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Symmetric Fish Patterns on Regular Periodic Polyhedra

Douglas Dunham
University of Minnesota Duluth
Duluth, Minnesota USA

Outline

- Some previously designed patterned (closed) polyhedra
- Triply periodic polyhedra
- Hyperbolic geometry and regular tessellations
- Relation between periodic polyhedra and regular tessellations
- A fish pattern on the $\{4, 6|4\}$ polyhedron
- A fish pattern on the $\{6, 4|4\}$ polyhedron
- A fish pattern on the $\{6, 6|3\}$ polyhedron
- A Pattern of fish on a $\{3, 8\}$ polyhedron
- Future research

Previously Designed Patterned Polyhedra

- M.C. Escher (1898–1972) created at least 3 such polyhedra.
- In 1977 Doris Schattschneider and Wallace Walker placed Escher patterns on each of the Platonic solids and the cuboctahedron.
- Schattschneider and Walker also put Escher patterns on rotating rings of tetrahedra, which they called "kaleidocycles".
- In 1985 H.S.M. Coxeter showed how to place 18 Escher butterflies on a torus.

Triply Periodic Polyhedra

- A *triply periodic polyhedron* is a (non-closed) polyhedron that repeats in three different directions in Euclidean 3-space.
- We will mostly discuss *regular skew polyhedra*, triply periodic polyhedra that are "flag-transitive" — there is a symmetry of the polyhedron that takes any vertex, edge containing that vertex, and face containing that edge to any other such (vertex, edge, face) combination. These are natural analogs of the Platonic solids.
- In 1926 John Petrie and H.S.M. Coxeter proved that there are exactly three regular skew polyhedra, which Coxeter called $\{4, 6|4\}$, $\{6, 4|4\}$, and $\{6, 6|3\}$, where $\{p, q|r\}$ denotes a polyhedron made up of p-sided regular polygons meeting q at a vertex, and with regular r-sided holes.

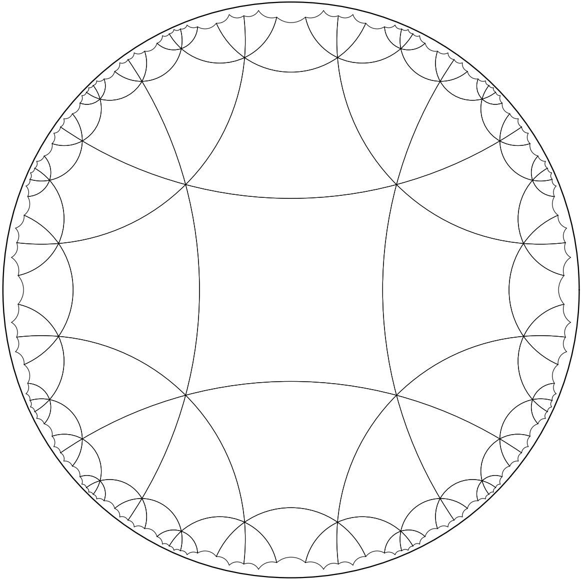
Hyperbolic Geometry and Regular Tessellations

- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.
- Thus we must use *models* of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.
- One such model is the *Poincaré disk model*. The hyperbolic points in this model are represented by interior point of a Euclidean circle — the *bounding circle*. The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (with diameters as special cases).
- This model is appealing to artests since (1) angles have their Euclidean measure (i.e. it is conformal), so that motifs of a repeating pattern retain their approximate shape as they get smaller toward the edge of the bounding circle, and (2) it can display an entire pattern in a finite area.

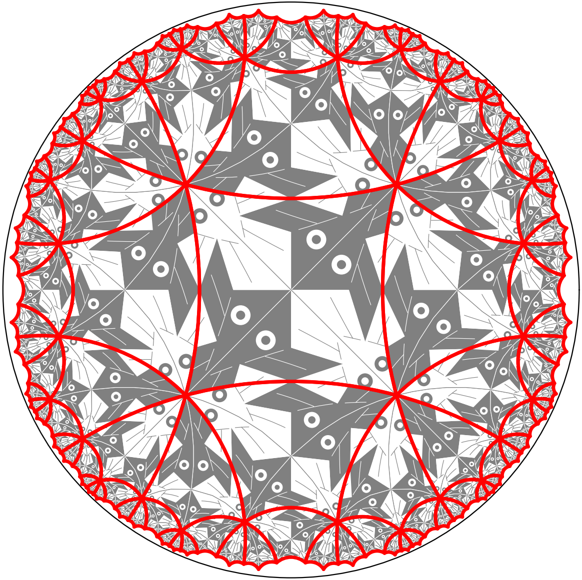
Repeating Patterns and Regular Tessellations

- A *repeating pattern* in any of the 3 "classical geometries" (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or *motif*.
- The *regular tessellation*, $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{6, 4\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

The Regular Tessellation $\{4, 6\}$



The tessellation $\{4, 6\}$ superimposed on the pattern of angular fish of the title slide pattern



Relation between periodic polyhedra and regular tessellations — a 2-Step Process

- (1) Some triply periodic polyhedra approximate triply periodic minimal surfaces (TPMS's).

  As a bonus, some triply periodic polyhedra contain embedded Euclidean lines which are also lines embedded in the corresponding TPMS. The backbones of our fish lie along those lines and form skew rhombi for regular skew polyhedra. If these skew rhombi are spanned by "soap films", one obtains the corresponding TPMS.
- (2) As a minimal surface, a TPMS has negative curvature (except for isolated points of zero curvature), and so its universal covering surface also has negative curvature and thus has the same large-scale geometry as the hyperbolic plane.

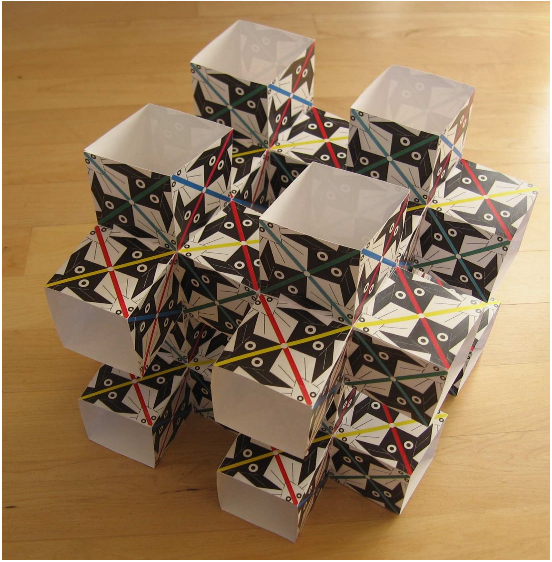
  So the polygons of the triply periodic polyhedron can be transferred to the polygons of a corresponding regular tessellation of the hyperbolic plane, and similarly a pattern on such a polyhedron can be "lifted" to a *universal covering pattern* in the hyperbolic plane.

A Fish Pattern on the $\{4, 6|4\}$ Polyhedron

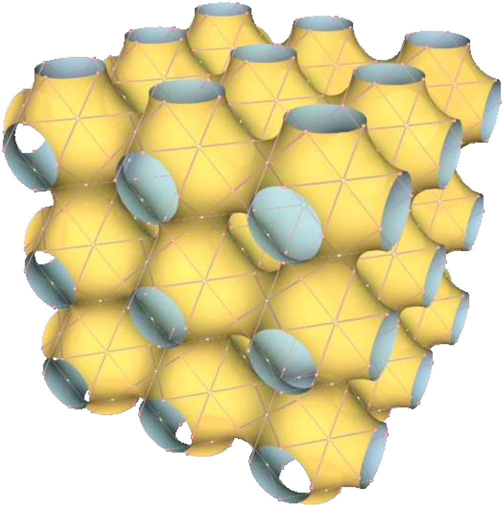
The $\{4, 6|4\}$ polyhedron is easiest to understand. It consists of invisble "hub" cubes connected by "strut" cubes on all 6 faces of the hubs. We show the 2-step relation between the patterned $\{4, 6|4\}$ polyhedron and its "universal covering pattern" as follows:

- The pattern of the Title Slide, which we have seen.
- Schwarz's P-surface, the TPMS that is approximated by the $\{4, 6|4\}$ polyhedron, showing the embedded lines that correspond to the backbone lines of the fish.
- A close-up of Schwarz's P-surface showing the skew rhombi.
- A close-up of one of the vertices of the Title Slide polyhedron.
- The hyperbolic "universal covering pattern" of the Title Slide polyhedron.

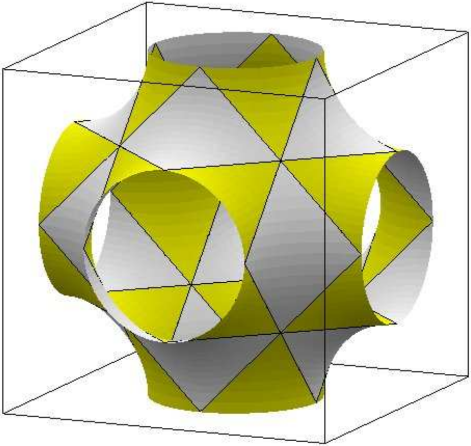
The triply periodic polyhedron of the Title Slide
— showing colored embedded lines and skew rhombi



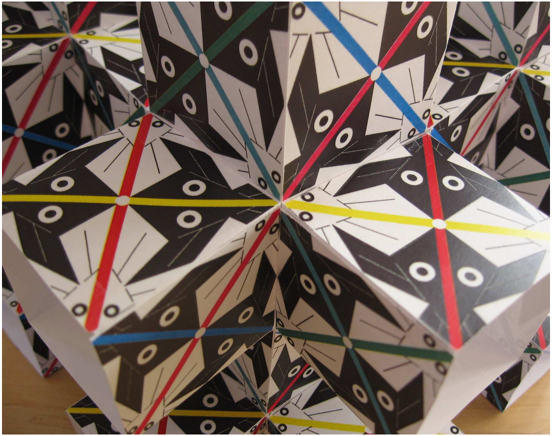
Schwarz's P-surface — approximated by the previous triply periodic polyhedron, and showing corresponding embedded lines



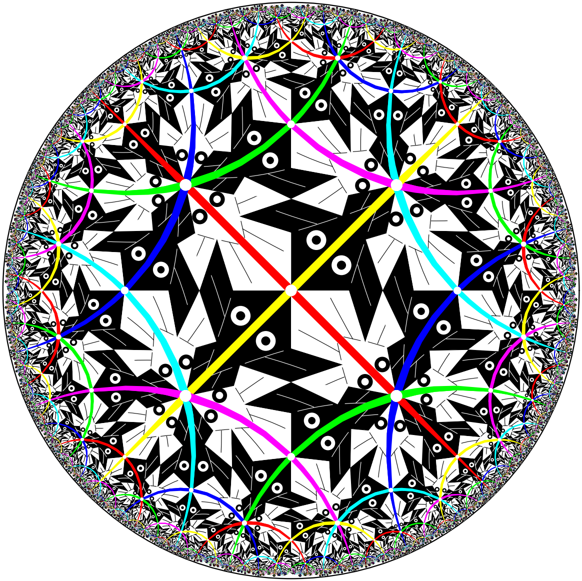
A close-up of Schwarz's P-surface showing corresponding embedded lines and "skew rhombi"



A close-up of a vertex of the Title Slide polyhedron



The pattern of the Title Slide "lifted" to its hyperbolic "universal covering pattern" — showing the embedded lines as hyperbolic lines, which bound the "skew rhombi".

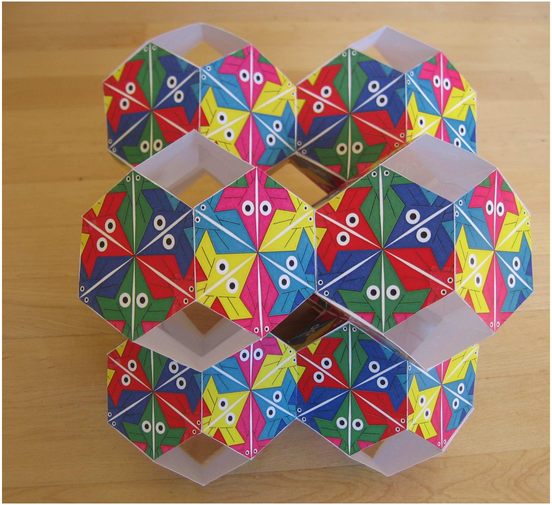


A Fish Pattern on the $\{6, 4|4\}$ Polyhedron

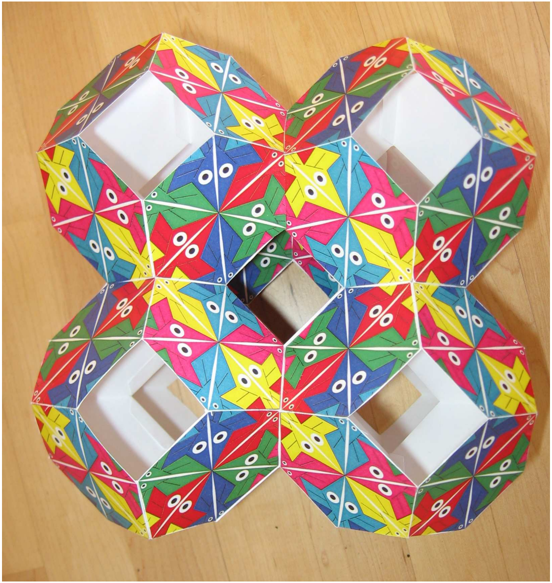
The $\{6, 4|4\}$ polyhedron is dual to the $\{4, 6|4\}$ polyhedron which we just saw. In fact the backbone lines of the fish can be taken to be the same lines in 3-space for both polyhedra. Thus they both approximate the same TPMS, Schwarz's P-surface. The $\{4, 6|4\}$ polyhedron consists of truncated octahedra in a cubic lattice arrangement and connected on their (invisible) square faces. For this polyheron we show:

- The pattern of fish on the $\{6, 4|4\}$ polyhedron.
- A top view of the patterned polyhedron that shows how fish of a single color line up along backbone lines.
- The hyperbolic "universal covering pattern" of the patterned polyhedron.

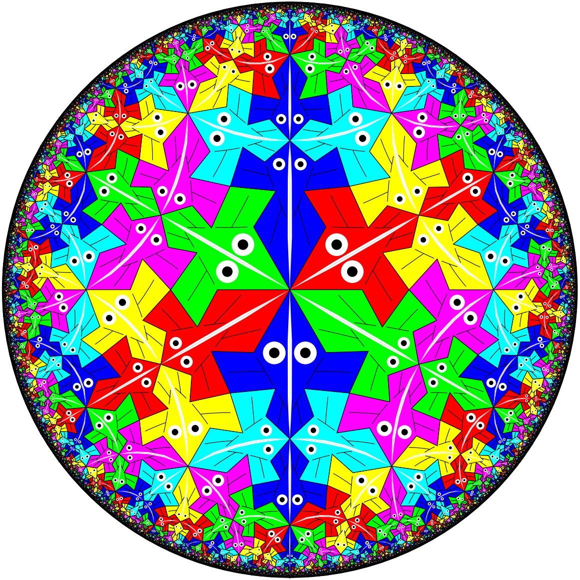
The Pattern of Fish on the $\{6,4|4\}$ Polyhedron



A top view of the fish on the $\{6,4|4\}$ polyhedron — showing fish along embedded lines



The hyperbolic universal covering pattern of fish — a version of Escher's Circle Limit I pattern with 6-color symmetry

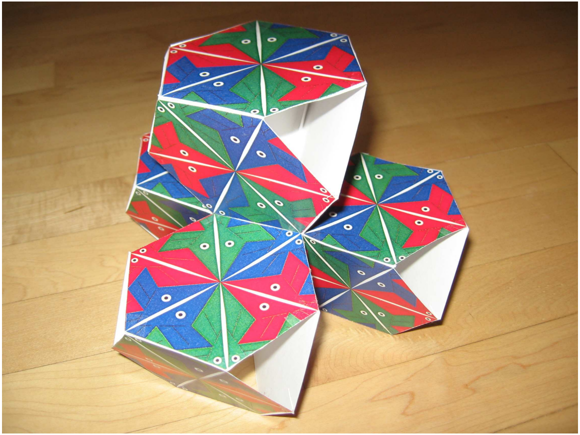


A Pattern of Fish on the $\{6, 6|3\}$ Polyhedron

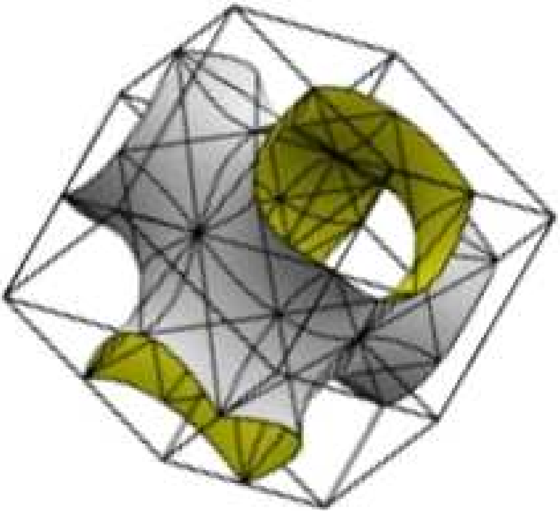
The $\{6, 6|3\}$ polyhedron is self-dual. It consists of truncated tetrahedra, four of which share (invisible) equilateral triangular faces with an invisible small regular tetrahedron. The embedded backbone lines of the fish also form skew rhombi (but different than for the $\{4, 6|4\}$ and $\{6, 4|4\}$ polyhedra). If we span these skew rhombi with "soap films", we obtain the corresponding TPMS, Schwarz's D-surface which has the topology of a thickened diamond lattice. For this polyheron we show:

- The pattern of fish on the $\{6, 6|3\}$ polyhedron.
- A "construction unit" of Schwarz's D-surface within a rhombic dodecahedron. Since rhombic dodecahedra tile space, this gives the entire D-surface.
- A top view of the patterned polyhedron that shows a vertex.
- The hyperbolic "universal covering pattern" of the patterned polyhedron.

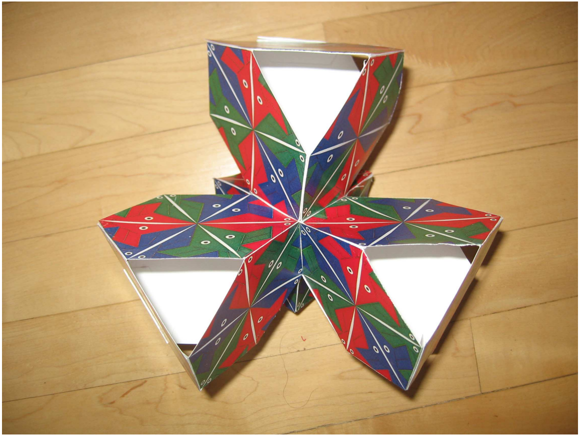
The Pattern of Fish on the $\{6,6|3\}$ Polyhedron — showing an invisible tetrahedral hub with 4 truncated tetrahedral "struts"



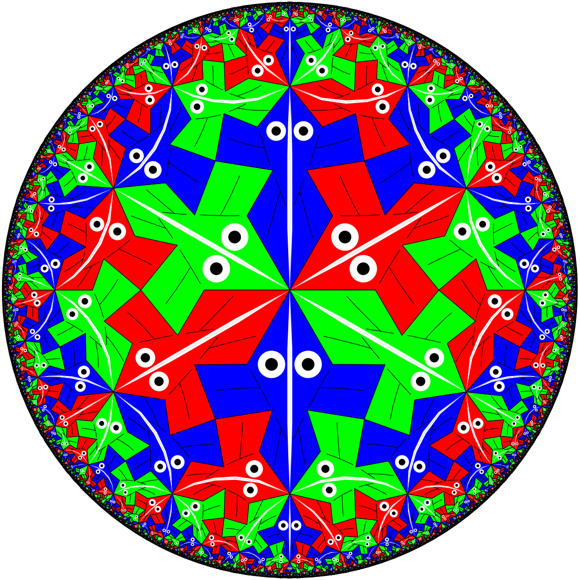
A piece of Schwarz's D-surface showing embedded lines



A top view of the fish on the $\{6, 6|3\}$ polyhedron — showing a vertex



The corresponding universal covering pattern of fish — based on the $\{6, 6\}$ tessellation

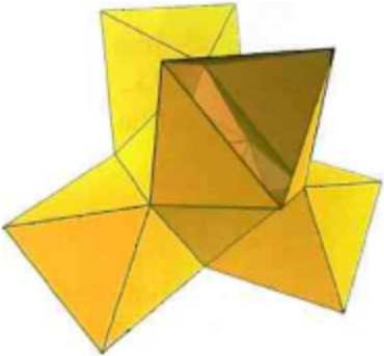


A Fish Pattern on a $\{3, 8\}$ Polyhedron

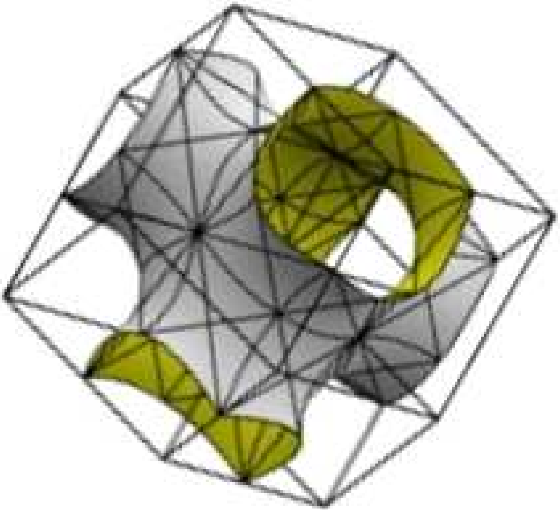
We show a fish pattern on a triply periodic $\{3, 8\}$ polyhedron, which while not regular, is *uniform*: there is a symmetry of the polyhedron that takes any vertex to any other vertex. It consists of regular octahedral "hubs" with regular octahedral "struts" on alternate faces of the hubs. The fish pattern is inspired by Escher's hyperbolic print *Circle Limit III*, which is based on the regular $\{3, 8\}$ tessellation. This polyhedron, like the $\{3, 8\}$ polyhedron is also an approximation to Schwarz's D-surface. The red, green, and yellow fish swim along the embedded lines of the D-surface (the blue fish swim in loops around the "waists"). We show:

- A piece of the triply periodic polyhedron consisting of a hub with 4 octahedral struts.
- The corresponding piece of Schwarz's D-surface within a rhombic dodecahedron.
- Escher's *Circle Limit III* pattern with the $\{3, 8\}$ equilateral triangle tessellation superimposed — the universal covering pattern.
- The patterned polyhedron.

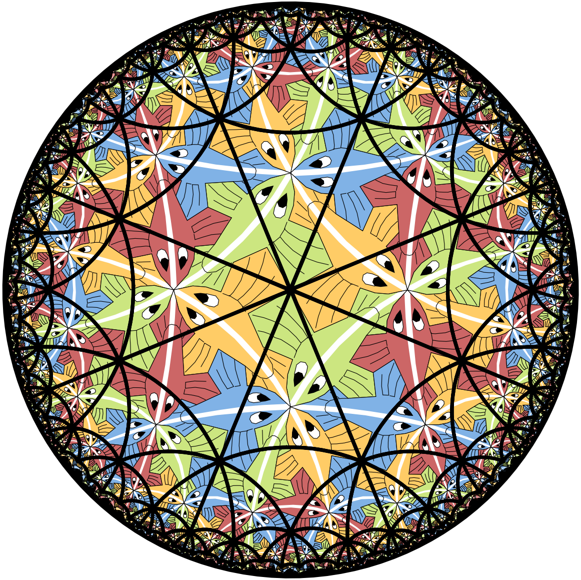
A piece of the triply periodic polyhedron showing a hub and four struts



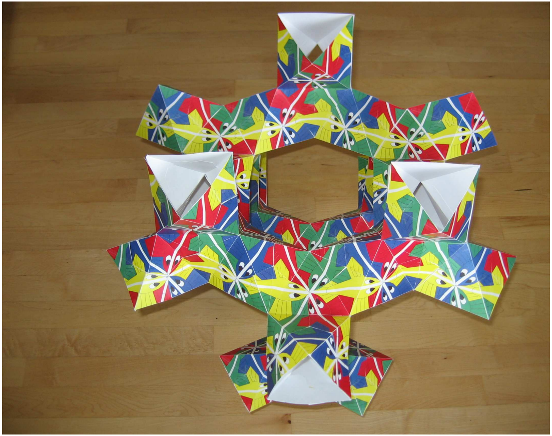
A corresponding piece of Schwarz's D-surface showing embedded lines



Escher's Circle Limit III pattern with the equilateral triangle tessellation superimposed — the universal covering pattern



The patterned polyhedron



A top view of the patterned polyhedron

Future Work

- Put other patterns on the regular skew polyhedra, exploiting their embedded lines.
- Place patterns on non-regular, uniform triply periodic polyhedra.
- Put patterns on non-uniform triply periodic polyhedra — especially those that more closely approximate triply periodic minimal surfaces.
- Draw patterns on TPMS's — the gyroid, for example.

Thank You!

Gyuri and *all* the other Symmetry Festival organizers

Contact Information:
Doug Dunham
Email: `ddunham@d.umn.edu`
Web: `http://www.d.umn.edu/~ddunham`