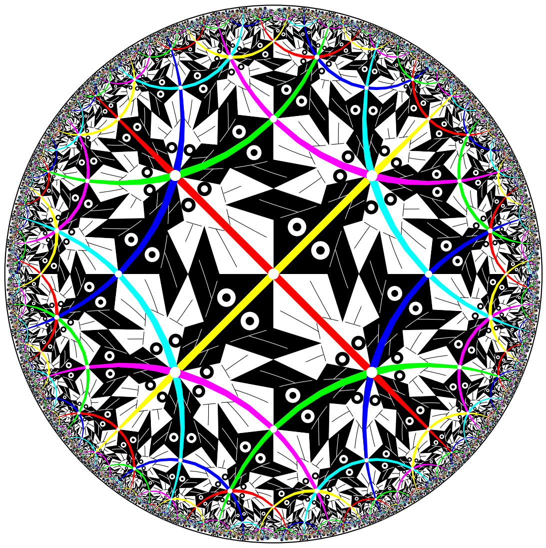

Figure 5: The "universal covering pattern" for Figure 1.

patterned polyhedron of Figure 1. In the next section we discuss fish patterns on infinite skew polyhedra.

3. ANGULAR FISH ON INFINITE SKEW POLYHEDRA

In Figure 1 we have shown a pattern of angular fish on the infinite skew polyhedron $\{4,6\,|\,4\}$. In this section we also show patterns of angular fish on the other two infinite skew polyhedra: $\{6,4\,|\,4\}$ and $\{6,6\,|\,3\}$.

The $\{6,4\,|\,4\}$ polyhedron is the dual of the $\{4,6\,|\,4\}$ polyhedron in which each vertex is replaced by a hexagon. The $\{4,6\,|\,4\}$ polyhedron is based on the bi-truncated, cubic, space-filling tessellation by truncated octahedra. The $\{6,4\,|\,4\}$ polyhedron divides space into two sets of truncated octahedra which are connected by their square faces. Figure 6 shows the polyhedron decorated with angular fish. Figure 7 shows a top view of the backbone lines of the fish on the $\{6,4\,|\,4\}$ polyhedron, which are the same lines as the backbone lines of the fish in Figure 1 — not surprising since the polyhedra are duals. Thus the TPMS corresponding to the $\{6,4\,|\,4\}$ polyhedron is the same as for the $\{4,6\,|\,4\}$ poly-

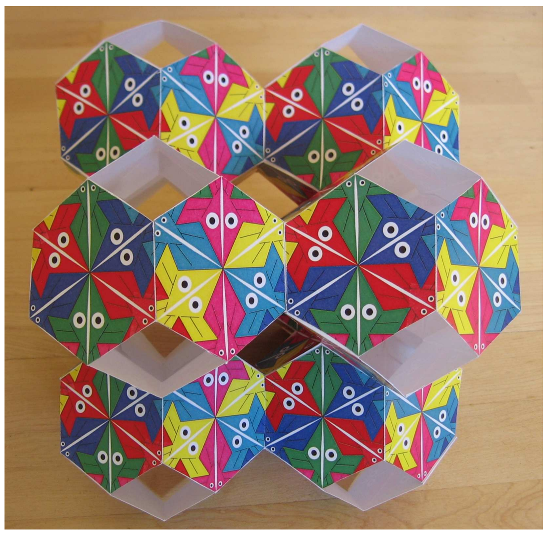

Figure 6: The $\{6,4\,|\,4\}$ polyhedron decorated with angular fish.

hedron: the Schwarz P-surface. Figure 8 shows

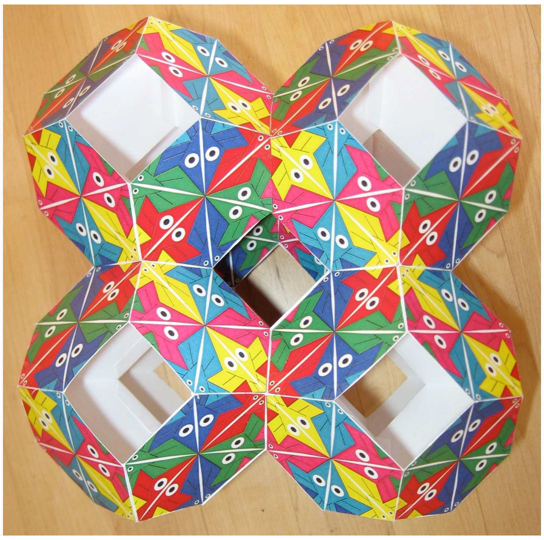

Figure 7: A top view of the $\{6,4 \mid 4\}$ polyhedron.

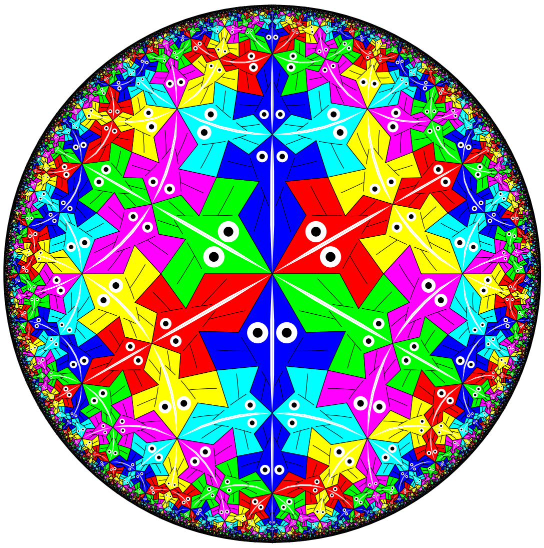

Figure 8: The "universal covering pattern" for Figures 6 and 7.

the universal covering pattern for the pattern shown in Figures 6 and 7.

The self-dual $\{6,6 \mid 3\}$ polyhedron is more difficult to understand than the other two infinite skew polyhedra. It is formed from truncated tetrahedra with their triangular faces removed. Such "missing" triangular faces from four truncated tetrahedra are then placed in a tetrahedral arrangement (around a small invisible tetrahedron). Figure 9 shows a side view of a $\{6,6 \mid 3\}$ polyhedron decorated with angular fish. Figure 10 shows a top view looking down at one of the vertices (where six hexagons meet). The corresponding TPMS is Schwarz's D-surface, where D stands for Diamond [2]. The Schwarz D-surface divides space into two congruent parts, each with the shape of a thickened diamond lattice. It also has embedded Euclidean lines, which correspond to the backbone lines of the fish. Figure 11 shows the corresponding universal covering pattern based on the $\{6,6\}$ tessellation.

In the next section we discuss patterns on $\{3,8\}$ polyhedra.

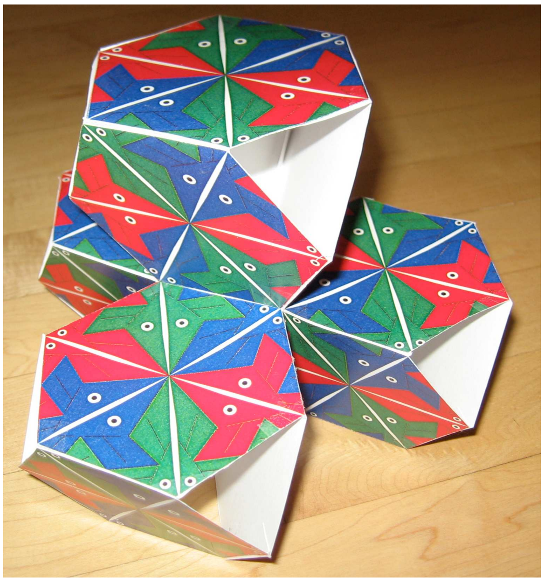

Figure 9: The $\{6,6 \mid 3\}$ polyhedron decorated with angular fish.

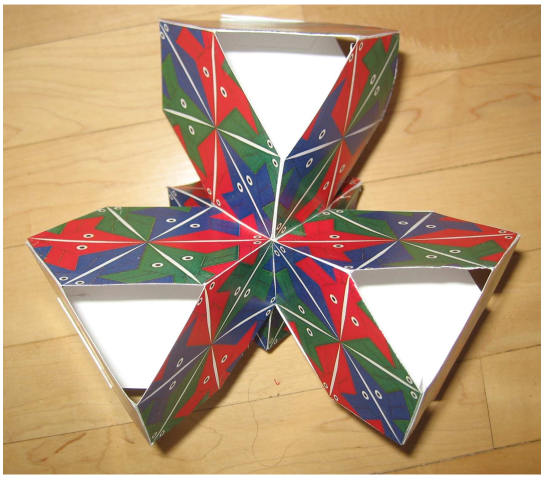

Figure 10: A top view of a pattern of fish shown in Figure 9.

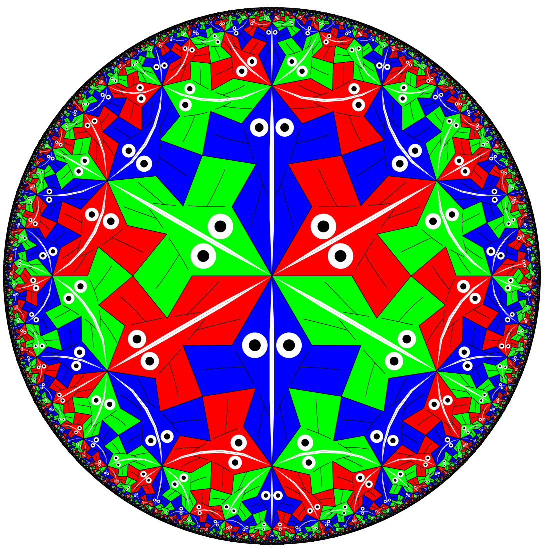

Figure 11: The universal covering pattern for Figures 9 and 10.

4. PATTERNS ON $\{3,8\}$ POLYHEDRA

In this section we start by examining butterfly patterns on two different $\{3,8\}$ Polyhedra, so we can see visually that the Schläfli symbol is not enough to specify a triply periodic minimal surface. Then we end by considering a fish pattern on one of those polyhedra.

The butterfly pattern was inspired by Escher's Regular Division Drawing 70, which is shown in Figure 12. Figure 13 shows the corresponding hyperbolic universal covering pattern for the patterns on the first two polyhedra that we will discuss.

Figure 12: Escher's Regular Division Drawing 70.

The first $\{3,8\}$ polyhedron has the shape of a diamond lattice, like the Schwarz D-surface, and thus the intermediate TPMS is that surface. This polyhedron is made up of parts of regular octahedra of two types: "hub" octahedra and "strut" octahedra. Each hub octahedron has four strut octahedra placed on alternate faces of that hub, so four hub triangles are covered by struts and four are exposed. Each strut connects two hubs to opposite faces of the strut, which are covered by the hubs, leaving six exposed triangle faces. Thus eight equilateral triangles meet at each vertex, and this a $\{3,8\}$ polyhedron. However it

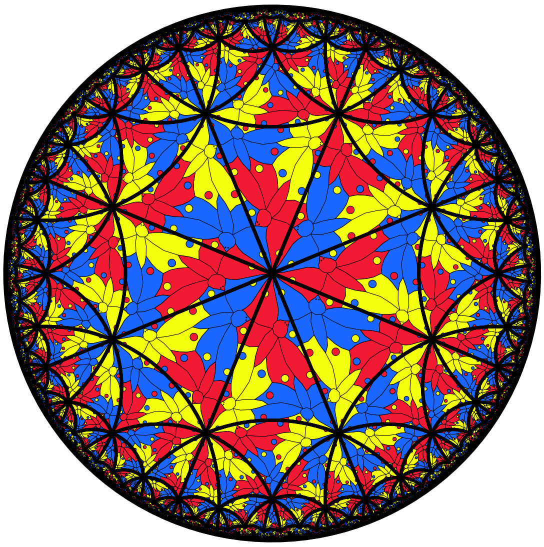

Figure 13: A butterfly pattern based on the $\{3,8\}$ tessellation.

is not as regular as the infinite skew polyhedra since there is no symmetry of the polyhedron that maps a hub triangle face to a strut triangle face (and vice versa). Figure 14 shows this polyhedron covered with butterflies.

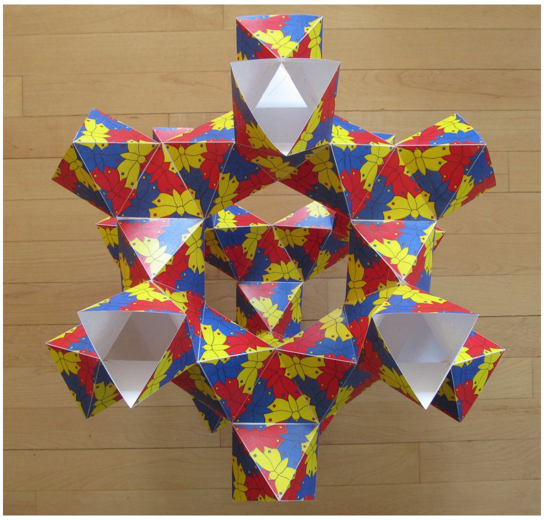

Figure 14: Butterflies on a $\{3,8\}$ diamond lattice polyhedron.

The second $\{3,8\}$ polyhedron is composed of skew cubes with alternating chiralities centered at cubic lattice points and connected by their (missing) square faces. Thus the intermediate TPMS is the Schwarz P-surface. Figure 15 shows that polyhedron with a butterfly pattern.

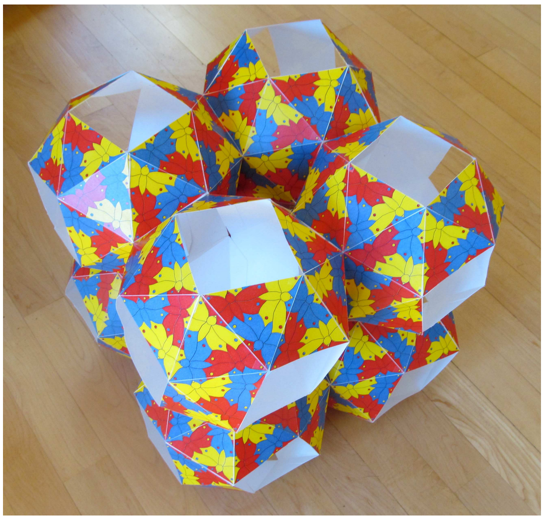

Figure 15: Butterflies on a $\{3,8\}$ polyhedron made of snub cubes.

Finally we show a pattern of fish on the "diamond lattice" $\{3,8\}$ polyhedron that was inspired by Escher's hyperbolic pattern *Circle Limit III*. Figure 16 shows one view of that polyhedron. Figure 17 shows Escher's *Circle Limit III* pattern with the $\{3,8\}$ tessellation superimposed. Figure 18 is a view of the polyhedron down a 3-fold rotation axis.

5. CONCLUSIONS

We have discussed some of the theory of triply periodic polyhedra. In particular we have concentrated on triply periodic polyhedra that are uniform (all veritces are congruent) and composed of copies of a regular p-sided polygon. We have also shown how those polyhedra can be decorated with patterns that are related to repeating patterns of the hyperbolic plane. In particular, we have shown angular fish patterns on each of the three most regular triply periodic polyhedra, the infinite skew polyhedra of Coxeter and Petrie. We have also shown an Escher-inspired butterfly pattern on two different $\{3,8\}$ polyhedra. Finally we have shown a fish pattern on a $\{3,8\}$ polyhedron that was inspired by Escher's hyperbolic print *Circle Limit III*.

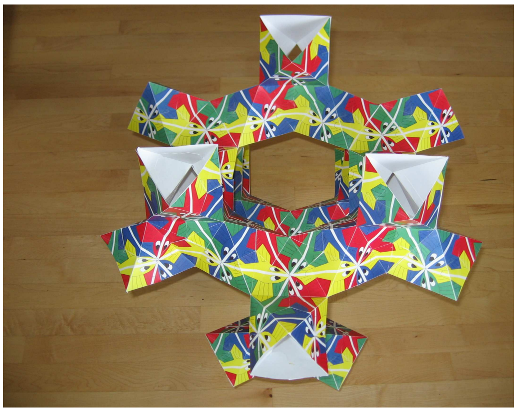

Figure 16: A fish pattern on the diamond lattice $\{3,8\}$ polyhedron.

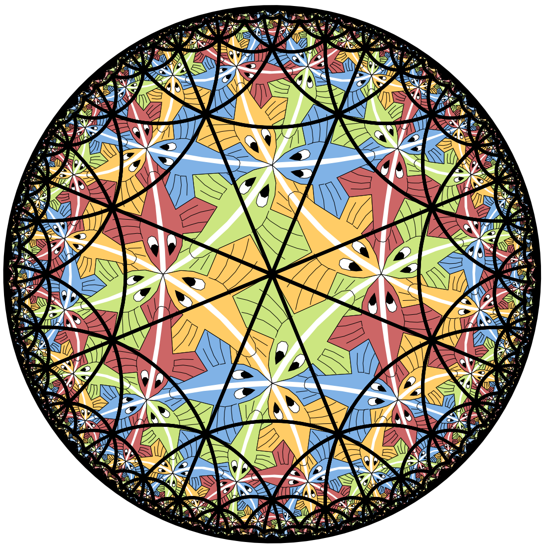

Figure 17: Escher's *Circle Limit III* with $\{3,8\}$ superimposed.

Figure 18: The polyhedron showing a 3-fold rotation axis.

In the future we hope to investigate other less regular triply periodic polyhedra. We would also like to place other Escher-inspired patterns on triply periodic polyhedra, either regular or less regular.

ACKNOWLEDGMENTS

I would like to thank the many students who have worked over the years on the software used to create the patterns shown in this paper.

REFERENCES

[1] H. S. M. Coxeter. Regular Skew Polyhedra in Three and Four Dimensions. *Proc. London Math. Soc.*, (2): 33–62, 43 1937.

[2] A. Schoen. Triply Periodic Minimal Surfaces (TPMS). http://schoengeometry.com/e-tpms.html (accessed 16 June, 2014).

[3] D. Schattschneider. *M.C. Escher: Visions of Symmetry*. Harry N. Abrams, New York, NY, 2004.

ABOUT THE AUTHOR

1. Douglas Dunham is a Professor in the Department of Computer Science at the University of Minnesota Duluth. His research interests include repeating patterns, hyperbolic geometry, and the art of M.C. Escher. He can be reached at his email address: ddunham@d.umn.edu or via his postal address: Department of Computer Science, University of Minnesota, 1114 Kirby Drive, Duluth, Minnesota, 55812-3036, USA.