Enumerations of Hyperbolic Truchet Tiles

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Abstract

Sébastien Truchet was a pioneer in applying combinatorics to the study of regular patterns. He enumerated the patterns that could be formed from square tiles that were divided by a diagonal into a black and a white triangle Following Truchet, others have created Truchet-like tilings composed of circular arcs and other motifs. These patterns are all based on Euclidean tessellations, usually the tiling by squares. In this paper we pose corresponding enumeration questions about hyperbolic Truchet tilings and show some sample patterns.

1. Introduction

About 300 years ago the French Dominican Father Sébastien Truchet enumerated Euclidean patterns that could be formed by using square tiles that are divided into two $45°$ equilateral triangles, one black and one white. The goal of this paper is to try to enumerate corresponding patterns in the hyperbolic plane. Figure 1 shows a hyperbolic Truchet pattern.

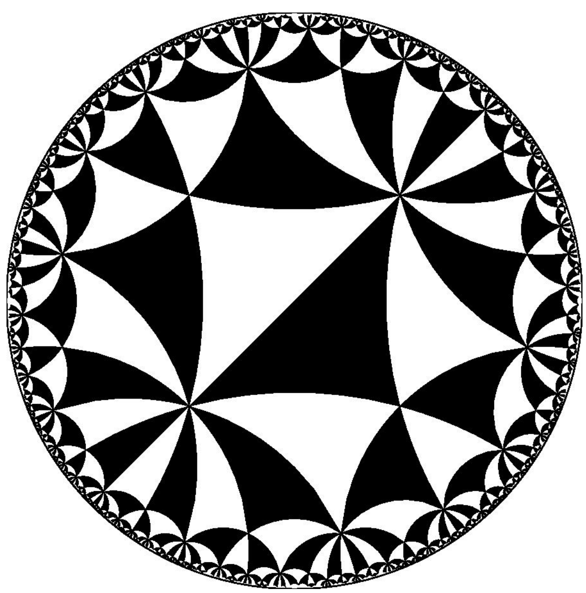


Figure 1: A hyperbolic Truchet tiling based on the {4,6} grid.

We begin with a short history of Truchet tilings. Then we review hyperbolic geometry and regular tessellations, upon which both Euclidean and hyperbolic tilings are based. Next we examine hyperbolic patterns based on “square” grids, which are most directly related to Truchet’s tilings. More generally we show how a p-sided polygon for can be subdivided by triangles for $p \neq 4$. We also investigate p-sides tiles decorated with circular arcs. Finally, we show sample patterns and indicate possible directions of further research.

2. A Short History of Truchet Tilings

Sébastien Truchet was born in Lyon, France in 1657, and became a Dominican Father as an adult. In addition to Truchet tilings, he is well known for his work in typography and the "Roman Du Roi" typeface that is an ancestor of "Times New Roman", in particular. Truchet also designed many French canals and invented sundials, weapons, and special implements for transporting trees (from Wikipedia [11]). He published his work on tilings "Memoir sur les Combinaisons" in the *Memoires de l'Académie Royale des Sciences* in 1704 [10]. In this paper Truchet considered all possible pairs of juxtaposed squares divided by a diagonal into a black and a white triangle. This was most likely the first published systematic enumeration of simple tile motifs. In the mid 1700's, Pierre Simon Fournier created Truchet patterns based on more complex motifs [2]. In 1942 M.C. Escher enumerated 2×2 tiles of squares formed from squares containing simple motifs, thus extending Truchet's idea of 2×1 tiles (see the section *Other experiments in regular division*, pages 44–52 of [8]). In 1987 Truchet's treatise was translated into English (by Pauline Bouchet), with some history and comments on Truchet's theory (by Cyril Smith) in a *Leonardo* paper which also reproduced Truchet's figures [9]. The Smith-Bouchet paper re-ignited interest in Truchet's tilings, and also introduced the "circular arc" Truchet tile, which has been popular with other pattern creators. Since then Browne [1], Lord and Ranganathan [3], Reimann [5, 6], and Rhode [7] have extended Truchet's ideas to other 2-dimensional motifs and to 3-dimensional patterns.

3. Hyperbolic Geometry and Regular Tessellations

Truchet used the Euclidean tessellation by squares for his tiling patterns. Others have also used the other two regular Euclidean tessellations, by equilateral triangles and by regular hexagons, as a basis for their Truchet-like tilings. In this paper, we show how to extend Truchet tilings to the hyperbolic plane, which has an infinite number of regular tessellations.

It has been known for more than a century that there is no smooth embedding of the hyperbolic plane into Euclidean 3-space. Thus we must rely on models of hyperbolic geometry. Specifically, we use the *Poincaré disk* model, whose (hyperbolic) points are represented by Euclidean points within a bounding circle. Hyperbolic lines are represented by (Euclidean) circular arcs orthogonal to the bounding circle (including diameters). The hyperbolic measure of an angle is the same as its Euclidean measure in the disk model (i.e the model is *conformal*), but equal hyperbolic distances correspond to ever-smaller Euclidean distances as figures approach the edge of the disk, as can be seen in Figure 1.

There is a *regular tessellation,* $\{p, q\}$, of the hyperbolic plane by regular p-sided polygons, which we call *p-gons*, with q of them meeting at each vertex, provided $(p-2)(q-2) > 4$. If $(p-2)(q-2) = 4$, one obtains three Euclidean tessellations: the square grid $\{4, 4\}$, the hexagon grid $\{6, 3\}$, and the equilateral triangle grid $\{3, 6\}$. Figure 2 shows the regular hyperbolic tessellation $\{4, 6\}$, and Figure 3 shows that tessellation superimposed on the Figure 1 pattern.

4. Hyperbolic Truchet Patterns Based on "Squares"

The simplest Euclidean Truchet tiling is the one created by translations of the *basic square* — a square divided into a black and a white isosceles right triangle by a diagonal, as shown in Figure 4 on the left. There is another Truchet tiling obtained by rotating the basic squares about its vertices, so that the $45°$ vertices meet at alternate vertices of the $\{4, 4\}$ grid, as shown on the right of Figure 4. These are patterns A and D of Truchet's *Memoir* [9] and the only ones adhering to the map-coloring principle: no triangles of the same color share and edge.

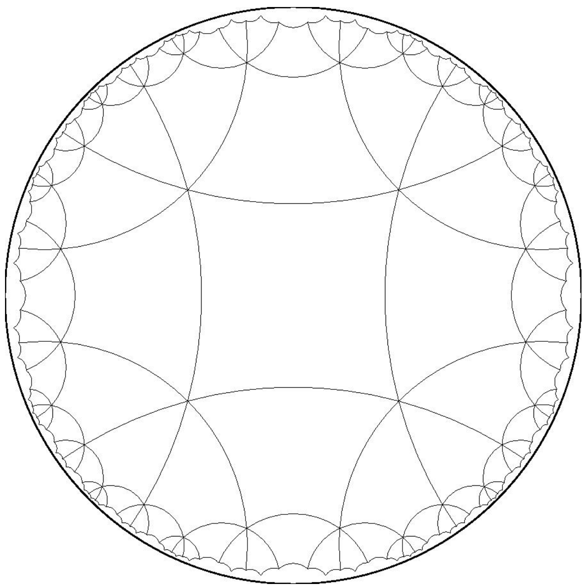


Figure 2: The $\{4,6\}$ tessellation

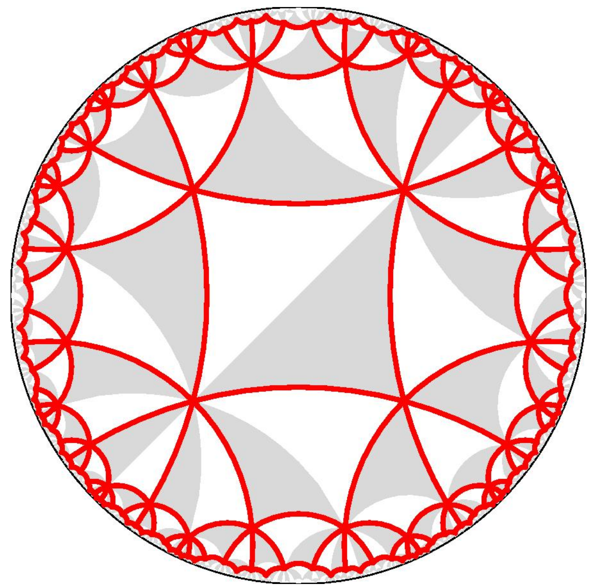
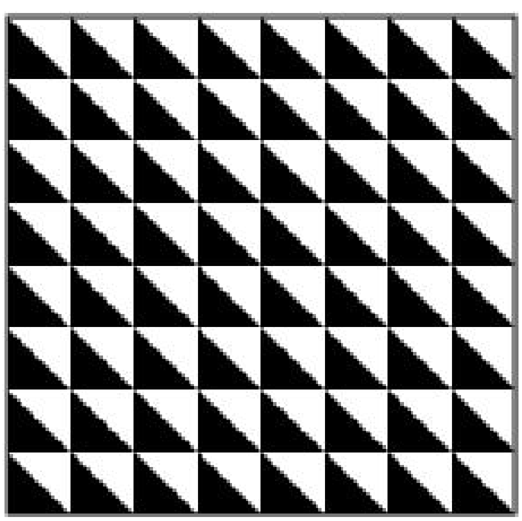


Figure 3: The $\{4, 6\}$ superimposed on the Figure 1 pattern.



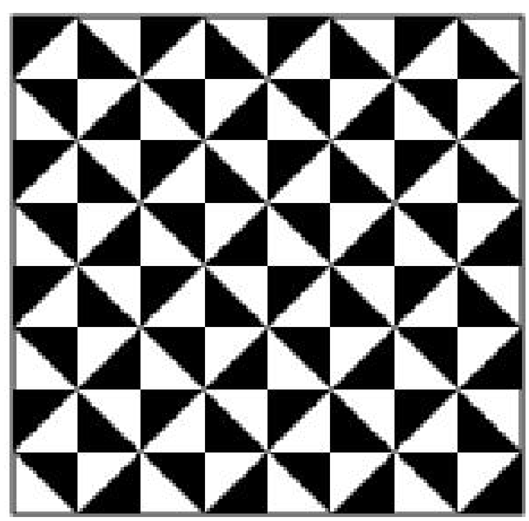


Figure 4: (a) A "translation" Truchet tiling, (b) A "rotation" Truchet tiling.

In the hyperbolic plane, if one translates a decorated 4-gon of a $\{4, q\}$ to the next 4-gon to the right, then upward, then to the left, etc., in a counter-clockwise manner about a q-vertex, the decorated 4-gon will return to its orginal position after q steps. However, the decoration will be rotated by an angle of $q\pi/2$. Therefore, to obtain a consistent tiling by a decorated 4-gon, $q\pi/2$ must be a multiple of 2π, i.e. q must be divisible by 4. Figure 5 shows the "smallest" hyperbolic example with $q = 8$.

If we apply the rotation construction in the hyperbolic case, the base angles of the black and white isosceles triangles meet at some of the vertices of $\{4, q\}$ and the vertex angles of the isosceles triangles meet at the other vertices of $\{4, q\}$. In this case q must be even to satisfy the map-coloring principle. Figure 1 shows the pattern when $q = 6$; Figure 6 shows the result when $q = 8$. In Figures 5 and 6 small circles have been placed at the vertex angles of the black and white isosceles triangles to illustrate the differences between the hyperbolic "translation" and "rotation" patterns. Truchet did not restrict himself to the map-coloring principle, allowing triangles of the same color to share an edge. Figure 7 shows such a pattern, F in Truchet's Plate 1 of his *Memoir* [9], which mixes "translation" and "rotation" edge matchings. Figure 8 shows a hyperbolic version of this pattern based on the $\{4, 6\}$ tessellation, which has large, alternately colored hexagons (since $q = 6$) instead of the squares of pattern F.

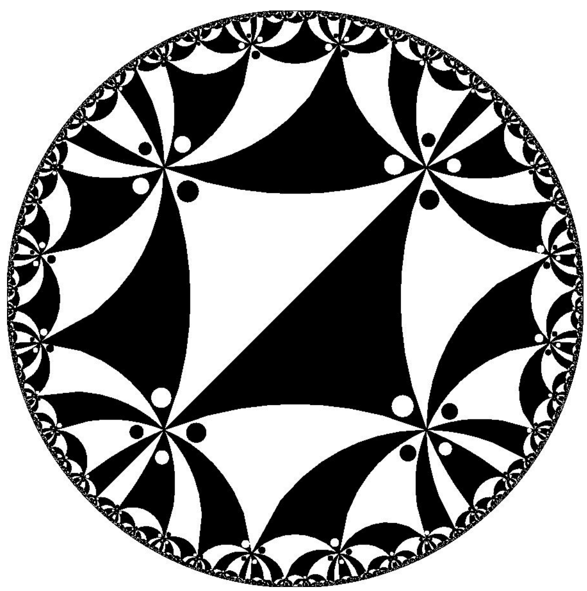


Figure 5: A "translation" Truchet pattern based on the $\{4, 8\}$ tessellation.

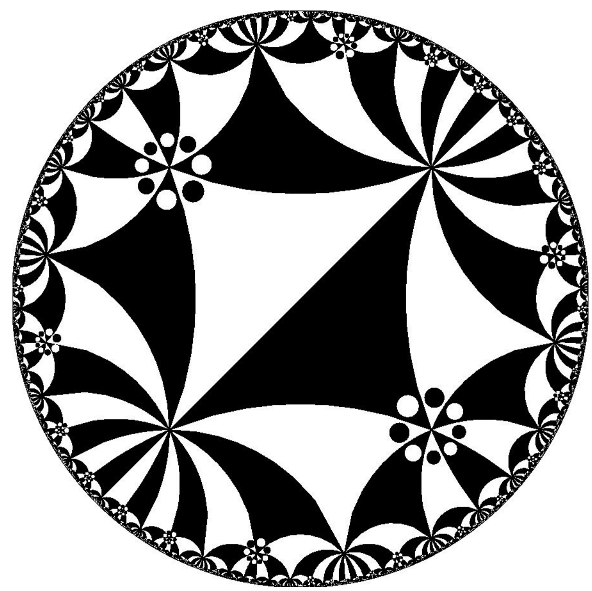


Figure 6: A "rotation" Truchet pattern based on the $\{4, 8\}$ tessellation.

5. Truchet Tiles with Multiple Triangles per p-gon

In his *Memoir*, Truchet considered rectangles composed of two basic squares (each divided into a black and white triangle). Each square could be given one of four orientations, and the second square could be placed adjacent to each of the four edges of the first square, giving 64 different rectangles. However, many pairs of rectangles are equivalent by rotation, yielding 10 inequivalent rectangles — shown in Truchet's Table 1 [9]. There are only six inequivalent rectangles if reflections are allowed, but Truchet did not consider them. Truchet constructed 24 patterns from his rectangles, six on each of Plates 1, 2, 3, and 4 of his *Memoir*. He labeled those patterns with the letters A through Z and &, omitting J, K, and W (we have seen A, D, and F above).

Though it is natural to tile the Euclidean plane by rectangles, it is more difficult to tile the hyperbolic plane by "rectangles" — quadrilaterals with congruent opposite sides. Instead, we divide the p-gons of a

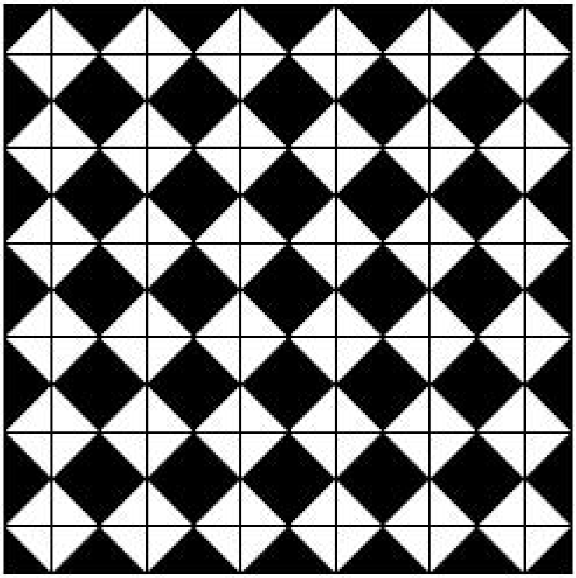


Figure 7: Truchet's pattern F, which does not adhere to the map-coloring principle.

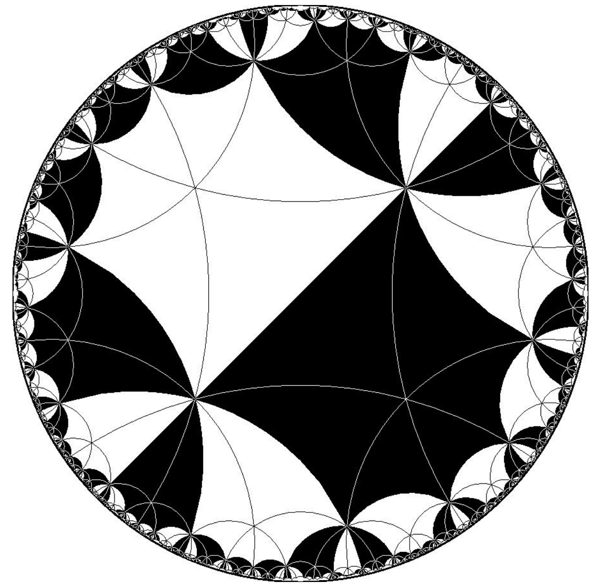


Figure 8: A hyperbolic Truchet pattern corresponding to Truchet's pattern F.

$\{p, q\}$ divided into black and white $\frac{\pi}{p}$ - $\frac{\pi}{q}$ - $\frac{\pi}{2}$ *basic triangles* by radii and apothems, since p-gons easily tile the hyperbolic plane. To satisfy the map-coloring principle, the basic triangles in the p-gon should alternate black and white, and that p-gon should be rotated about the midpoints of the edges to extend the pattern. There are two such patterns for any p and q, one obtained from the other by interchanging black and white. Figure 9 shows such a pattern based on the $\{4, 6\}$ tessellation — probably a better hyperbolic analog to Truchet's pattern A of Plate 1 than Figure 5 above.

If we do not require the pattern to be map-colored, there are many more possibilities. There are $N_2(2p)$ possible ways to fill a p-gon with black and white basic triangles, where $N_k(n)$ is the number of different n-bead necklaces that can be made using beads of k colors, and is given by [12]:

$$N_k(n) = \frac{1}{n} \sum_{d|n} \varphi(d) k^{n/k}$$

where $\varphi(d)$ is Euler's totient function (which gives the number of positive integers less than or equal to d and relatively prime to it). This can be seen as follows: we consider the perimeter of the p-gon to be the necklace, and the two basic triangles adjacent to each edge as "beads" ($2p$ beads total) of one of two colors. If we consider our "necklaces" to be equivalent by reflection across a diameter or apothem of the p-gon, there are fewer possibilities, given by $B_k(n)$ the number of n-bead "bracelets" made with k colors of beads [12]. It seems to be a difficult problem to enumerate all the ways such a p-gon pattern of triangles could be extended across each of its edges, though an upper bound would be $(2p)^p N_2(2p)$.

Figure 10 shows a $\{4, 6\}$ pattern with pairs of black and white triangles adjacent across apothems, analogous to Truchet's pattern E of Plate 1. Figure 11, also based on $\{4, 6\}$, uses the same triangles within the 4-gon as Figure 10, but extended differently across the 4-gon edges. Like Figure 8, it is analogous to Truchet's Pattern F of Plate 1.

Finally, we show patterns based on p-gons with $p \neq 4$. Figure 12 shows a tiling generated by alternating pairs of black and white basic triangles within a 6-gon. white triangles; it is analogous to Truchet's pattern N on Plate 2. Figure 13 shows a tiling generated by a symmetric arrangement of basic triangles within a 5-gon. These two patterns are not related to any patterns in Truchet's *Memoir*.

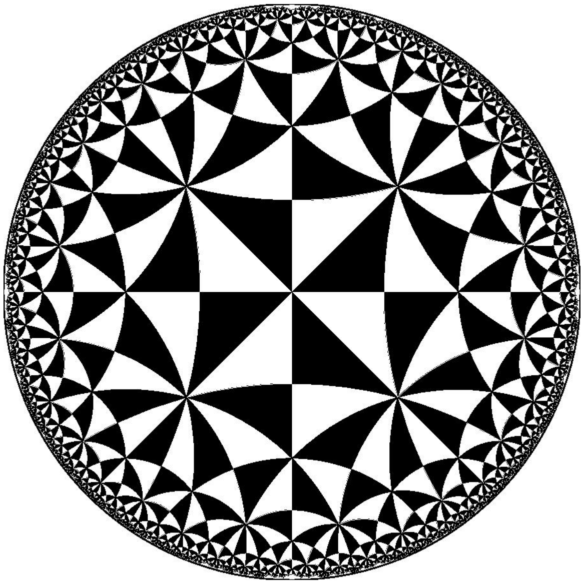


Figure 9: A pattern generated by alternate black and white triangles in a 4-gon.

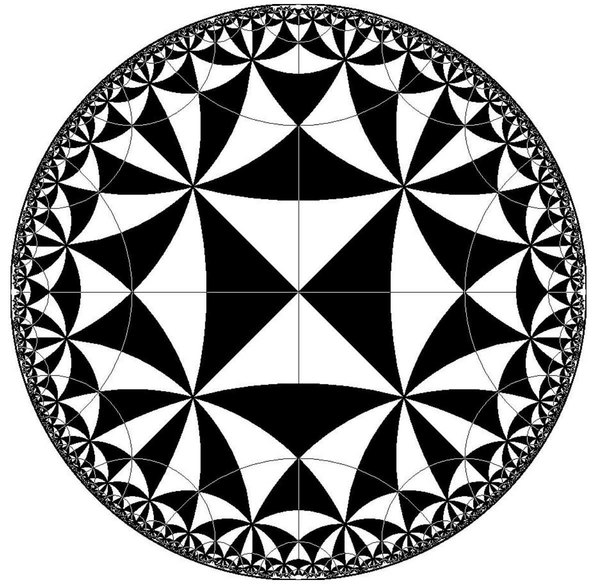


Figure 10: A pattern generated by paired black and white triangles in a 4-gon.

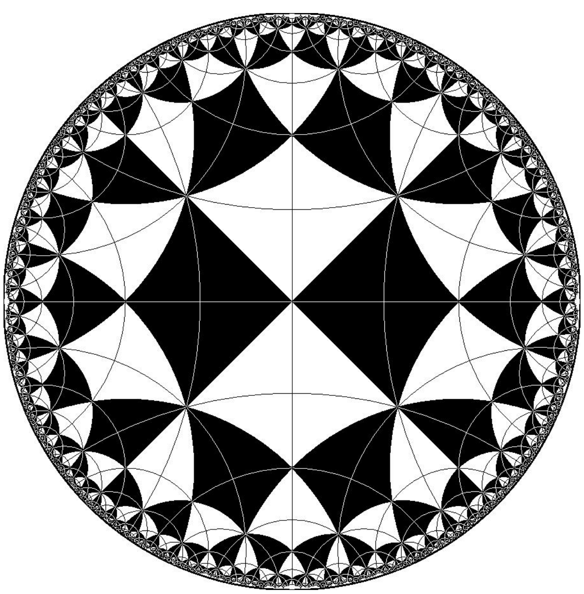


Figure 11: Another pattern generated by paired black and white triangles in a 4-gon.

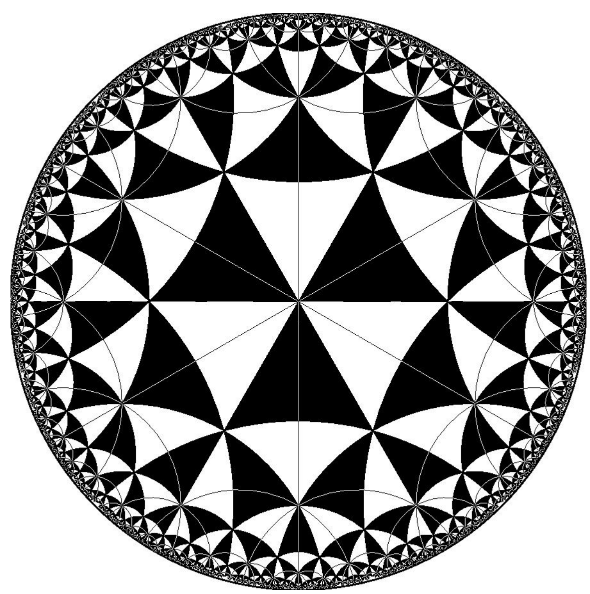


Figure 12: A simple $\{6, 4\}$ pattern.

6. Patterns with Other Motifs

Other designers have used motifs other than the triangularly divided square to make their Truchet-like patterns. One choice, first described by Smith is a motif consisting of two quarter arcs of circles with each arc connecting the midpoints of two adjacent edges of the square [9]. Such patterns can be regular, random, or even carefully arranged so as to spell words [5]. Figure 14 shows a hyperbolic pattern based on two-arcs motif (superimposed on the underlying $\{4, 6\}$ tessellation).

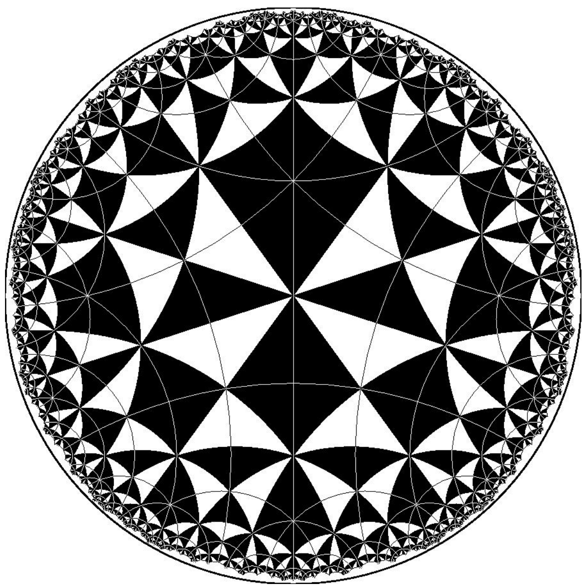


Figure 13: A new $\{5, 4\}$ Truchet-like tiling.

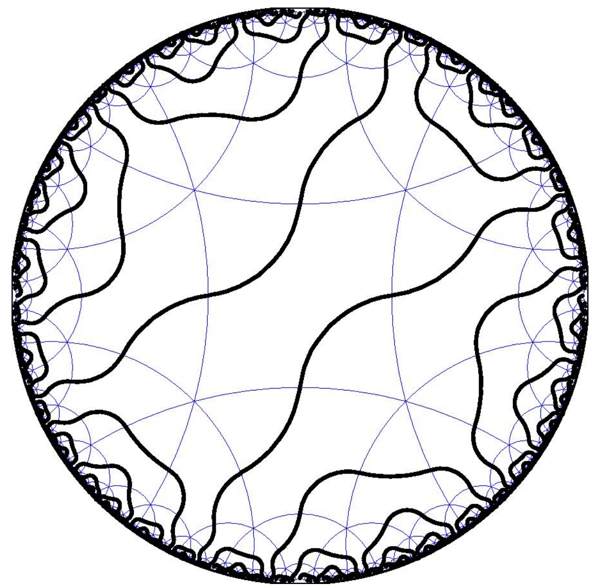


Figure 14: A hyperbolic Truchet arc pattern on a $\{4, 6\}$ grid.

One can generalize the "arcs" motif to $2n$-gons: there would be n non-intersecting arcs connecting the midpoints of the edges of the $2n$-gon. The number of possible $2n$-gon tiles is the same as the number of ways to connect $2n$ points on a circle with non-intersecting chords. It is the Catalan number $C(n) = 2n!/[n!(n+1)!]$ as noted for Sloane's sequence A000108 [4]. As is the case with the triangle-decorated p-gons, the number of possible patterns is bounded above by $(2n)^{2n}C(n)$, though it seems difficult to get an exact enumeration.

7. Future Work

We have shown some Truchet patterns in the hyperbolic plane based on the regular $\{p, q\}$ tessellations. We have also noted some combinatorial results on the number of possible tiles for "square", triangle-decorated p-gon, and arc Truchet patterns. But there are other questions that remain to be answered about the the possible number of patterns that can be formed in a regular way from such tiles. These questions seem to be difficult.

Since some Truchet patterns have black-white color symmetry, it would also seem natural to investigate the coloring of Truchet tilings with more than two colors. Another direction of future research would be to create Truchet patterns on hyperbolic Archimedean tessellations.

Acknowledgments

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