ISAMA 2012 Chicago IL

Patterns on Triply Periodic Uniform Polyhedra

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Previously Designed Patterned Polyhedra

- M.C. Escher (1898–1972) created at least 3 such polyhedra.
- In 1977 Doris Schattschneider and Wallace Walker placed Escher patterns on each of the Platonic solids and the cuboctahedron.
- Schattschneider and Walker also put Escher patterns on rotating rings of tetrahedra, which they called “kaleidocycles”.
- In 1985 H.S.M. Coxeter showed how to place 18 Escher butterflies on a torus.
Triply Periodic Polyhedra

A *triply periodic polyhedron* is a (non-closed) polyhedron that repeats in three different directions in Euclidean 3-space.

We will consider the special case of *uniform* triply periodic polyhedra which have the same vertex figure at each vertex — i.e. there is a symmetry of the polyhedron that takes any vertex to any other vertex.

We will mostly discuss a specialization of uniform triply periodic polyhedra: *regular* triply periodic polyhedra which are “flag-transitive” — there is a symmetry of the polyhedron that takes any vertex, edge containing that vertex, and face containing that edge to any other such (vertex, edge, face) combination.

In 1926 John Petrie and H.S.M. Coxeter proved that there are exactly three regular triply periodic polyhedra, which Coxeter denoted \(\{4,6\mid 4\}\), \(\{6,4\mid 4\}\), and \(\{6,6\mid 3\}\), where \(\{p,q\mid r\}\) denotes a polyhedron made up of \(p\)-sided regular polygons meeting \(q\) at a vertex, and with regular \(r\)-sided holes.
Inspirations for this Work

- Two papers by Steve Luecking at ISAMA 2011:
  - Building a Sherk Surface from Paper Tiles
  - Sculpture From a Space Filling Saddle Pentahedron
- Bead sculptures that approximate three triply periodic minimal surfaces (TPMS) by Chern Chuang, Bih-Yaw Jin, and Wei-Chi Wei at the 2012 Joint Mathematics Meeting Art Exhibit.

As we will see, some TPMS's are related to triply periodic polyhedra.
Hyperbolic Geometry and Regular Tessellations

- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.

- Thus we must use models of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.

- One such model is the Poincaré disk model. The hyperbolic points in this model are represented by interior point of a Euclidean circle — the bounding circle. The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (with diameters as special cases).

- This model is appealing to artists since (1) angles have their Euclidean measure (i.e. it is conformal), so that motifs of a repeating pattern retain their approximate shape as they get smaller toward the edge of the bounding circle, and (2) it can display an entire pattern in a finite area.
A *repeating pattern* in any of the 3 “classical geometries” (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or *motif*.

The *regular tessellation*, \( \{p, q\} \), is an important kind of repeating pattern composed of regular \( p \)-sided polygons meeting \( q \) at a vertex.

If \((p - 2)(q - 2) < 4\), \( \{p, q\} \) is a spherical tessellation (assuming \( p > 2 \) and \( q > 2 \) to avoid special cases).

If \((p - 2)(q - 2) = 4\), \( \{p, q\} \) is a Euclidean tessellation.

If \((p - 2)(q - 2) > 4\), \( \{p, q\} \) is a hyperbolic tessellation. The next slide shows the \( \{6, 4\} \) tessellation.

Escher based his 4 “Circle Limit” patterns, and many of his spherical and Euclidean patterns on regular tessellations.
The Regular Tessellation \{4, 6\}
The tessellation \{4, 6\} superimposed on the pattern of angular fish of the title slide pattern.
Relation between periodic polyhedra and regular tessellations — a 2-Step Process

- (1) Some triply periodic polyhedra approximate TPMS’s. As a bonus, some triply periodic polyhedra contain embedded Euclidean lines which are also lines embedded in the corresponding TPMS.

- (2) As a minimal surface, a TPMS has negative curvature (except for isolated points of zero curvature), and so its universal covering surface also has negative curvature and thus has the same large-scale geometry as the hyperbolic plane. So the polygons of the triply periodic polyhedron can be transferred to the polygons of a corresponding regular tessellation of the hyperbolic plane.

- We show this relationship in the next slides.
The triply periodic polyhedron of the Title Slide — showing colored embedded lines
Schwarz’s P-surface — approximated by the previous triply periodic polyhedron, and showing corresponding embedded lines
The pattern of the Title Slide “unfolded” onto a repeating pattern of the hyperbolic plane — showing the embedded lines as hyperbolic lines.
Patterns on the \(\{4, 6|4\}\) Polyhedron

We show two patterns on the \(\{4, 6|4\}\) polyhedron:

- The pattern of the Title Slide, which we have seen. Here we show a close-up of one of the vertices.
- A pattern of angels and devils, inspired by Escher. We show both the patterned polyhedron and the corresponding pattern in the hyperbolic plane.
A close-up of a vertex of the Title Slide polyhedron
Angels and Devils on the \( \{4, 6\mid 4\} \) polyhedron
The corresponding Angels and Devils pattern in the hyperbolic plane
A Pattern of Fish on the \( \{6, 4\mid 4\} \) Polyhedron
A top view of the fish on the \( \{6, 4|4\} \) polyhedron — showing fish along embedded lines
The corresponding hyperbolic pattern of fish — a version of Escher’s Circle Limit I pattern with 6-color symmetry
A Pattern of Fish on the \{6, 6|3\} Polyhedron
A top view of the fish on the $\{6, 6|3\}$ polyhedron — showing a vertex
The corresponding hyperbolic pattern of fish — based on the \( \{6,6\} \) tessellation
Patterns of Fish on a \(\{3, 8\}\) Polyhedron

Using a uniform triply periodic \(\{3, 8\}\) polyhedron, we show a pattern of fish inspired by Escher’s hyperbolic print *Circle Limit III*, which is based on the regular \(\{3, 8\}\) tessellation. This polyhedron is related to Schwarz's D-surface, a TRMS with the topology of a thickened diamond lattice, which has embedded lines. The red, green, and yellow fish swim along those lines (the blue fish swim in loops around the “waists”). We show:

- A piece of the triply periodic polyhedron.
- A corresponding piece of the patterned polyhedron.
- A piece of Schwarz’s D-surface showing embedded lines.
- Escher’s *Circle Limit III* with the equilateral triangle tessellation superimposed.
- A large piece of the patterned polyhedron.
- A top view of the large piece.
A piece of the triply periodic polyhedron
A corresponding piece of the patterned polyhedron
A piece of Schwarz’s D-surface showing embedded lines
Escher’s Circle Limit III with the equilateral triangle tessellation superimposed
A large piece of the patterned polyhedron
A top view of the large piece
Future Work

- Put other patterns on the regular triply periodic polyhedra, exploiting their embedded lines.
- Place patterns on non-regular, uniform triply periodic polyhedra.
- Put patterns on non-uniform triply periodic polyhedra — especially those that more closely approximate triply periodic minimal surfaces.
- Draw patterns on TPMS’s — the gyroid, for example.
Thank You!

Nat, Steve, Bob, and the other ISAMA organizers