Creating Repeating Patterns with Color Symmetry

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Abstract

M.C. Escher's repeating patterns have two distinguishing features: they interlock without gaps or overlaps, and they are colored in a regular way. In this paper we will discuss this second characteristic, which is usually called color symmetry. We will first discuss the history and theory of color symmetry, then show several patterns that exhibit color symmetry.

Introduction

Figure 1 below shows a pattern with color symmetry in the style of the Dutch artist M.C. Escher's "Circle Limit" patterns. Escher was a pioneer in creating patterns that were colored symmetrically, using two colors

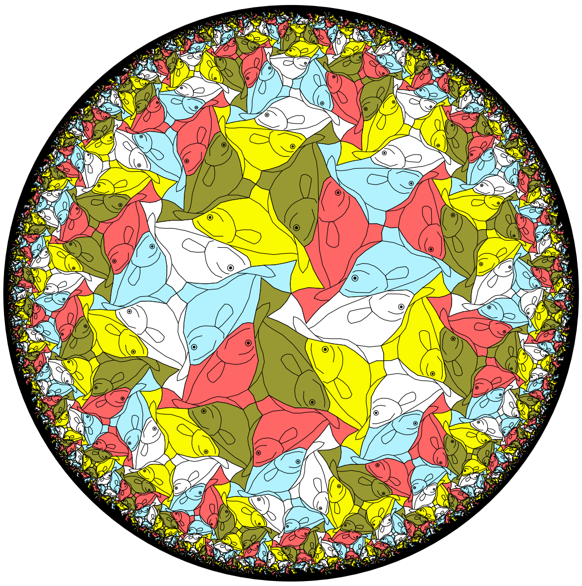


Figure 1: A pattern with 5-color symmetry.

(black-white) and n colors for $n > 2$. In the next section, we review the history and theory of color symmetry. Then we briefly discuss regular tessellations and hyperbolic geometry, since such tessellations provide a framework for repeating patterns, and hyperbolic geometry allows for many different kinds of repeating patterns and thus many kinds of color symmetry. Next, we explain how to implement color symmetry in common programming languages. With that background, we show patterns of fish like those of Figure 1 that illustrate these concepts. Then we discuss Escher's hyperbolic print *Circle Limit III* and related patterns which have an additional color restriction. Finally, we suggest directions for future research.

Color Symmetry: History and Theory

Others created patterns with color symmetry before Escher, but he was quite prolific at making such patterns and, the use of color symmetry was one of the hallmarks of his work. As early as 1921 Escher created a pattern with 2-color (black-white) symmetry shown in Figure 2, 15 years before the theory of such patterns was elucidated [Woods36]. Figure 3 shows a hyperbolic pattern with 2-color symmetry. Starting in the mid

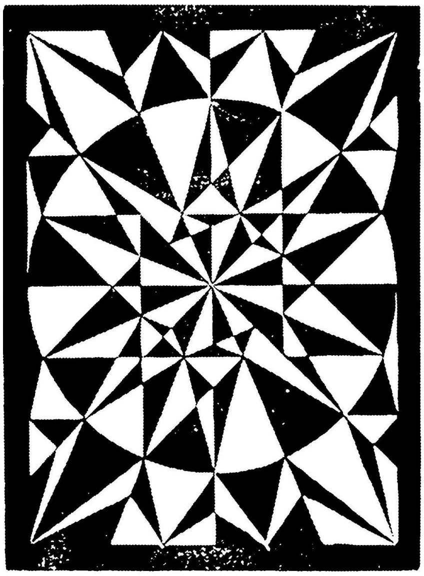


Figure 2: An Escher pattern with 2-color symmetry.

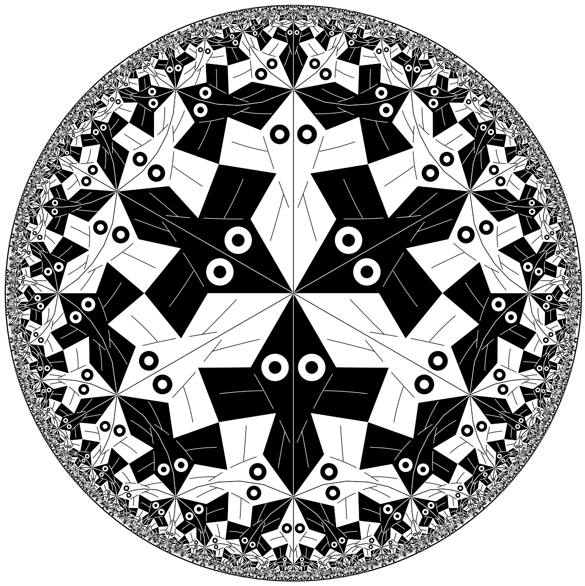


Figure 3: A hyperbolic pattern of angular fish with 2-color symmetry.

1920's, Escher drew patterns with 3-color symmetry, and in 1938 he created Regular Division Drawing 20 (Figure 6 below, and the inspiration for Figure 1), a pattern of fish, with 4-color symmetry. From 1938 to 1942 Escher developed his own theory of repeating patterns, some of which had 3-color symmetry [Schattschneider04]. This was two decades before mathematicians defined their own theory of n-color symmetry for $n > 2$ [Van der Waerden61].

In order to understand color symmetry, it is necessary to understand symmetries without respect to color. A *repeating pattern* is a pattern made up of congruent copies of a basic subpattern or *motif*. A *symmetry* of a repeating pattern is an isometry (a distance preserving transformation) that takes the pattern onto itself so that each copy of the motif is mapped to another copy of the motif. For example, reflection across the vertical diameter in Figure 3 is a symmetry of that pattern, as are reflections across the diameters that make 60 degree angles with the vertical diameter. Also rotation by 180 degrees about the center is a symmetry of the pattern in Figure 2. A *color symmetry* of a pattern of colored motifs is a symmetry of the uncolored pattern that takes all motifs of one color to motifs of a single color — that is, it permutes the colors of the motifs. This concept is sometimes called *perfect* color symmetry. Escher required that colored patterns adhere to the *map-coloring principle*: motifs that share an edge must be different colors (but motifs of the same color can share a vertex), and we will follow that principle also. For example, reflection about the horizontal or vertical axis through the center of Figure 2 is almost a color symmetry of that pattern since it interchanges black and white (there are a few small pieces that do not quite correspond). In Figure 1, a counter-clockwise rotation about the center by 72 degrees is a color symmetry in which red → yellow → blue → brown → white → red. Black remains fixed since it is used as an outline/detail color. Note that if a symmetry of an uncolored pattern has period k, then the period of the color permutation it induces must divide k. In group theory terms, this means that the mapping from symmetries to color permutations is a homomorphism. So, in Figure 1, since rotation about the center by 72 degrees has period 5 (and the five central fish must be different colors by the map-coloring principle), the color permutation also has period 5 since 5 is a prime. In general any rotation of prime period k would induce a color permutation of period k. See [Schwarzenberger84] for an account of the development of the theory of color symmetry.

Regular Tessellations and Hyperbolic Geometry

One important kind of repeating pattern is the *regular tessellation*, denoted $\{p, q\}$, of the hyperbolic plane by regular p-sided polygons meeting q at a vertex. Actually this definition also works for the sphere and in the Euclidean plane. It is necessary that $(p - 2)(q - 2) > 4$ to obtain a hyperbolic tessellation. If $(p-2)(q-2) = 4$, the tessellation is Euclidean and there are three possibilities: the tessellation by squares $\{4, 4\}$, by regular hexagons $\{6, 3\}$, and by equilateral triangles $\{3, 6\}$. If $(p-2)(q-2) < 4$, the tessellation is spherical and there are five possibilities, corresponding to the five Platonic solids. Escher made extensive use of the Euclidean tessellations as a framework for his Regular Division Drawings [Schattschneider04]. He used $\{6, 4\}$ in his construction of his hyperbolic patterns *Circle Limit I* and *Circle Limit IV*; he used $\{8, 3\}$ for *Circle Limit II* and *Circle Limit III*. Figure 4 shows $\{6, 4\}$ in red superimposed on *Circle Limit I*; Figure 5 shows $\{8, 3\}$ in blue superimposed on *Circle Limit II*. Figures 1 and 3 are based on the tessellations

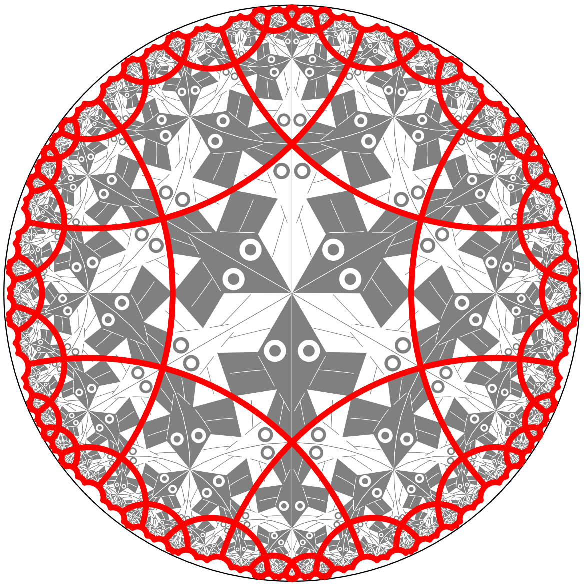


Figure 4: The $\{6, 4\}$ tessellation (red) superimposed on the *Circle Limit I* pattern.

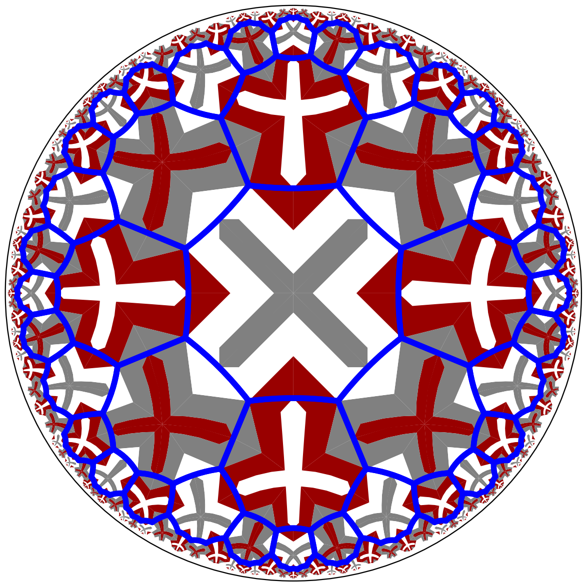


Figure 5: The $\{8, 3\}$ tessellation (blue) superimposed on the *Circle Limit II* pattern.

$\{5, 4\}$ and $\{6, 6\}$ respectively.

Since there are infinitely many solutions to the inequality $(p - 2)(q - 2) > 4$, there are infinitely many different hyperbolic patterns. Escher undoubtedly would have created many more such patterns if it had not required so much tedious hand work. In this age of computers, this is not a problem, so we can investigate many such patterns with many kinds of color symmetry.

The patterns of Figures 1, 3, 4, and 5 are drawn in the Euclidean plane, but they could also be interpreted as repeating patterns in the *Poincaré disk model* of hyperbolic geometry. In this model, hyperbolic points in this model are just the (Euclidean) points within a Euclidean bounding circle. Hyperbolic lines are represented by circular arcs orthogonal to the bounding circle (including diameters). Thus, the backbone lines of the fish lie along hyperbolic lines in Figure 3, as do the edges of $\{6, 4\}$ in Figure 4. The hyperbolic measure of an angle is the same as its Euclidean measure in the disk model — we say such a model is *conformal*. Equal hyperbolic distances correspond to ever smaller Euclidean distances toward the edge of the disk. Thus, all the fish in Figure 1 are hyperbolically the same size, as are all the fish in Figure 3.

A reflection in a hyperbolic line is an inversion in the circular arc representing that line (or just Euclidean reflection across a diameter). As in Euclidean geometry, any isometry can be built up from at most three reflections. For example, successive reflections across intersecting lines produces a rotation about the intersection point by twice the angle between the lines in both Euclidean and hyperbolic geometry. This can best be seen in Figure 3 which has 120 degree rotations about the meeting points of noses; there are also 90 degree rotations about the points where the trailing edges of fin tips meet. In Figure 1, there are 72 degree rotations about the tails and 90 degree rotations about the dorsal fins. For more on hyperbolic geometry see [Greenberg08].

Implementation of Color Symmetry

Symmetries of of uncolored patterns in both Euclidean and hyperbolic geometry [Dunham86] can be implemented as matrices in many programming languages. To implement permutations of the colors, it is convenient to use integers to represent the colors and arrays to represent the permutations. The representation of permutations by cycles or matrices seems less useful. In Figure 1, we let 0 ↔ black, 1 ↔ white, 2 ↔ red, 3 ↔ yellow, 4 ↔ blue, and 5 ↔ brown. If α is the color permutation induced by counter clockwise rotation about the center by 72 degrees,

$$\alpha = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

in two-line notation. This is easily implemented as an array in some common programming languages as:

$$\alpha[0] = 0, \;\; \alpha[1] = 2, \;\; \alpha[2] = 3, \;\; \alpha[3] = 4, \;\; \alpha[4] = 5, \;\; \alpha[5] = 1.$$

It is then easy to multiply permutations α and β to obtain their product γ as follows:

```
for i ← 0 to nColors - 1
        γ[i] = β[α[i]]
```

It is also easy to obtain the inverse of a permutations α as follows:

```
for i ← 0 to nColors - 1
        α⁻¹[α[i]] = i
```

It is useful to "bundle" the matrix representing a symmetry with its color permutation as an array into a single "transformation" structure (or class in an object oriented language).

Patterns Based on Escher's Notebook Drawing 20

Escher's Notebook Drawing 20, Figure 6, seems to be the first of his repeating patterns with 4-color symmetry. It is based on the Euclidean square tessellation $\{4, 4\}$. It requires four colors since there is a meeting point of three fish near the fish mouths and thus needs at least 3 colors, and the number of colors must divide 4. It was the inspiration for the hyperbolic pattern of Figure 1, which as noted above requires at least five colors since it has rotation points of prime period five, and as can be seen, five colors suffice. Figure 7 shows a related pattern with 4-fold rotations at the tails and 5-fold rotations at the dorsal fins — the reverse of Figure 1. We present two more patterns in this family. Figure 8 is based on the the $\{5, 5\}$ tessellation. In order to obtain 4-colored pattern as in Escher's Notebook Drawing 20, it is necessary that four divides p and q — Figure 9 shows such a pattern of distorted fish based on $\{8, 4\}$.

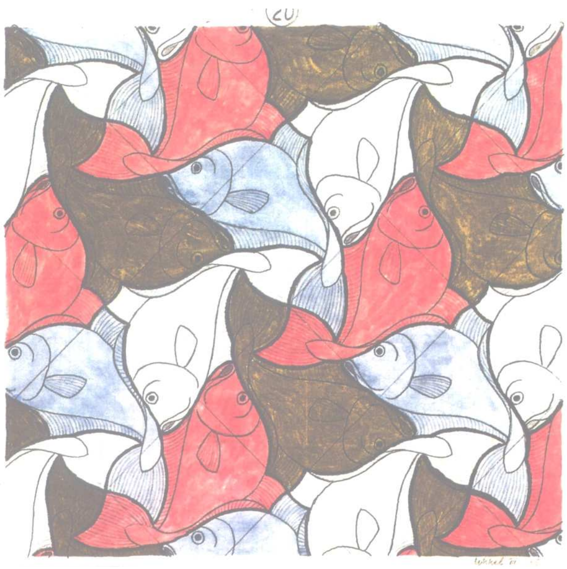


Figure 6: Escher's Notebook Drawing 20.

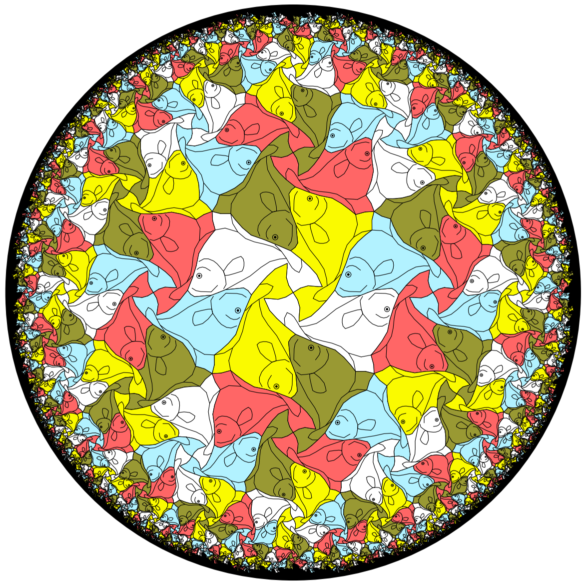


Figure 7: A fish pattern based on the $\{4, 5\}$ tessellation.

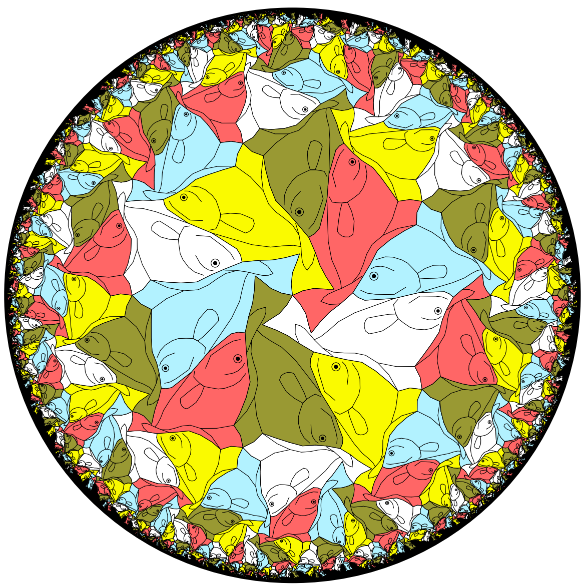


Figure 8: A fish pattern based on the $\{5, 5\}$ tessellation.

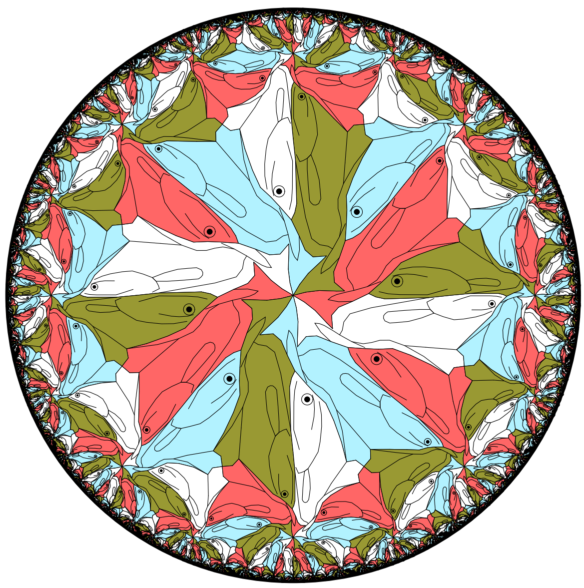


Figure 9: A pattern of distorted fish based on the $\{8, 4\}$ tessellation.

The Color Symmetry of *Circle Limit III* and Related Patterns

Escher's print *Circle Limit III* is probably his most attractive and intriguing hyperbolic pattern. Figure 10 shows a computer rendition of that pattern. Escher wanted to design a pattern in which the fish along a backbone line were all the same color. This is nominally an added restriction to obtaining a symmetric coloring. However in the case of *Circle Limit III*, four colors are required anyway. Certainly at least three colors are required since three fish meet at left rear fin tips. But three colors are not enough to achieve color symmetry. A contradiction will arise if we assume that we can re-color some of the fish in *Circle Limit III* using only three colors and while maintaining color symmetry. To see this, focus on the yellow fish to the upper right of the center of the circle (with its right fin at the center). Red and blue fish meet at its left fin. There are two possibilities for coloring the fish that meet its nose: (1) use two colors, with all the "nose" fish colored yellow and all the "tail" fish colored red (to preserve the coloring of the fish at the yellow fish's left fin), or (2) use three colors, with yellow and red fish as in *Circle Limit III*, and the green fish of *Circle Limit III* being colored blue instead (since yellow, red, and blue are the three colors). In both cases the color symmetry requirement would imply that there were at least three different colors for the four fish around the center of the circle. In case (1), the green lower right fish of the four center fish would be red instead since it is a "tail" fish at the nose of the yellow "focus" fish, and the upper left fish of the four center fish would be blue, since the "nose" fishes at the tail of the "focus" fish are blue. In case (2), the only way that we can use three colors is to have fish along the same backbone line be the same color. In this case the green lower right fish of the four center would have to be blue, and the upper left fish of the four center fish would be red. So in both cases the central four fish would be colored by at least three different colors, and thus must use at least four colors to have color symmetry (since the center is a 4-fold rotation point), a contradiction to the assumption that *Circle Limit III* could be 3-colored.

Figure 11 shows a pattern based on the $\{10, 3\}$ tessellation and related to *Circle Limit III*, but with five fish meeting at right fins. Like *Circle Limit III*, Figure 11 also satisfies Escher's additional restriction that

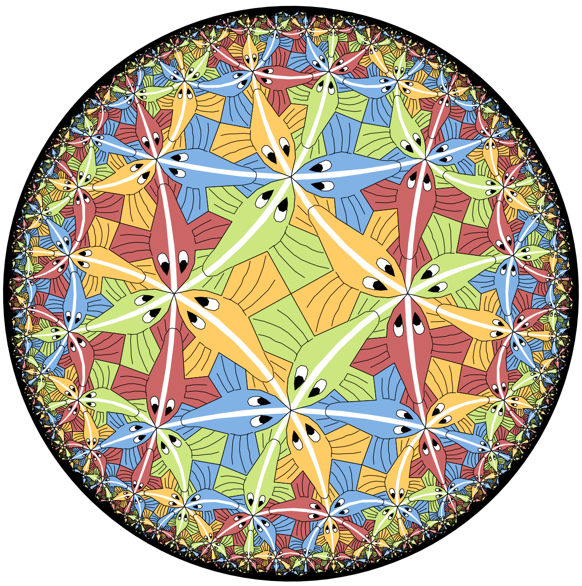


Figure 10: A computer rendition of the *Circle Limit III* pattern.

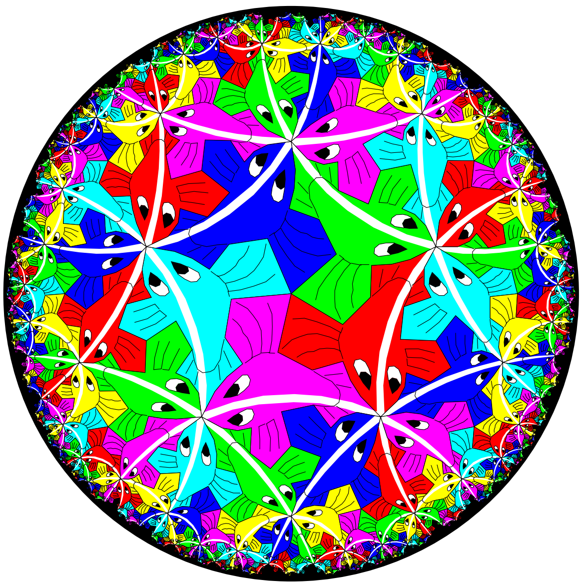
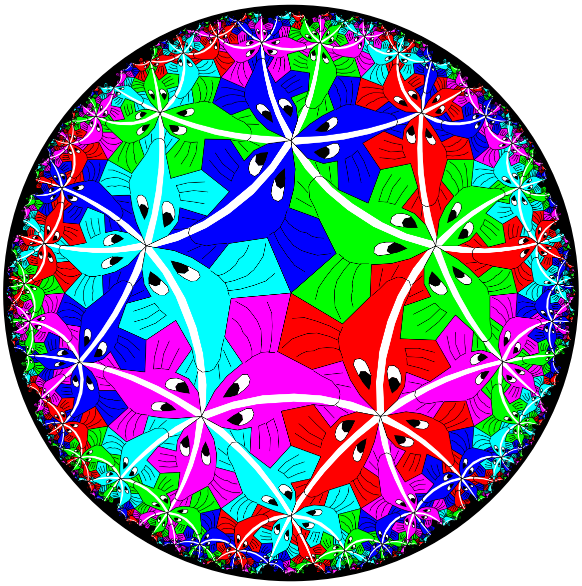


Figure 11: A *Circle Limit III* like pattern based on the $\{10, 3\}$ tessellation.

fish along the same backbone line be the same color. Certainly five colors are needed to color this pattern, since there is a 5-fold rotation point in the center, but it can be proved that a sixth color, yellow, is actually needed to satisfy Escher's restriction.

If Escher's restriction is removed, it turns out that the pattern of Figure 11 can be 5-colored, as is shown in Figure 12. If the fish are "symmetric", with five fish meeting at both the right and left fins, then the pattern can be colored with only five colors and still adhere to Escher's restriction that fish along each backbone line be the same color. Figure 13 shows such a pattern. For more on patterns related to *Circle Limit III* see [Dunham09].



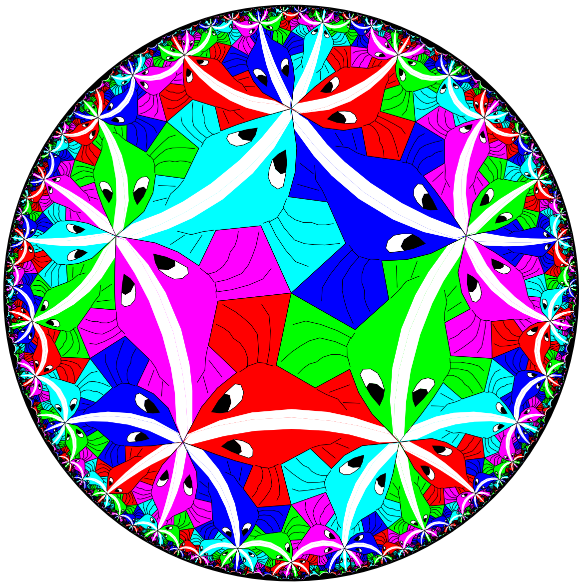


Figure 12: A 5-coloring of the pattern of Figure 11. **Figure 13:** A 5-colored "symmetric" fish pattern.

Conclusions and Future Work

Except for Escher's patterns, I determined the colorings of all the patterns of Figures "by hand", which was usually a trial and error process. It seems to be a difficult problem to automate the process of coloring a pattern symmetrically — i.e. with color symmetry while adhering to the map-coloring principle. And even if that is possible, it would seem to be even harder if we add Escher's restriction that fish alone each backbone line be the same color for *Circle Limit III* like patterns.

Acknowledgments

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