Use of Models of Hyperbolic Geometry in the Creation of Hyperbolic Patterns

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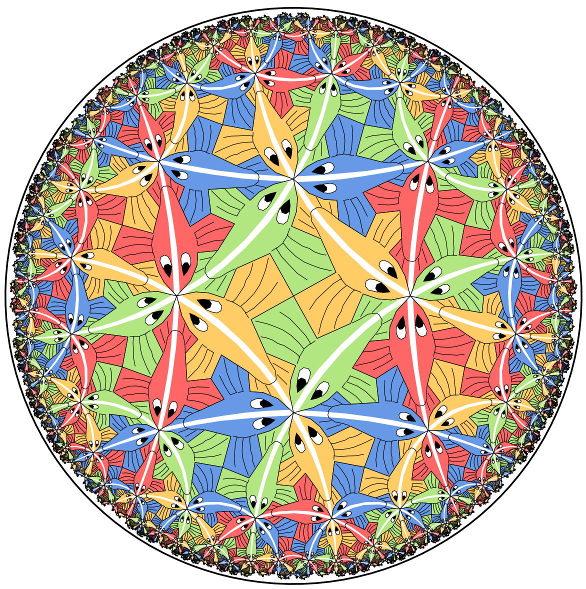
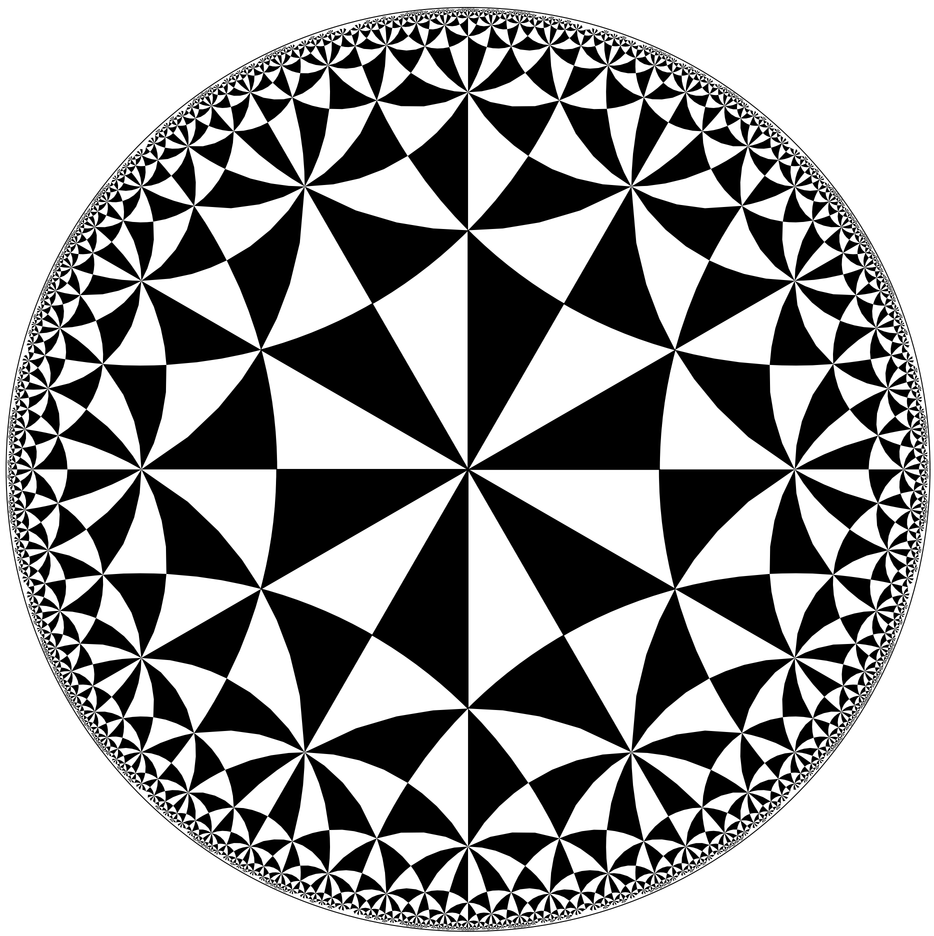


Outline

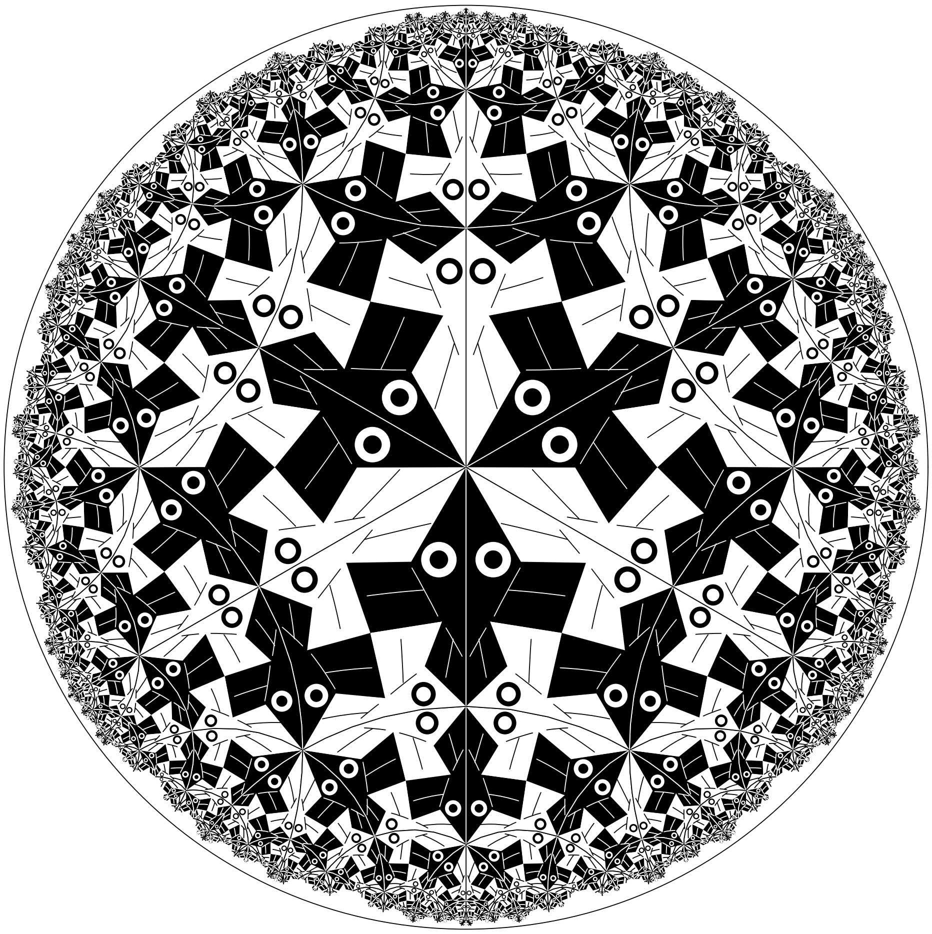
- History of hyperbolic art
- The use of the Poincaré circle model
- The use of the Weierstrass model
- The use of the Klein model
- Conclusions and future work

A Hyperbolic Triangle Pattern



- Used by mathematicians since the late 1800's.
- Used in H.S.M. Coxeter's 1957 paper "Crystal symmetry and its generalizations", in *Transactions of the Royal Society of Canada*, (3), 51 (1957), pp. 1–13.
- The inspiration for M.C. Escher's "Circle Limit" patterns.

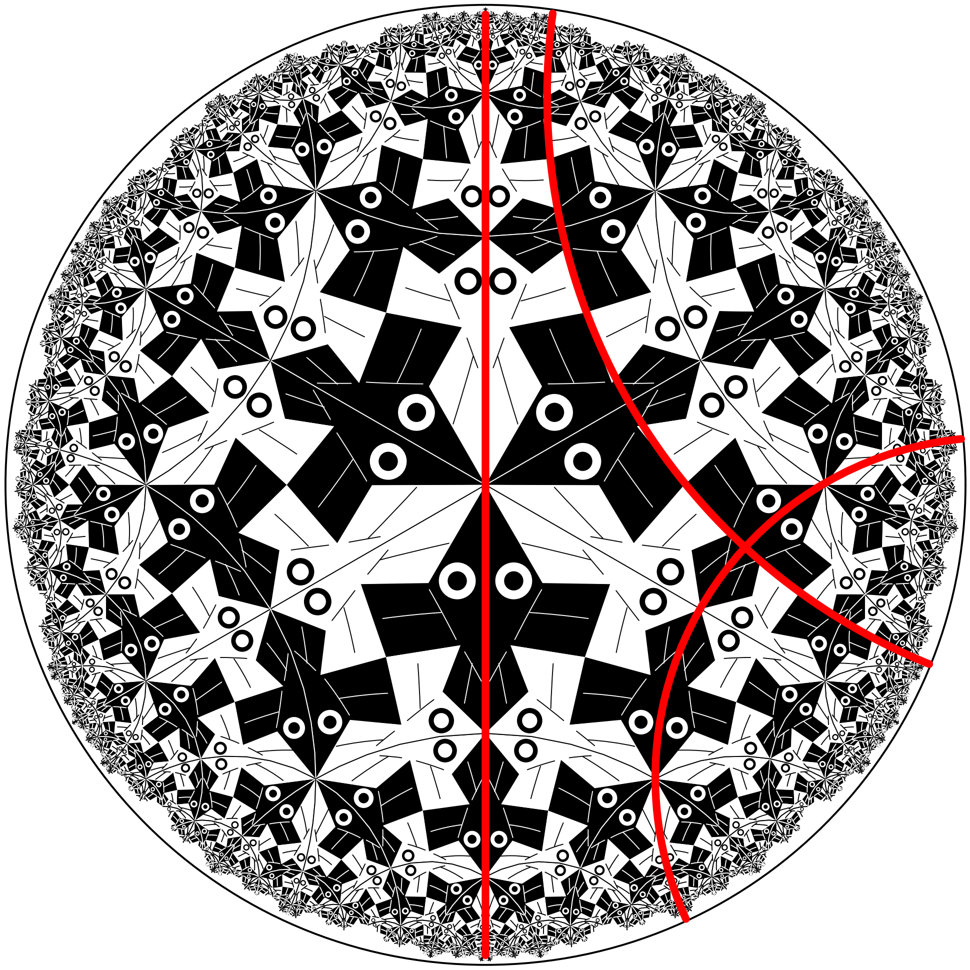
A Rendition of Escher's *Circle Limit I* Pattern



Escher: Shortcomings of *Circle Limit I:*

"There is no continuity, no 'traffic flow', no unity of colour in each row ..."

The Poincaré Circle Model



Points: points within the **bounding circle,** which is usually taken to be the unit circle in the xy-plane.

Lines: circular arcs perpendicular to the bounding circle (including diameters as special cases).

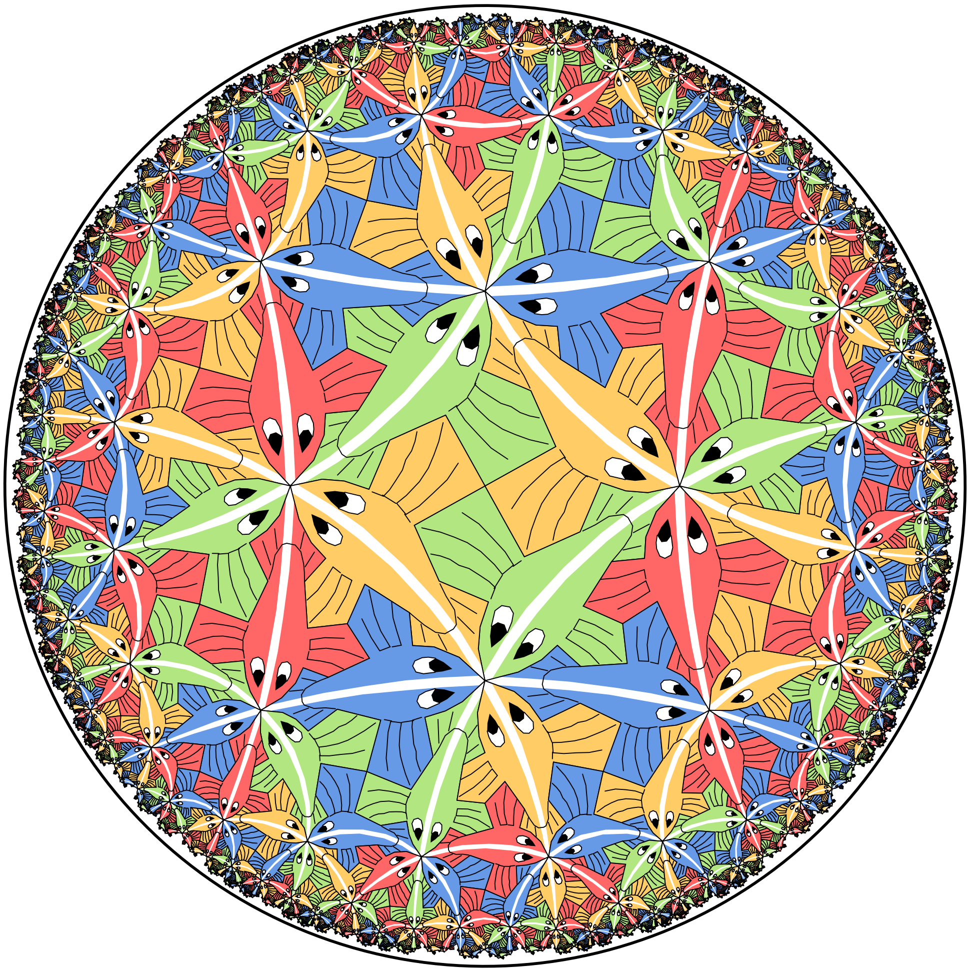
Note that it satisfies the hyperbolic parallel axiom: Given a line and a point not on that line, there is more than one line through the point that does not intersect the given line.

Artists and the Poincaré Circle Model

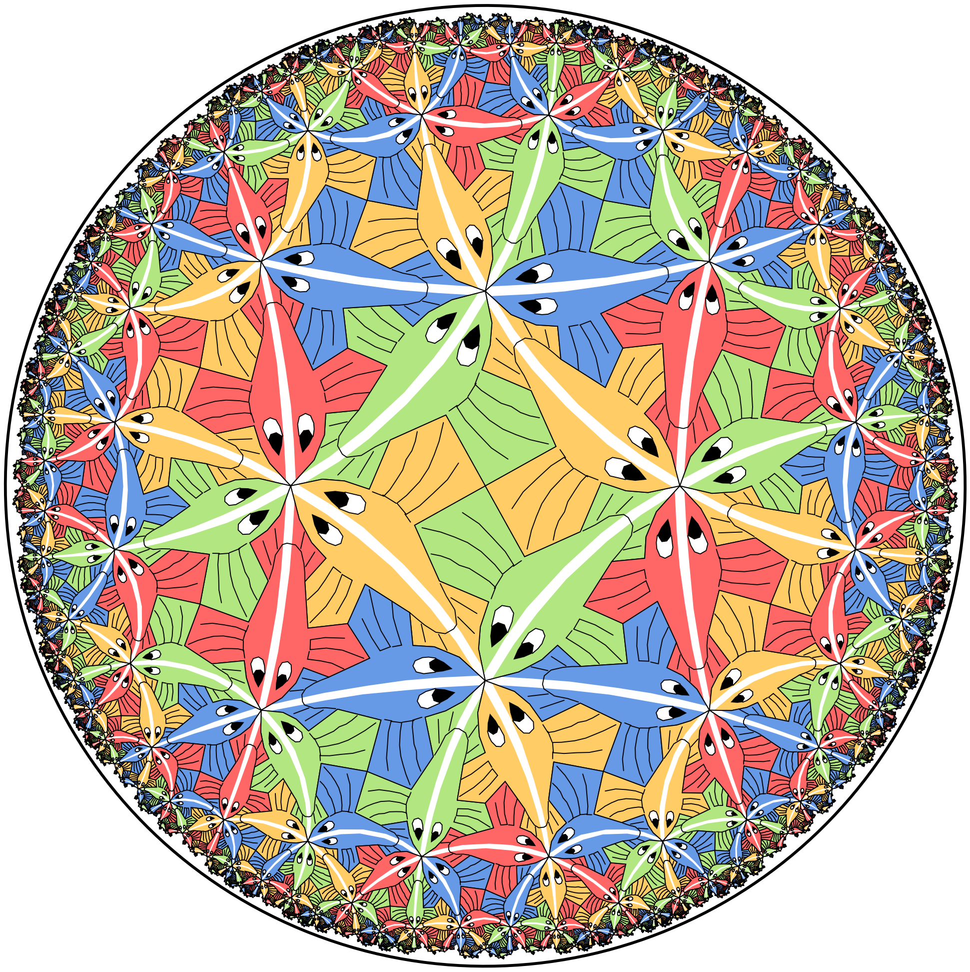
Artists (and others) preferred the Poincaré Circle Model for two reasons:

1. It could be used to display an entire pattern in a finite region (unlike the Poincaré half-plane model).
2. It is **conformal** — angles have their Euclidean measure (unlike the Klein model) — which means that small copies of the motif near the bounding circle retain the approximate shape of the original, and thus remain recognizable.

A Computer Rendition of *Circle Limit III*



Equidistant Curves and *Circle Limit III*



- **Equidistant Curves:** circular arcs *not* perpendicular to the bounding circle (including chords as special cases).

  For each hyperbolic line and a given hyperbolic distance, there are two equidistant curves, one on each side of the line, all of whose points are that distance from the given line.

Repeating Patterns

A **repeating pattern** is a pattern made up of congruent copies of a basic subpattern or **motif**.

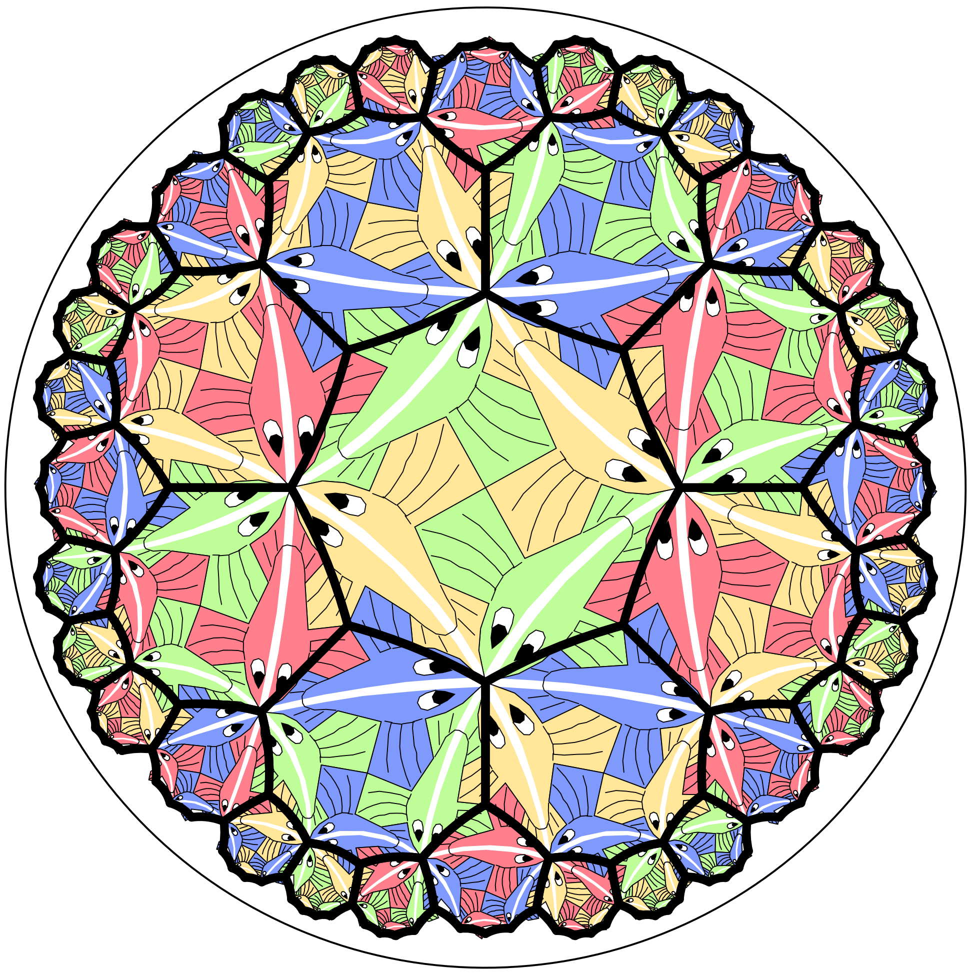
If we disregard color, the motif is one of the triangles in the triangle tessellation that Coxeter used.

In *Circle Limit I*, the motif can be taken to be half of a white fish together with half of an adjacent black fish.

In *Circle Limit III*, the motif can be taken to be one of the fish — disregarding color.

The Regular Tessellations {p,q}

Important repeating patterns are the ***regular tessellations,*** $\{\boldsymbol{p},\boldsymbol{q}\}$ of the hyperbolic plane by regular p-sided polygons meeting q at a vertex provided $(p-2)(q-2) > 4$.



The tessellation $\{8,3\}$ superimposed on the *Circle Limit III* pattern.

All four of Escher's "Circle Limit" patterns and many of his Euclidean and spherical patterns are based on regular tessellations.

Symmetries

A **symmetry** of a pattern is an isometry (distance preserving transformation) that takes the pattern onto itself.

Reflections across the sides of the triangles of the triangle tessellation are symmetries of that pattern. In the Poincaré model, reflections across the circular arcs representing hyperbolic lines are inversions in those arcs. In the hyperbolic plane, any isometry can be built from a sequence of at most three reflections.

In the *Circle Limit I* pattern, there are reflection symmetries across the backbone lines of the fish.

If we disregard color in the *Circle Limit III* pattern, there are rotation symmetries about the fin tips and about the meeting points of the noses. In hyperbolic geometry, as in Euclidean geometry, successive reflections across intersection lines produces a rotation about the intersection point by twice the angle of intersection.

Color Symmetry

We say a pattern has **n-color symmetry** if it is colored by n colors and every symmetry of the uncolored pattern permutes the colors. Usually we also require that the set of symmetries act transitively on the colors, so that they all get exchanged. This is called *perfect color symmetry*.

The triangle tessellation has 2-color symmetry.

The *Circle Limit I* pattern does not have color symmetry since the black fish are not congruent to the white fish.

The *Circle Limit III* pattern has 4-color symmetry.

Weierstrass Model of Hyperbolic Geometry

- **Points:** points on the upper sheet of a hyperboloid of two sheets: $x^2 + y^2 - z^2 = -1$, $z \geq 1$.

- **Lines:** the intersections of a Euclidean plane through the origin with this upper sheet (and so each one is one branch of a hyperbola).

  A line can be represented by its **pole**, a 3-vector $\begin{bmatrix} \ell_x \\ \ell_y \\ \ell_z \end{bmatrix}$ on the dual hyperboloid $\ell_x^2 + \ell_y^2 - \ell_z^2 = +1$, so that the line is the set of points satisfying $x\ell_x + y\ell_y - z\ell_z = 0$.

The Relation Between the Models

Stereographic projection from the Weierstrass model onto the Poincaré disk in the xy-plane toward the point $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$,

Given by the formula: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x/(1+z) \\ y/(1+z) \\ 0 \end{bmatrix}$.

A Special Reflection Matrix

The pole corresponding to the hyperbolic line perpendicular to the x-axis and through the point $\begin{bmatrix} \sinh d \\ 0 \\ \cosh d \end{bmatrix}$ is given by $\begin{bmatrix} \cosh d \\ 0 \\ \sinh d \end{bmatrix}$.

The matrix *Ref* representing reflection of Weierstrass points across that line is given by:

$$Ref = \begin{bmatrix} -\cosh 2d & 0 & \sinh 2d \\ 0 & 1 & 0 \\ -\sinh 2d & 0 & \cosh 2d \end{bmatrix}$$

where d is the the hyperbolic distance from the line (or point) to the origin.

The General Reflection Matrix

In general, reflection across a line whose nearest point to the origin is rotated by angle θ from the x-axis is given by:

$$Rot(\theta) Ref\, Rot(-\theta)$$

where, as usual,

$$Rot(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

.

The Pattern Generation Process

The main idea of the pattern generation process is to transform the motif about the hyperbolic plane by symmetries of the desired pattern. In practice we only apply a finite number of symmetries — enough to give the idea of what the complete (infinite) pattern would look like.

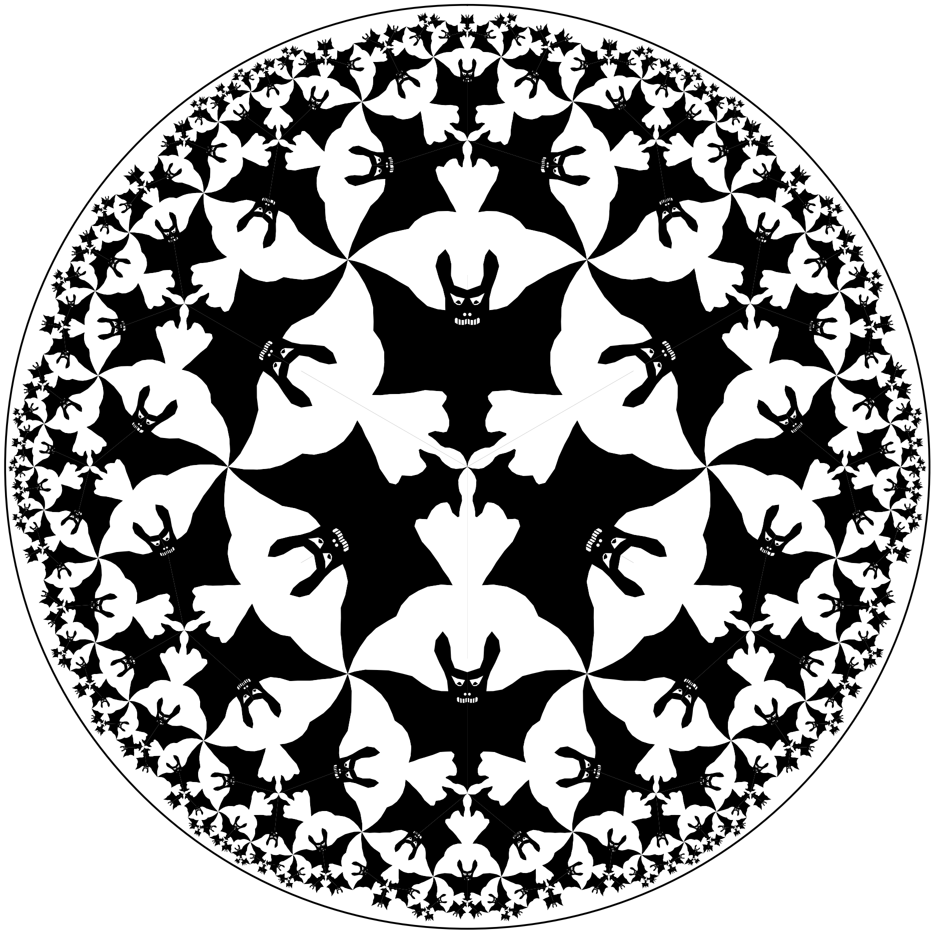
The individual transformations take place as follows:

1. First project all the points of the motif up from the Poincaré model onto the Weierstrass model, obtaining 3-vectors.

2. Then multiply each of the 3-vectors by the 3x3 matrix representing the transformation, obtaining transformed 3-vectors.

3. Finally project the transformed 3-vectors back to the Poincaré model and draw the transformed motif.

A Sample Pattern



A computer rendition of Escher's *Circle Limit IV* pattern.

The Klein Model

As in the Poincaré model, the **points** of the Klein model are the interior points of the unit circle.

The **lines** of the Klein model are represented by chords of the circle.

A chord in the Klein model corresponds to the circular arc in in the Poincaré model with the same endpoints. The chord and the arc represent the same hyperbolic line in their respective models.

The Relation Between the Poincaré and Klein Models

The transformation from the Poincaré model to the Klein model is given by:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x/(1+z) \\ 2y/(1+z) \end{bmatrix}$$

.

The inverse transformation is given by:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x/(1+\sqrt{1-x^2-y^2}) \\ y/(1+\sqrt{1-x^2-y^2}) \end{bmatrix}$$

.

Approximating Poincaré Line Segments

We can use the Klein model to find a sequence of short Euclidean line segments that approximates the circular arc representing a hyperbolic line segment in the Poincaré model as follows:

1. First map the endpoints of the circular arc from the Poincaré to the Klein model. The new points represent the endpoints of the same hyperbolic line segment in the Klein model.
2. Divide the line segment in the Klein model into many short segments (which is easy to do).
3. Finally map all the endpoints of the short segments back to the Poincaré model and connect them by Euclidean line segments, and thus obtain a good approximation to the circular arc.

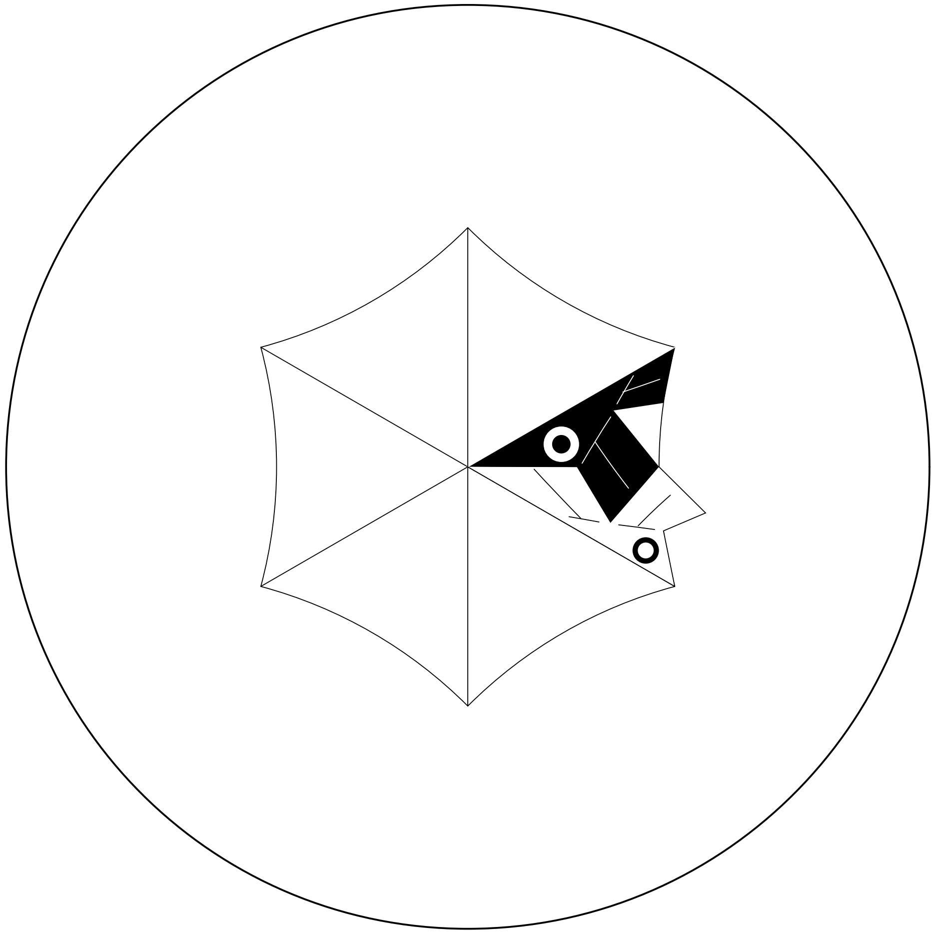
Transforming Motifs Between Tessellations

We can use the same technique to transform a pattern based on a $\{p, q\}$ tessellation to one based on a $\{p', q'\}$ tessellation as follows. We assume that the motif for the $\{p, q\}$ pattern is contained within the central p-gon in the bounding circle. The central p-gon is divided into p isoceles triangles with angles $2\pi/p$, π/q, and π/q. We further assume that the motif is contained in the isoceles triangle to the right of the origin and symmetric about the x-axis.

Then one can transform the original motif to one based on a $\{p', q'\}$ tessellation as follows:

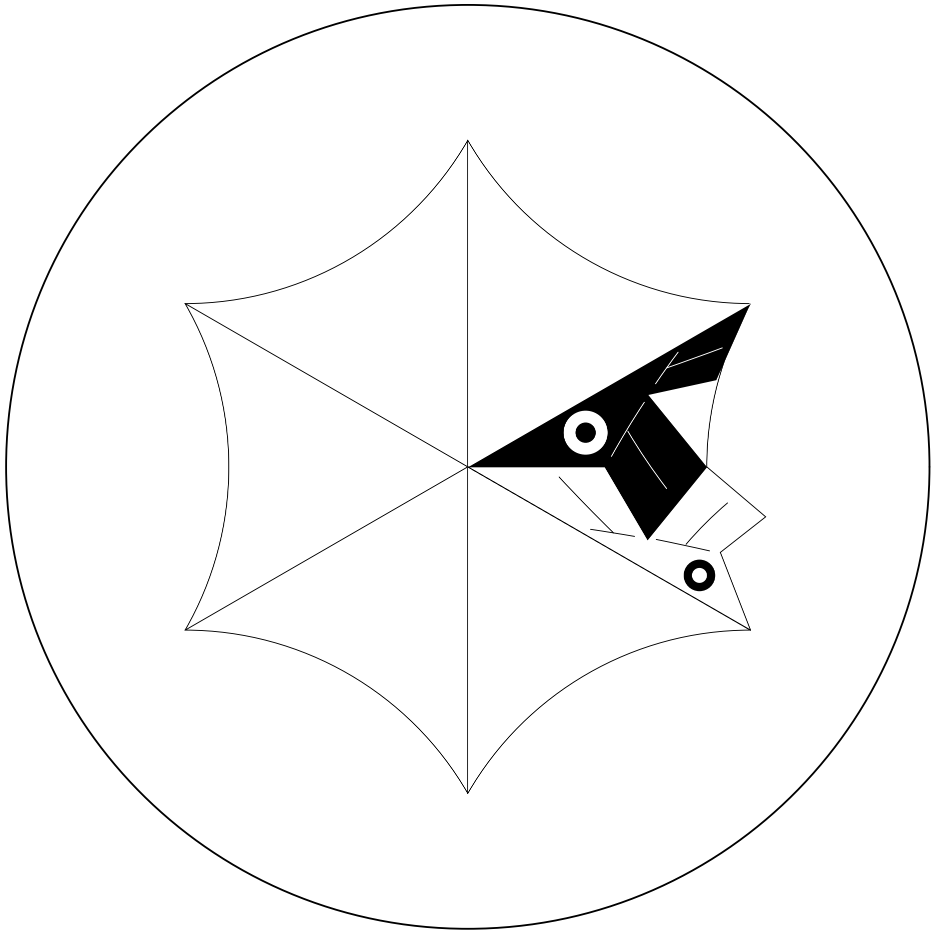
1. First map all the points of the motif from the Poincaré to the Klein model, obtaining a motif within a Euclidean isoceles triangle.
2. Then apply a (differential) scaling to that isoceles triangle to obtain the Euclidean isoceles triangle corresponding to the $\{p', q'\}$ tessellation in the Klein model.
3. Finally map all the points of the motif back to the Poincaré model, obtaining the motif for the $\{p', q'\}$ pattern.

A Motif for a $\{6,4\}$ Pattern



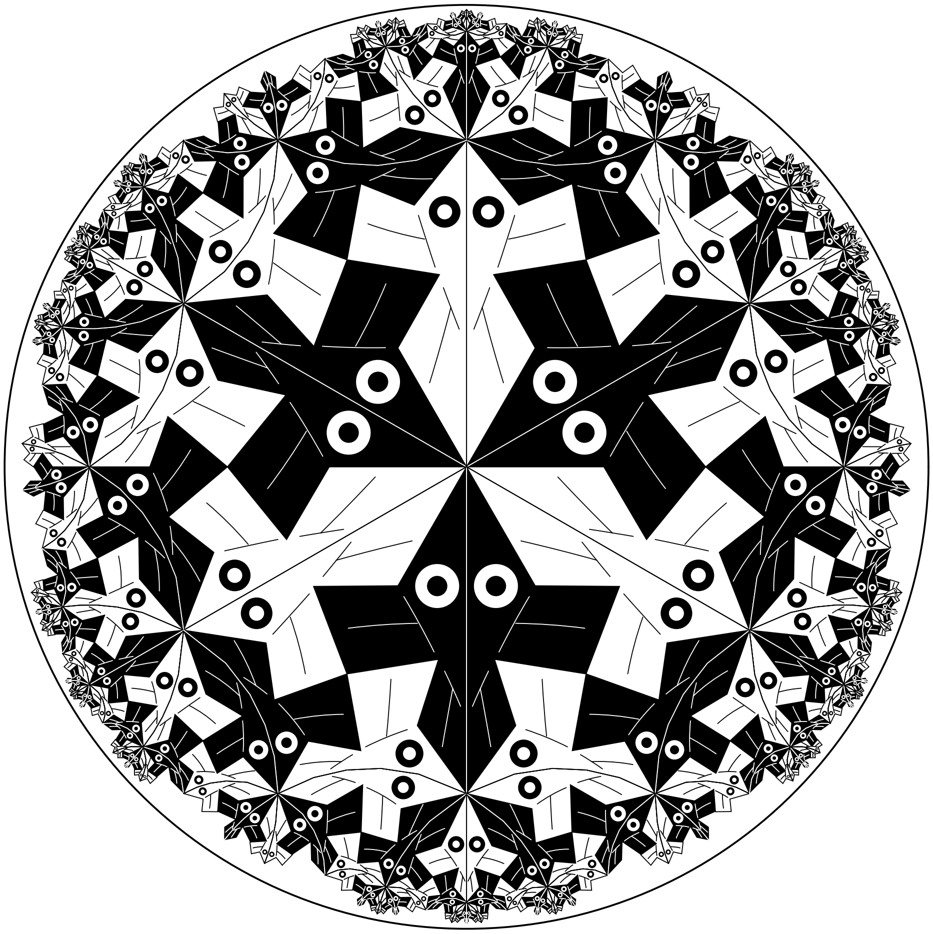
A motif for the *Circle Limit I* pattern within the isoceles triangle corresponding to the $\{6, 4\}$ tessellation.

The Transformed Motif for a $\{6,6\}$ Pattern



The motif for a pattern based on the $\{6, 6\}$ tessellation.

The Resulting $\{6,6\}$ Pattern



Note that this pattern has 2-color symmetry — the black and white fish are congruent.

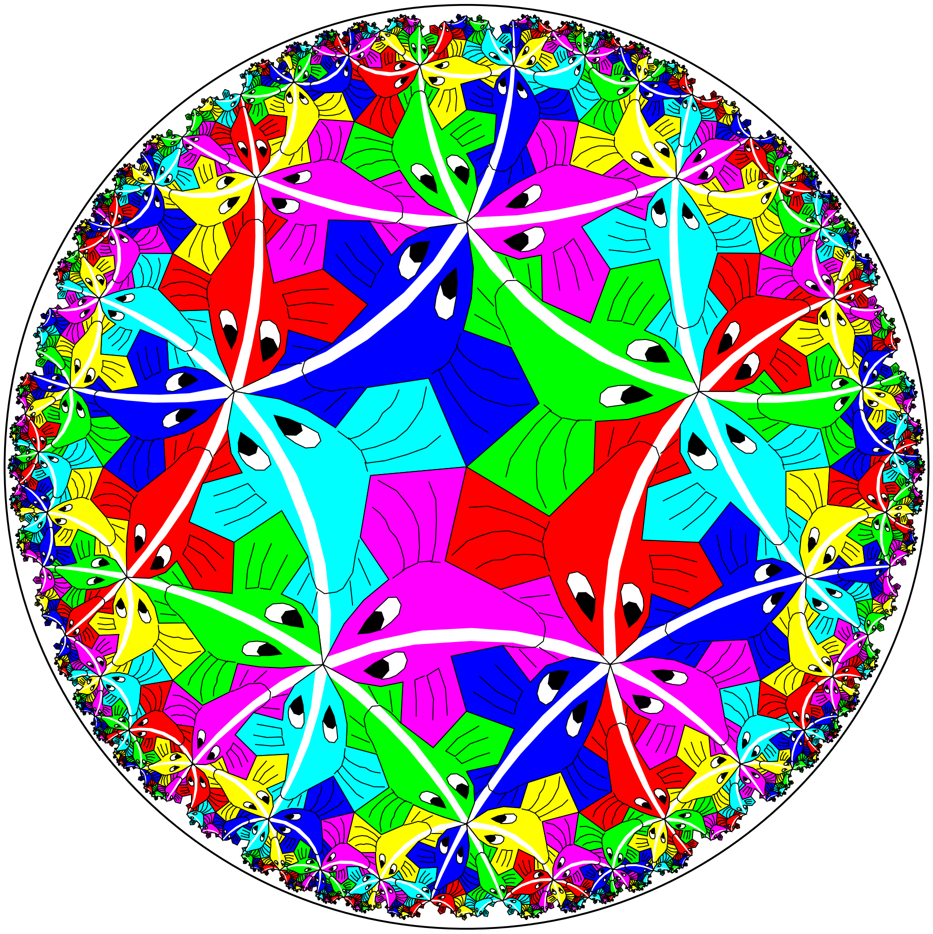
Another Example: *Circle Limit II*

A rendition of Escher's *Circle Limit II* pattern, which is based on the $\{8,3\}$ tessellation.

A Transformed Pattern Based on {10,3}

A 5-arm cross pattern related to *Circle Limit II*, based on the {10,3} tessellation.

Another Transformed Pattern



Another *Circle Limit III* fish pattern based on the {10,3} tessellation.

Conclusions

We have shown how different models of hyperbolic geometry can be used in the process of creating repeating patterns of the hyperbolic plane — specifically:

- The Poincaré model is probably best suited for the display of such patterns, since (1) it displays the patterns in a finite area, and (2) it is conformal, so that the motifs retain the same approximate shape even as they become small.
- The Weierstrass model is useful since its points can be represented as 3-vectors and transformations can be represented as 3x3 matrices.
- The Klein model is useful for transforming motifs based on one tessellation to those based on another tessellation. It is also useful for approximating hyperbolic lines in the Poincaré model.

Future Work

- A model of hyperbolic geometry that is intermediate between the Poincaré model and the Klein model may be useful in transforming one *Circle Limit III* motif to another (based on different combinatorial indices — the numbers of fish meeting at fins, noses and tails).
- Extend the pattern-generation framework to include cases in which p or q is infinity. This may require using a different model of hyperbolic geometry so that transformations are represented in a uniform way.
- Find an algorithm to determine the minimum number of colors needed for a perfect coloring of a pattern based on the $\{p, q\}$ tessellation and the motif, while adhering to the map-coloring principle that adjacent motifs should not be the same color.