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Enumerations of Hyperbolic Truchet Tiles  

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Sébastian Truchet
Brief History of Truchet Tilings

- Sébastien Truchet was born in Lyon, France 1657, died 1729.
- Interests: mathematics, hydraulics, graphics, and typography.
- Also invented sundials, weapons, and methods for transporting large trees within the Versailles gardens.
- In 1704 he published “Memoir sur les Combinaisons” in *Memoires de l’Académie Royale des Sciences* enumerating possible pairs of juxtaposed squares divided by a diagonal into a black and a white triangle. The “Memoir” contained 7 plates, the first four showed 24 simple pattern, labeled A to Z and & (no J, K, W); the last three showed six more complicated patterns.
- In 1942 M.C. Escher enumerated $2 \times 2$ tiles of squares containing simple motifs, thus extending Truchet’s idea for $2 \times 1$ tiles.
- In 1987 Truchet’s “Memoir” was translated in English by Pauline Bouchard with comments and “circular arc” tiles by Cyril Smith in *Leonardo*, igniting renewed interest in these tilings.
Examples of Truchet Tilings

- Truchet triangle tilings
- Based on a square divided in two into a black and white triangle — 4 orientations.
- Either repeating patterns or random patterns.
Regular Truchet Tilings
A Random Truchet Tiling
Hyperbolic Geometry and Regular Tessellations

- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.

- Thus we must use models of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.

- One such model is the Poincaré disk model. The hyperbolic points in this model are represented by interior point of a Euclidean circle — the bounding circle. The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (with diameters as special cases).

- This model is appealing to artists since (1) angles have their Euclidean measure (i.e. it is conformal), so that motifs of a repeating pattern retain their approximate shape as they get smaller toward the edge of the bounding circle, and (2) it can display an entire pattern in a finite area.
Repeating Patterns and Regular Tessellations

- A *repeating pattern* in any of the 3 “classical geometries” (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or *motif*.

- The *regular tessellation*, \( \{p, q\} \), is an important kind of repeating pattern composed of regular \( p \)-sided polygons meeting \( q \) at a vertex.

- If \((p - 2)(q - 2) < 4\), \( \{p, q\} \) is a spherical tessellation (assuming \( p > 2 \) and \( q > 2 \) to avoid special cases).

- If \((p - 2)(q - 2) = 4\), \( \{p, q\} \) is a Euclidean tessellation.

- If \((p - 2)(q - 2) > 4\), \( \{p, q\} \) is a hyperbolic tessellation. The next slide shows the \( \{6, 4\} \) tessellation.

- Escher based his 4 “Circle Limit” patterns, and many of his spherical and Euclidean patterns on regular tessellations.
The Regular Tessellation \( \{4, 6\} \)
Underlying the Title Slide Image
The tessellation \( \{4, 6\} \) superimposed on the title slide pattern
Truchet’s “translation” tiling A.
Truchet’s “rotation” tiling D.
A hyperbolic “translation” Truchet tiling based on the \{4, 8\} tessellation.
A hyperbolic “rotation” Truchet tiling based on the \{4, 8\} tessellation.
A Non-Regular Hyperbolic Truchet Tiling
(based on the \(\{4, 5\}\) tessellation)
Random Hyperbolic Truchet Tilings
(One based on the \( \{4, 6\} \) tessellation)
Another Random Hyperbolic Truchet Tiling
(based on the \{4, 5\} tessellation)
Truchet’s pattern $F$, which does not adhere to the map-coloring principle
A hyperbolic Truchet pattern corresponding to Truchet’s pattern F (based on the \( \{4, 6\} \) tessellation)
Truchet Tiles with Multiple Triangles per $p$-gon

- Truchet considered $2 \times 1$ rectangles composed of two squares, which easily tile the Euclidean plane.

- Problem: it is more difficult to tile the hyperbolic plane by “rectangles” — quadrilaterals with congruent opposite sides.

- Solution: the $p$-gons of $\{p, q\}$ tile the hyperbolic plane.

- We divide the $p$-gons of a $\{p, q\}$ divided into black and white $\frac{\pi}{p} - \frac{\pi}{q} - \frac{\pi}{2}$ basic triangles by radii and apothems.

- To satisfy the map-coloring principle, the basic triangles should alternate black and white, giving only two possible tilings.

- If we don’t require map-coloring, there are $N_2(2p)$ possible ways to fill a $p$-gon with black and white basic triangles, where $N_k(n)$ is the number of $n$-bead necklaces using beads of $k$ colors:

$$N_k(n) = \frac{1}{n} \sum_{d|n} \varphi(d) k^{n/k}$$

where $\varphi(d)$ is Euler’s totient function.
If we consider our “necklaces” to be equivalent by reflection across a diameter or apothem of the $p$-gon, there are fewer possibilities, given by $B_k(n)$ the number of $n$-bead “bracelets” made with $k$ colors of beads. The value of $B_k(n)$ is $1/2$ that of $N_k(n)$ with added adjustment terms that depend on the parity of $n$.

It seems to be a difficult problem to enumerate all the ways such a $p$-gon pattern of triangles could be extended across each of its edges, though an upper bound would be $(2p)^p N_2(2p)$. 
A pattern generated by alternate black and white triangles in a 4-gon, a $p$-gon analog of Truchet’s pattern A.
A pattern generated by paired black and white triangles in a 4-gon, analogous to Truchet’s pattern E.
Another pattern generated by paired black and white triangles in a 4-gon, analogous to Truchet’s pattern F.
A pattern based on the \( \{6, 4\} \) tessellation, similar to Truchet’s pattern E.
A Truchet-like pattern based on the \{5, 4\} tessellation.
Truchet Tilings with other Motifs — Circular Arcs

- Based on a square with circular arcs connecting adjacent sides — 2 orientations.
- Either repeating patterns or random patterns.
A Random Truchet Arc Tiling
(based on the Euclidean \(\{4, 4\}\) tessellation by squares)
A Hyperbolic Arc Tile (based on the \{4,6\} tessellation)
A Hyperbolic Arc Pattern (based on the \(\{4, 6\}\) tessellation)
A Hyperbolic Arc Pattern of Circles (based on the \{4, 5\} tessellation)
We generalize Truchet arc patterns from Euclidean squares to $p$-gons by connecting the midpoints of the edges of a $2n$-gon ($p = 2n$ must be even).

The number of possible $2n$-gon tiles is the same as the number of ways to connect $2n$ points on a circle with non-intersecting chords. It is the Catalan number:

$$C(n) = \frac{2n!}{n!(n+1)!}$$

As is the case with the triangle-decorated $p$-gons, the number of possible patterns is bounded above by $(2n)^{2n} C(n)$, but again, it seems difficult to get an exact count.
Truchet Tilings with other Motifs — “Wasps”

Four wasps at the corners of a square — wasp motif designed by Pierre Simon Fournier (mid 1700’s)
A Truchet Pattern of Wasps
(based on the \{4, 5\} tessellation)
Future Work

- Investigate colored hyperbolic Truchet triangle patterns.
- Implement a hyperbolic circular arc tool in the program.
- Investigate more hyperbolic Truchet arc patterns with more arcs per $p$-gon.
Thank You!

Nat, ISAMA, the organizers at Columbia College