A Formula for the Intersection Angle of Backbone Arcs with the Bounding Circle for General *Circle Limit III* Patterns

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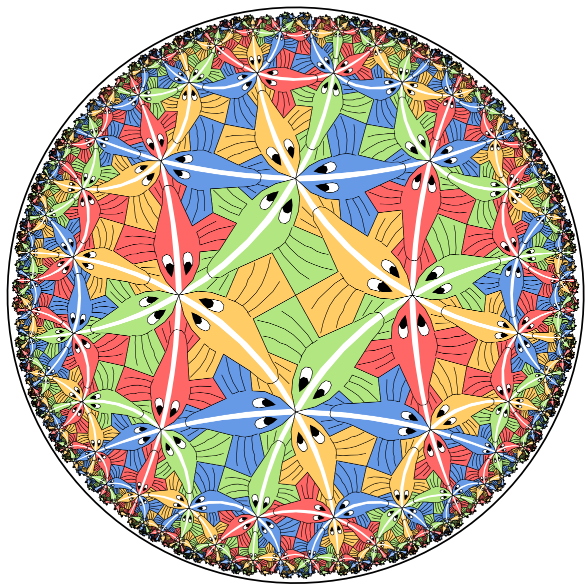
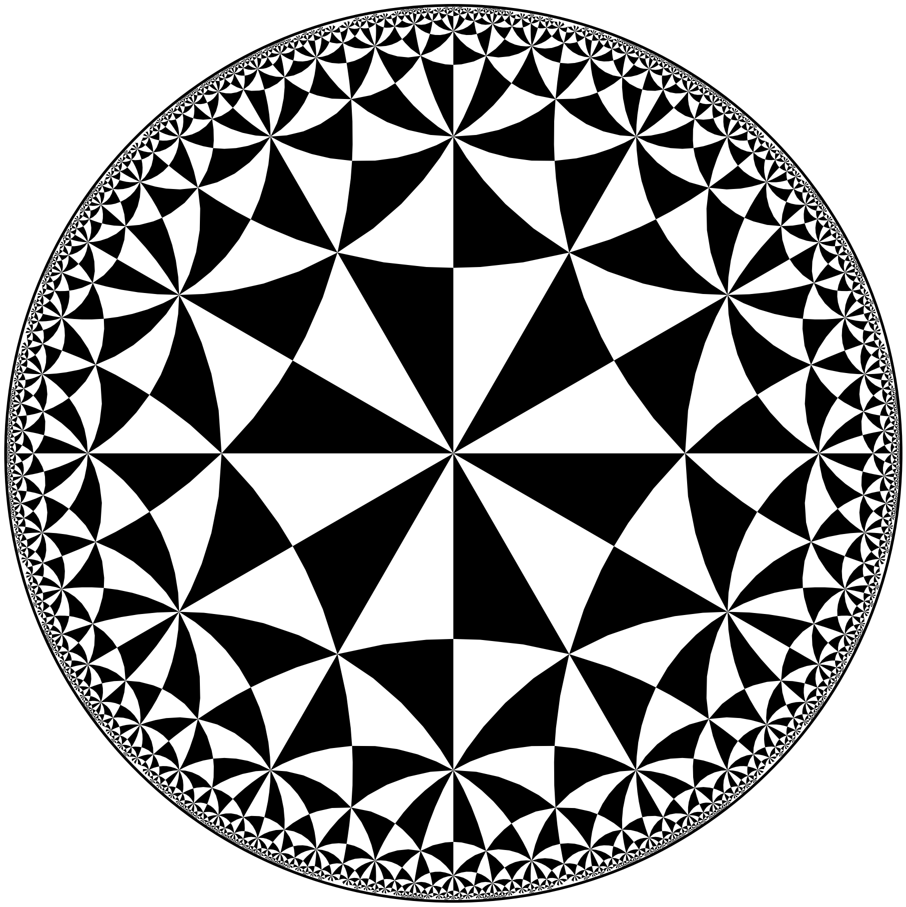


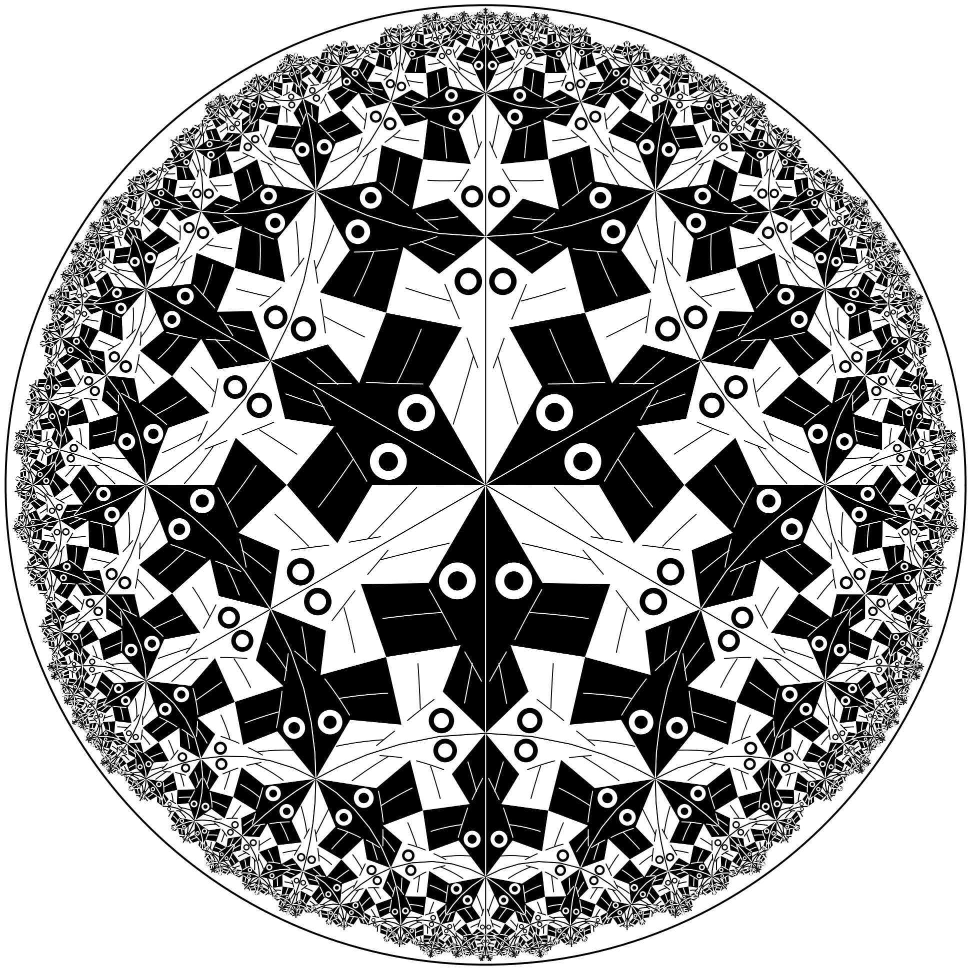
History - Outline

- Early 1958: H.S.M. Coxeter sends M.C. Escher a reprint containing a hyperbolic triangle tessellation.
- Later in 1958: Inspired by that tessellation, Escher creates *Circle Limit I*.
- Late 1959: Solving the "problems" of *Circle Limit I*, Escher creates *Circle Limit III*.
- 1979: In a *Leonardo* article, Coxeter uses hyperbolic trigonometry to calculate the "backbone arc" angle.
- 1996: In a *Math. Intelligencer* article, Coxeter uses Euclidean geometry to calculate the "backbone arc" angle.
- 2006: In a *Bridges* paper, D. Dunham introduces (p, q, r) "Circle Limit III" patterns and gives an "arc angle" formula for $(p, 3, 3)$.
- 2007: In a *Bridges* paper, Dunham shows an "arc angle" calculation in the general case (p, q, r).
- Later 2007: L. Tee derives an "arc angle" formula in the general case.

The hyperbolic triangle pattern in Coxeter's paper



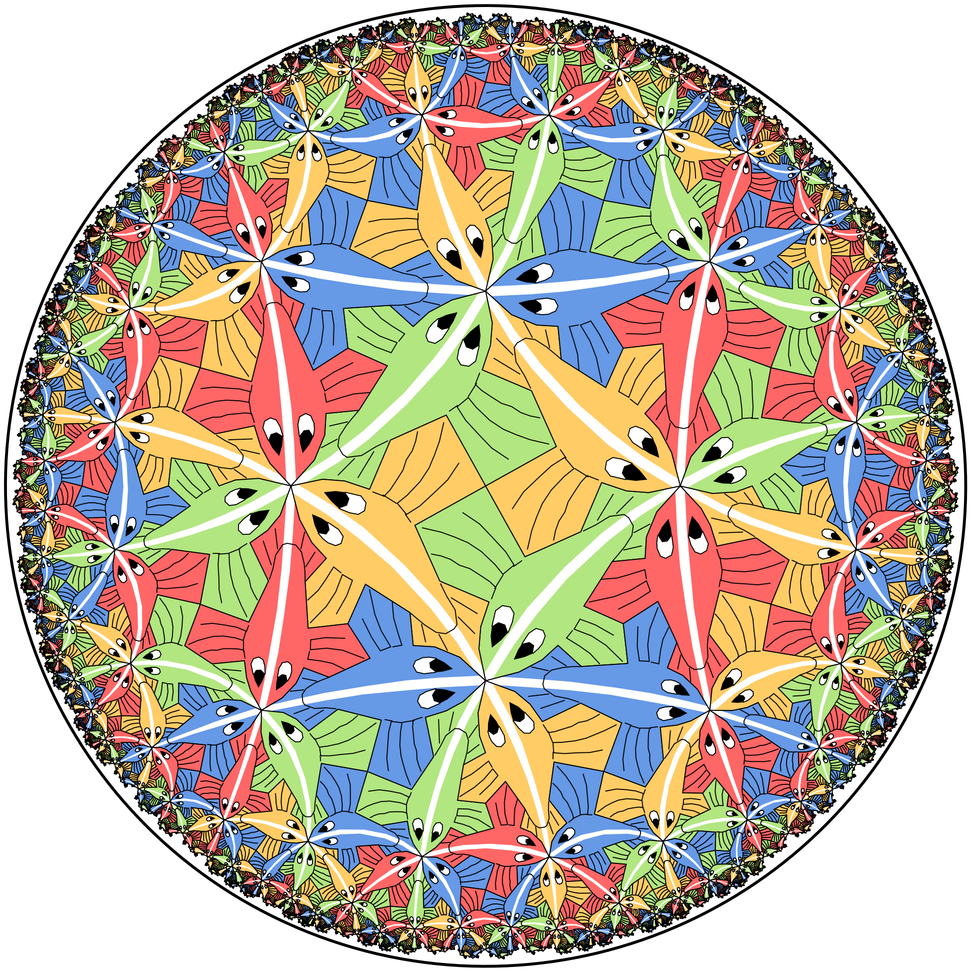
A Computer Rendition of *Circle Limit I*



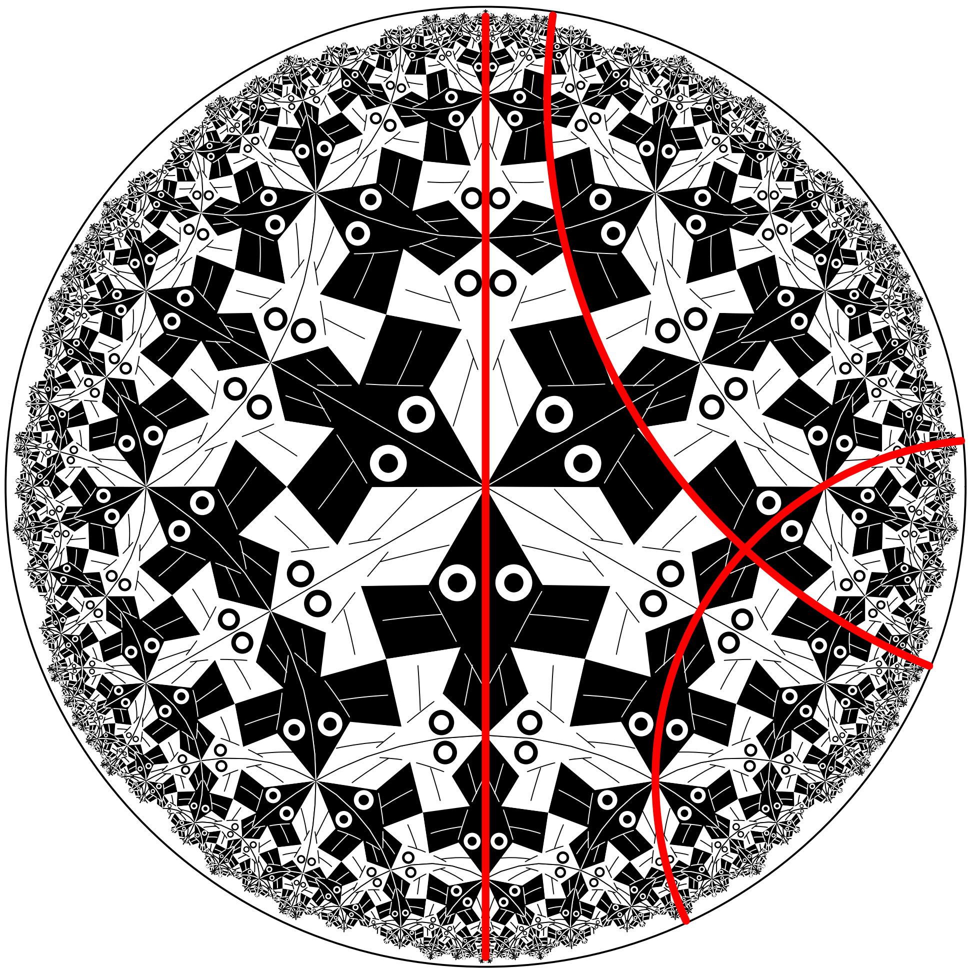
Escher: Shortcomings of *Circle Limit I*

"There is no continuity, no 'traffic flow', no unity of colour in each row ..."

A Computer Rendition of *Circle Limit III*



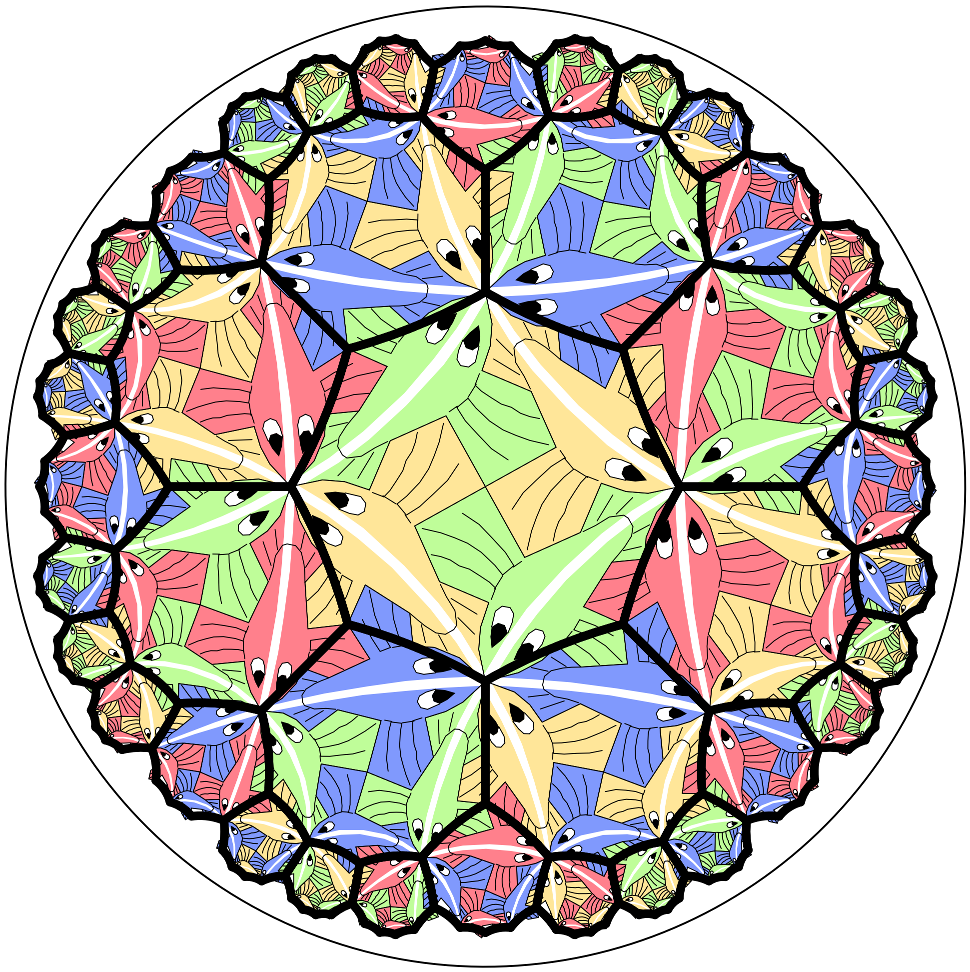
Poincaré Circle Model of Hyperbolic Geometry



- **Points:** points within the **bounding circle**
- **Lines:** circular arcs perpendicular to the bounding circle (including diameters as a special case)

The Regular Tessellations $\{m,n\}$

There is a *regular tessellation*, $\{m,n\}$ of the hyperbolic plane by regular *m*-sided polygons meeting *n* at a vertex provided $(m-2)(n-2) > 4$.

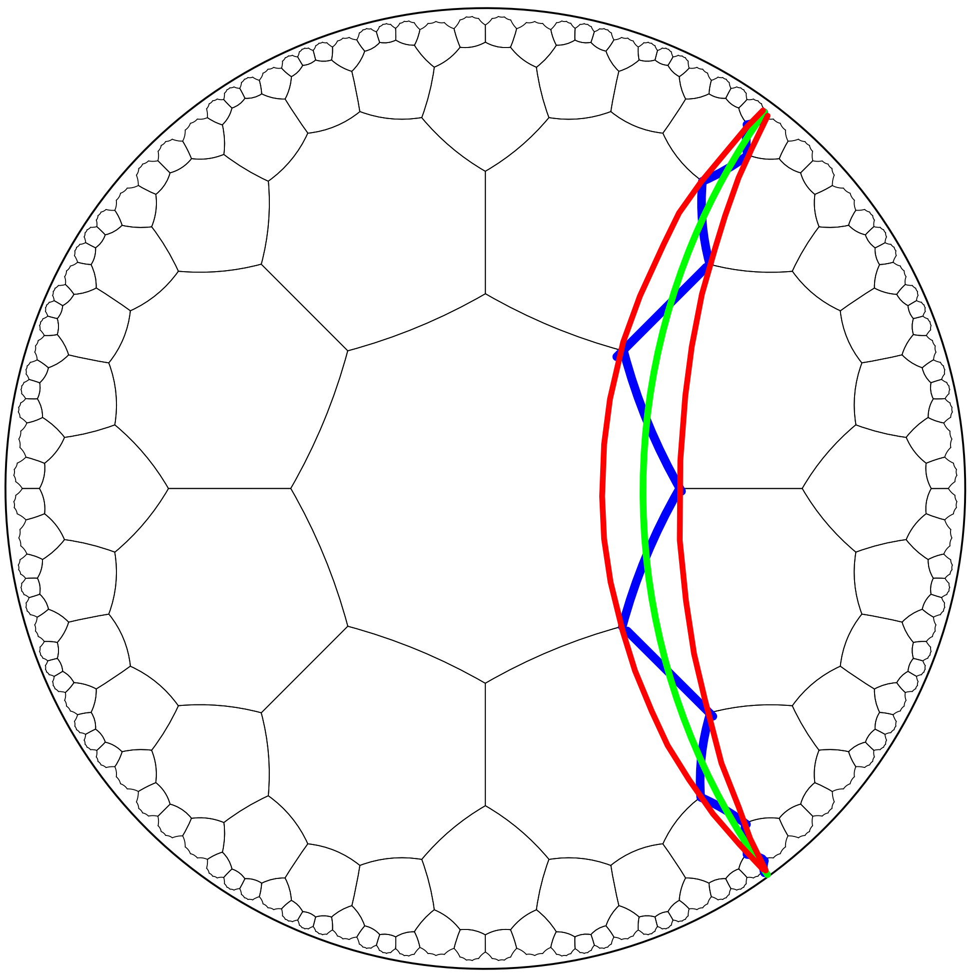


The tessellation $\{8,3\}$ superimposed on the *Circle Limit III* pattern.

Equidistant Curves and Petrie Polygons

For each hyperbolic line and a given hyperbolic distance, there are two *equidistant curves*, one on each side of the line, all of whose points are that distance from the given line.

A *Petrie polygon* is a polygonal path of edges in a regular tessellation traversed by alternately taking the left-most and right-most edge at each vertex.



A Petrie polygon (blue) based on the $\{8{,}3\}$ tessellation, and a hyperbolic line (green) with two associated equidistant curves (red).

Coxeter's *Leonardo* and *Intelligencer* Articles

In *Leonardo* 12, (1979), pages 19–25, Coxeter used hyperbolic trigonometry to determine that the angle ω that the backbone arcs make with the bounding circle is given by:

$$\cos\omega = (2^{1/4} - 2^{-1/4})/2 \quad \text{or} \quad \omega \approx 79.97^\circ$$

Coxeter derived the same result, using elementary Euclidean geometry, in *The Mathematical Intelligencer* 18, No. 4 (1996), pages 42–46.

A General Theory?

The cover of The Mathematical Intelligencer containing Coxeter's article, showed a reproduction of Escher's *Circle Limit III* print and the words "What Escher Left Unstated".

Also an anonymous editor, remarking on the cover, wrote:

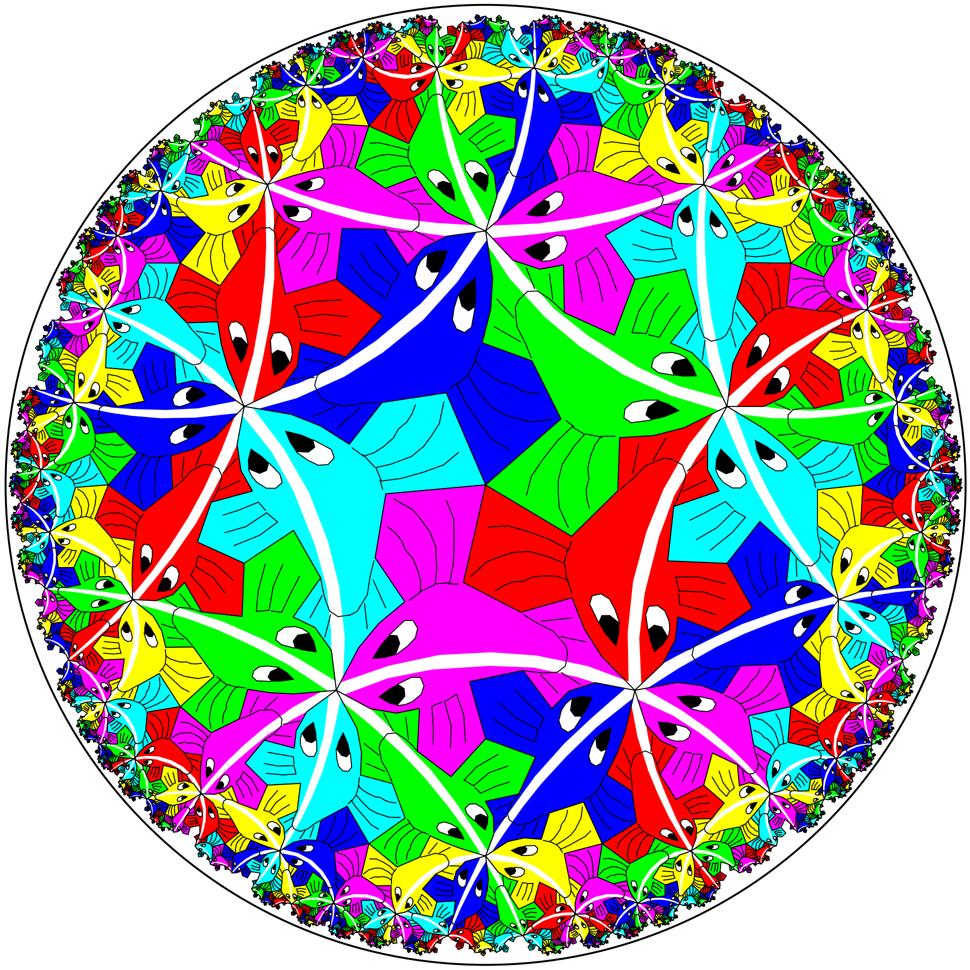
> **Coxeter's enthusiasm for the gift M.C. Escher gave him, a print of Circle Limit III, is understandable. So is his continuing curiosity. See the articles on pp. 35–46. He has not, however said of what general theory this pattern is a special case. Not as yet.**

A General Theory

We use the symbolism **(*p*,*q*,*r*)** to denote a pattern of fish in which ***p*** meet at right fin tips, ***q*** meet at left fin tips, and ***r*** fish meet at their noses. Of course ***p*** and ***q*** must be at least three, and ***r*** must be odd so that the fish swim head-to-tail.

The *Circle Limit III* pattern would be labeled (4,3,3) in this notation.

A (5,3,3) Pattern



Dunham's *Bridges 2006* Paper

In the *Bridges 2006 Conference Proceedings*, Dunham followed Coxeter's *Leonardo* article, using hyperbolic trigonometry to derive the more general formula that applied to (*p,3,3*) patterns:

$$\cos\omega = \frac{1}{2}\sqrt{1 - 3/4\cos^2(\frac{\pi}{2p})}$$

For *Circle Limit III*, *p* = *4* and $\cos\omega = \sqrt{\frac{3\sqrt{2}-4}{8}}$ **, which agrees with Coxeter's calculations.**

For the (*5,3,3*) pattern, $\cos\omega = \sqrt{\frac{3\sqrt{5}-5}{40}}$ **and** $\omega \approx 78.07^\circ$ **.**

Dunham's *Bridges 2006* Paper

In the *Bridges 2006 Conference Proceedings*, Dunham presented a 5-step process for calculating ω for a general (p, q, r) pattern. This calculation utilized the Weierstrass model of hyperbolic geometry and the geometry of a tessellation by "kites", any one of which forms a fundamental region for the pattern.

Weierstrass Model of Hyperbolic Geometry

- **Points:** points on the upper sheet of a hyperboloid of two sheets: $x^2 + y^2 - z^2 = -1$, $z \geq 1$.
- **Lines:** the intersection of a Euclidean plane through the origin with this upper sheet (and so is one branch of a hyperbola).

  A line can be represented by its **pole**, a 3-vector $\begin{bmatrix} \ell_x \\ \ell_y \\ \ell_z \end{bmatrix}$ on the dual hyperboloid $\ell_x^2 + \ell_y^2 - \ell_z^2 = +1$, so that the line is the set of points satisfying $x\ell_x + y\ell_y - z\ell_z = 0$.

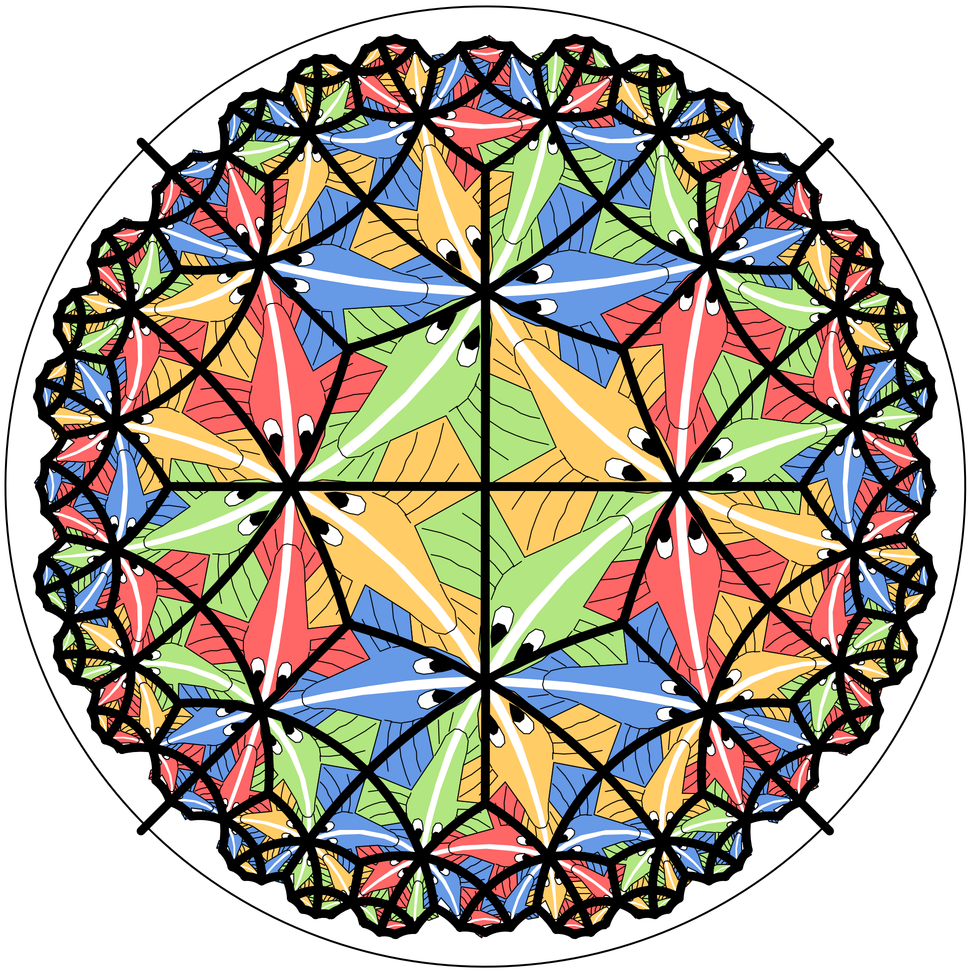
The Relation Between the Models

The models are related via stereographic projection from the Weierstrass model onto the (unit) Poincaré disk in the xy-plane toward the point $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$,

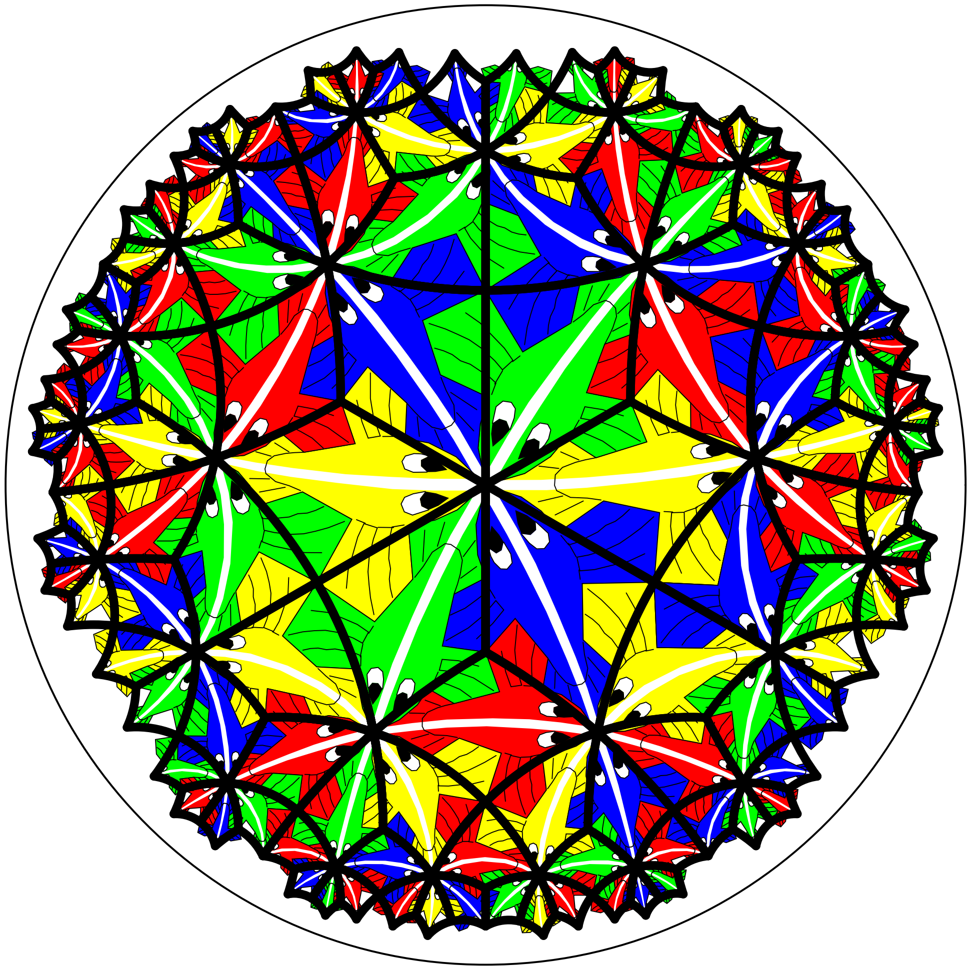
Given by the formula: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x/(1+z) \\ y/(1+z) \\ 0 \end{bmatrix}$.

The Kite Tessellation

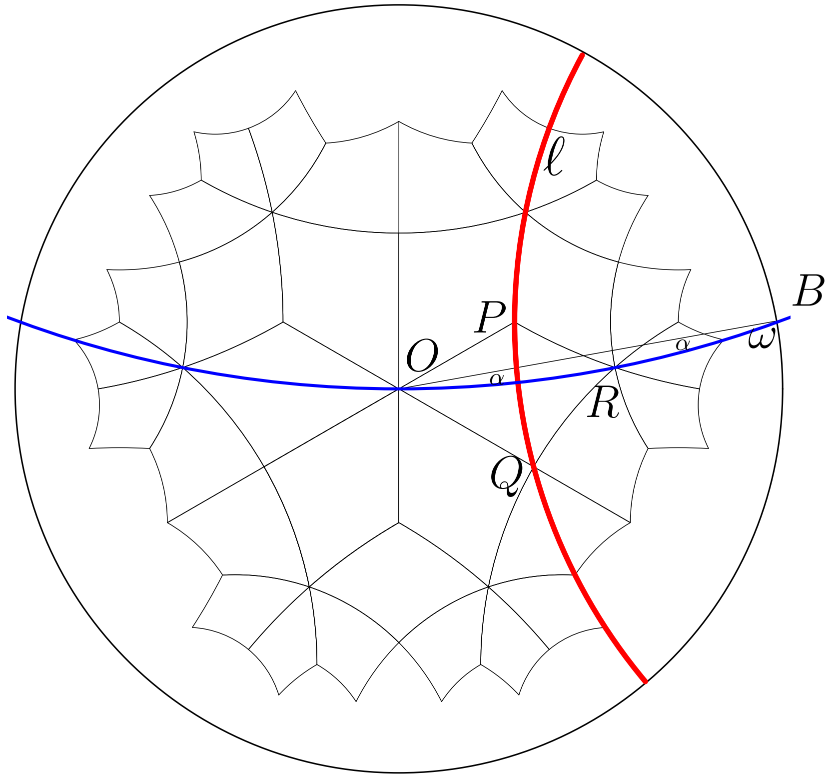
The **fundamental region** for a (p, q, r) pattern can be taken to be a ***kite*** — a quadrilateral with two opposite angles equal. The angles are $2\pi/p$, π/r, $2\pi/q$, and π/r.



A Nose-Centered Kite Tessellation



The Geometry of the Kite Tessellation

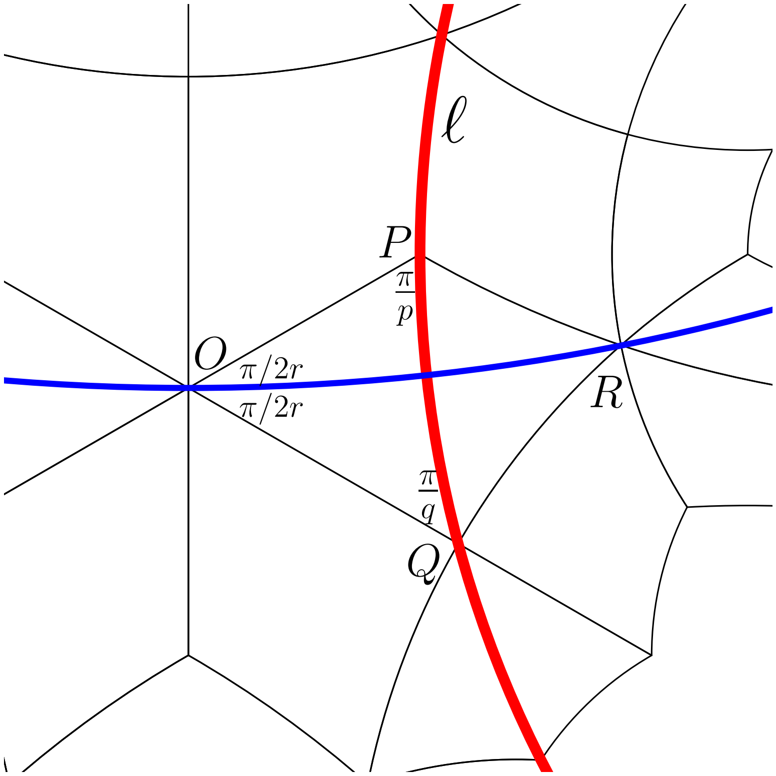


The kite $OPRQ$, its bisecting line, ℓ, the backbone line (equidistant curve) through O and R, and radius OB.

Outline of the Calculation

1. Calculate the Weierstrass coordinates of the points P and Q.
2. Find the coordinates of ℓ from those of P and Q.
3. Use the coordinates of ℓ to compute the matrix of the reflection across ℓ.
4. Reflect O across ℓ to obtain the Weierstrass coordinates of R, and thus the Poincaré coordinates of R.
5. Since the backbone equidistant curve is symmetric about the y-axis, the origin O and R determine that circle, from which it is easy to calculate ω, the angle of intersection of the backbone curve with the bounding circle.

Details of the Central Kite



1. The Weierstrass Coordinates of P and Q

From a standard trigonometric formula for hyperbolic triangles, the hyperbolic cosines of the hyperbolic lengths of the sides OP and OQ of the triangle OPQ are given by:

$$cosh(d_p) = \frac{\cos(\pi/q)\cos(\pi/r) + \cos\pi/p}{\sin(\pi/q)\sin(\pi/r)}$$

and

$$cosh(d_q) = \frac{\cos(\pi/p)\cos(\pi/r) + \cos\pi/q}{\sin(\pi/p)\sin(\pi/r)}$$

From these equations, we obtain the Weierstrass coordinates of P and Q:

$$P = \begin{bmatrix} \cos(\pi/2r)sinh(d_q) \\ \sin(\pi/2r)sinh(d_q) \\ cosh(d_q) \end{bmatrix} \quad Q = \begin{bmatrix} \cos(\pi/2r)sinh(d_p) \\ -\sin(\pi/2r)sinh(d_p) \\ cosh(d_p) \end{bmatrix}$$

2. The Coordinates of ℓ

The coordinates of the pole of ℓ are given by

$$\ell = \begin{bmatrix} \ell_x \\ \ell_y \\ \ell_z \end{bmatrix} = \frac{P \times_h Q}{|P \times_h Q|}$$

Where the hyperbolic cross-product $P \times_h Q$ is given by:

$$P \times_h Q = \begin{bmatrix} P_y Q_z - P_z Q_y \\ P_z Q_x - P_x Q_z \\ -P_x Q_y + P_y Q_x \end{bmatrix}$$

and where the norm of a pole vector V is given by: $|V| = \sqrt{(}V_x^2 + V_y^2 - V_z^2)$

3. The Reflection Matrix - A Simple Case

The pole corresponding to the hyperbolic line perpendicular to the x-axis and through the point $\begin{bmatrix} \sinh d \\ 0 \\ \cosh d \end{bmatrix}$ is given by $\begin{bmatrix} \cosh d \\ 0 \\ \sinh d \end{bmatrix}$.

The matrix *Ref* representing reflection of Weierstrass points across that line is given by:

$$Ref = \begin{bmatrix} -\cosh 2d & 0 & \sinh 2d \\ 0 & 1 & 0 \\ -\sinh 2d & 0 & \cosh 2d \end{bmatrix}$$

where d is the the hyperbolic distance from the line (or point) to the origin.

3. The Reflection Matrix - The General Case

In general, reflection across a line whose nearest point to the origin is rotated by angle θ from the x-axis is given by: $Rot(\theta) Ref\, Rot(-\theta)$ where, as usual,

$$Rot(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From ℓ we identify $\sinh d$ as ℓ_z, and $\cosh d$ as $\sqrt(\ell_x^2 + \ell_y^2)$, which we denote ρ. Then $\cos\theta = \frac{\ell_x}{\rho}$ and $\sin\theta = \frac{\ell_y}{\rho}$.

Further, $\sinh 2d = 2 \sinh d \cosh d = 2\rho\ell_z$ and $\cosh 2d = \cosh^2 d + \sinh^2 d = \rho^2 + \ell_z^2$.

Thus *Ref*$_\ell$, the matrix for reflection across ℓ is given by:

$$Ref_\ell = \begin{bmatrix} \frac{\ell_x}{\rho} & -\frac{\ell_y}{\rho} & 0 \\ \frac{\ell_y}{\rho} & \frac{\ell_x}{\rho} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -(\rho^2 + \ell_z^2) & 0 & 2\rho\ell_z \\ 0 & 1 & 0 \\ -2\rho\ell_z & 0 & (\rho^2 + \ell_z^2) \end{bmatrix} \begin{bmatrix} \frac{\ell_x}{\rho} & \frac{\ell_y}{\rho} & 0 \\ -\frac{\ell_y}{\rho} & \frac{\ell_x}{\rho} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. The Coordinates of R

We use *Ref*$_\ell$ to reflect the origin to R since the kite $OPRQ$ is symmetric across ℓ:

$$R = Ref_\ell \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\ell_x\ell_z \\ 2\ell_y\ell_z \\ \rho^2 + \ell_z^2 \end{bmatrix}$$

Then we project Weierstrass point R to the Poincaré model:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2\ell_x\ell_z}{1+\rho^2+\ell_z^2} \\ \frac{2\ell_y\ell_z}{1+\rho^2+\ell_z^2} \\ 0 \end{bmatrix}$$

5. The Angle ω

The three points $\begin{bmatrix} u \\ v \\ 0 \end{bmatrix}, \begin{bmatrix} -u \\ v \\ 0 \end{bmatrix}$, and the origin determine the (equidistant curve) circle centered at $w = (u^2 + v^2)/2v$ on the y-axis.

The y-coordinate of the intersection points of this circle, $x^2 + (y - w)^2 = w^2$, with the unit circle to be $y_{int} = 1/2w = v/(u^2 + v^2)$.

In the figure showing the geometry of the kite tessellation, the point B denotes the right-hand intersection point.

The central angle, α, made by the radius OB with the x-axis is the complement of ω (which can be seen since the equidistant circle is symmetric across the perpendicular bisector of OB).

Thus $y_{int} = \sin\alpha = \cos\omega$, so that

$$\cos\omega = y_{int} = v/(u^2 + v^2),$$

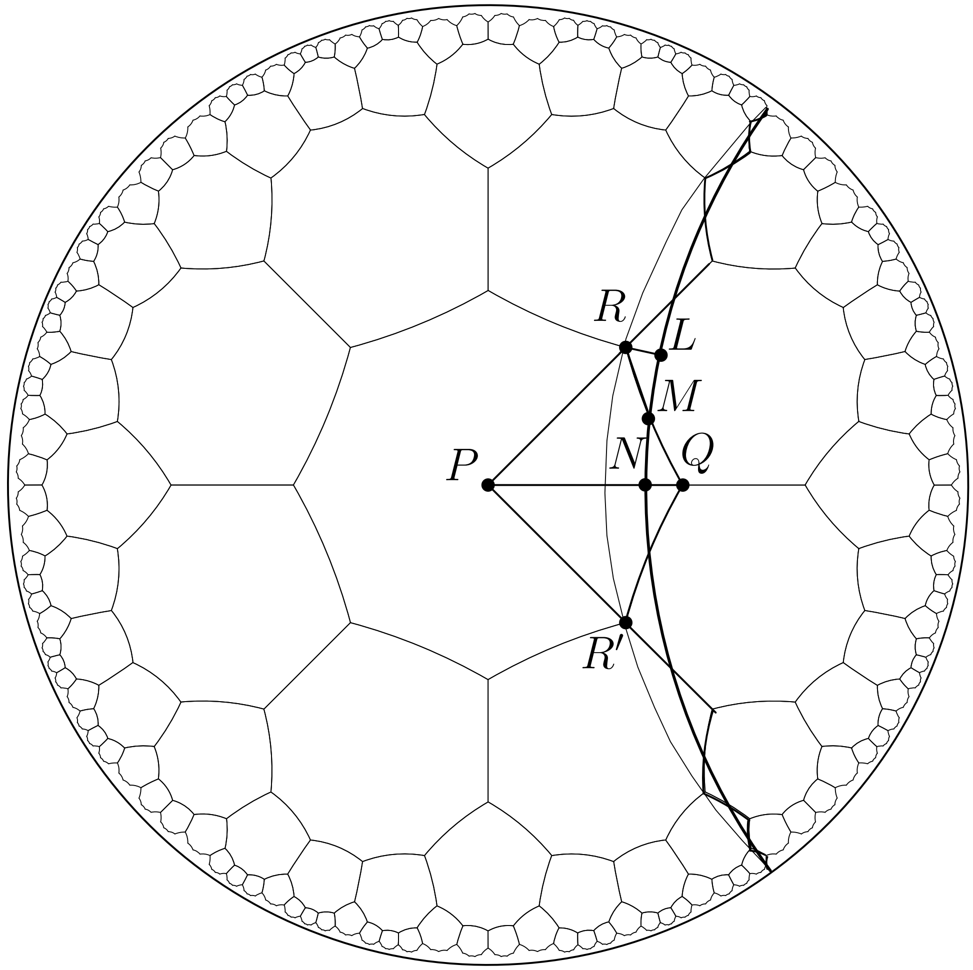
the desired result.

Luns Tee's Formula for ω

In mid-2007, Luns Tee used hyperbolic trigonometry to derive a general formula for ω, generalizing the calculations of Coxeter in the *Leonardo* article and Dunham in the 2006 *Bridges* paper.

As in those previous calculations, Tee based his computations on a fin-centered version of the (p, q, r) tessellation, with the central p-fold fin point labeled P, the opposite q-fold point labeled Q, and the nose point labeled R'.

A Diagram for Tee's Formula



The "backbone" equidistant curve is shown going through R and R'. The hyperbolic line through L, M and N has the same endpoints as that equidistant curve. The segments RL and QN are perpendicular to that hyperbolic line.

The Goal

By a well-known formula, the angle ω is given by:

$$\cos\omega = \tanh(RL)$$

Since RLM is a right triangle, by one of the formulas for hyperbolic right triangles, $\tanh(RL)$ is related to $\tanh(RM)$ by:

$$\tanh(RL) = \cos(\angle LRM)\tanh(RM)$$

But $\angle LRM = \frac{\pi}{2} - \frac{\pi}{2r}$ since the equidistant curve bisects $\angle PRQ = \frac{\pi}{r}$.

Thus

$$\cos(\angle LRM) = \cos(\frac{\pi}{2} - \frac{\pi}{2r}) = \sin(\frac{\pi}{2r})$$

and

$$\tanh(RL) = \sin(\frac{\pi}{2r})\tanh(RM) \quad (1)$$

so that our task is reduced to calculating $\tanh(RM)$.

A Formula for $\tanh(RM)$

To calculate $\tanh(RM)$, we note that as hyperbolic distances $RQ = RM + MQ$, so eliminating MQ from this equation will relate RM to RQ, for which there are formulas.

By the subtraction formula for $\cosh$

$$\cosh(MQ) = \cosh(RQ - RM) = \cosh(RQ)\cosh(RM) - \sinh(RQ)\sinh(RM)$$

Dividing through by $\cosh(RM)$ gives:

$$\cosh(MQ)/\cosh(RM) = \cosh(RQ) - \sinh(QR)\tanh(RM)$$

Also by a formula for hyperbolic right triangles applied to QMN and RML:

$$\cosh(MQ) = \cot(\angle QMN)\cot(\tfrac{\pi}{q}) \text{ and}$$
$$\cosh(RM) = \cot(\angle RML)\cot(\tfrac{\pi}{2} - \tfrac{\pi}{2r})$$

As opposite angles, $\angle QMN) = \angle RML$, so dividing the first equation by the second gives another expression for $\cosh(MQ)/\cosh(RM)$:

$$\cosh(MQ)/\cosh(RM) = \cot(\frac{\pi}{q})\cot(\frac{\pi}{2r})$$

Equating the two expressions for $\cosh(MQ)/\cosh(RM)$ gives:

$$\cosh(RQ) - \sinh(RQ)\tanh(RM) = \cot(\frac{\pi}{q})\cot(\frac{\pi}{2r})$$

Which can be solved for $\tanh(RM)$ in terms of RQ:

$$\tanh(RM) = (\cosh(RQ) - \cot(\frac{\pi}{q})\cot(\frac{\pi}{2r}))/\sinh(RQ) \quad (2)$$

The Final Formula

Another formula for general hyperbolic triangles, applied to QPR gives:

$$\cosh(RQ) = (\cos(\frac{\pi}{q})\cos(\frac{\pi}{r}) + \cos(\frac{\pi}{p}))/\sin(\frac{\pi}{q})\sin(\frac{\pi}{r})$$

We can calculate $\sinh(RQ)$ from this by the formula $\sinh^2 = \cosh^2 - 1$.

Plugging those values of $\cosh(RQ)$ and $\sinh(RQ)$ into equation (2), and inserting that result into equation (1) gives the final result:

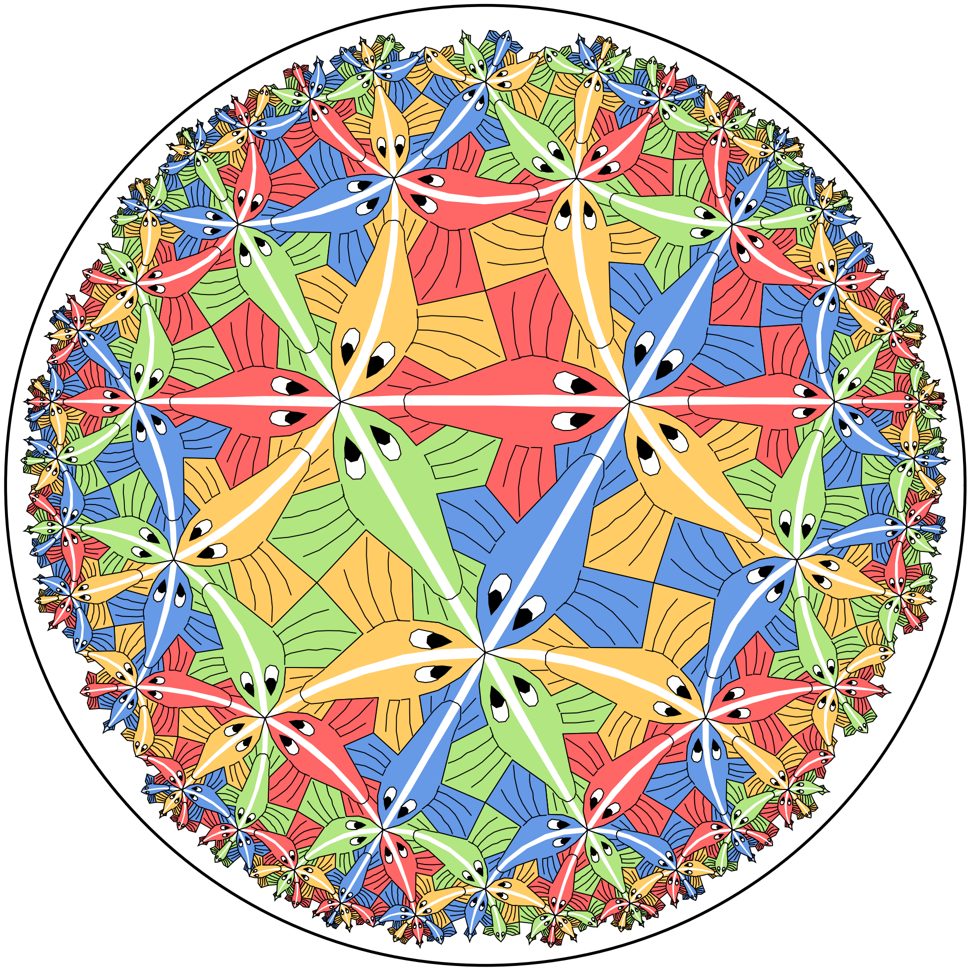
$$\cos(\omega) = \frac{\sin(\frac{\pi}{2r})\ (\cos(\frac{\pi}{p}) - \cos(\frac{\pi}{q}))}{\sqrt{(\ \cos(\frac{\pi}{p})^2 + \cos(\frac{\pi}{q})^2 + \cos(\frac{\pi}{r})^2 + 2\cos(\frac{\pi}{p})\cos(\frac{\pi}{q})\cos(\frac{\pi}{r}) - 1)}}$$

Comments

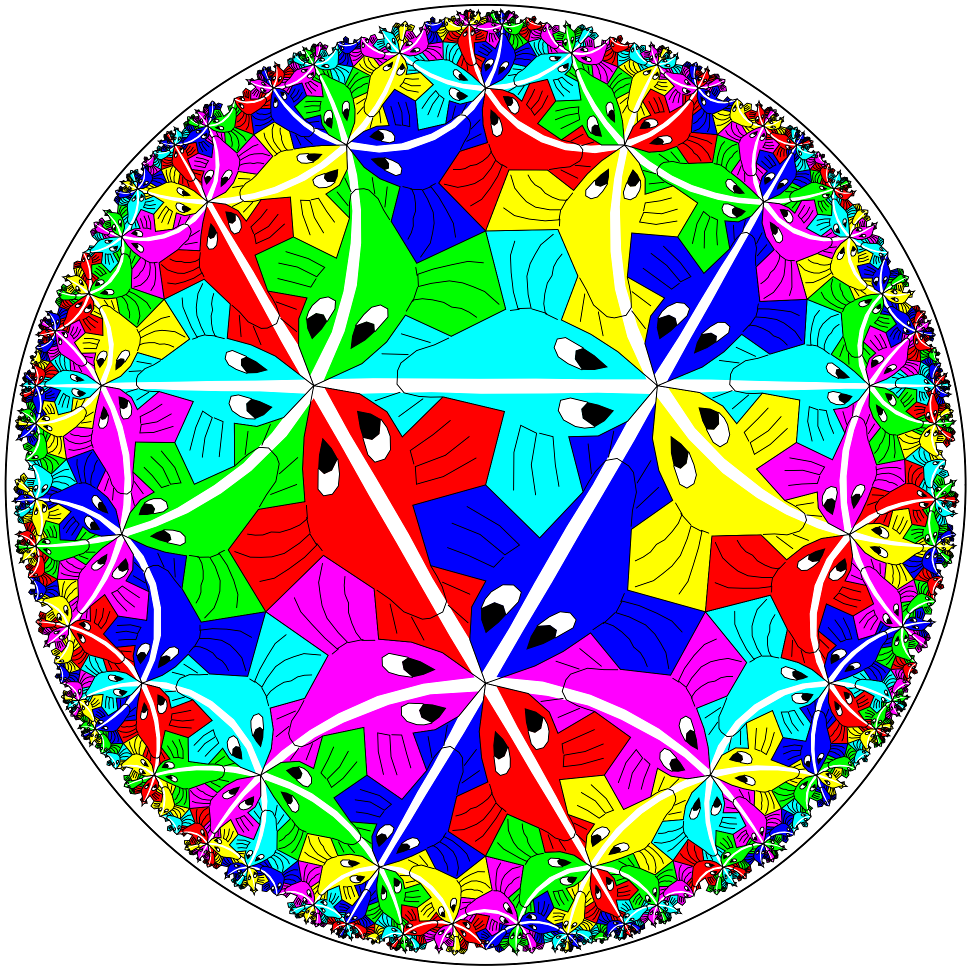
1. Substituting $q = r = 3$ into the formula and some manipulation gives the same formula as in Dunham's 2006 Brigdes paper.
2. If $p \neq q$, calculating $\tan\omega$ gives the following alternative formula:

$$\tan(\omega) = \cot(\frac{\pi}{2r})\sqrt{(}1+\frac{(4\cos(\frac{\pi}{p})\cos(\frac{\pi}{q}) + 2\cos(\frac{\pi}{r}) - 2)}{(\cos(\frac{\pi}{p}) - \cos(\frac{\pi}{q}))^2})$$

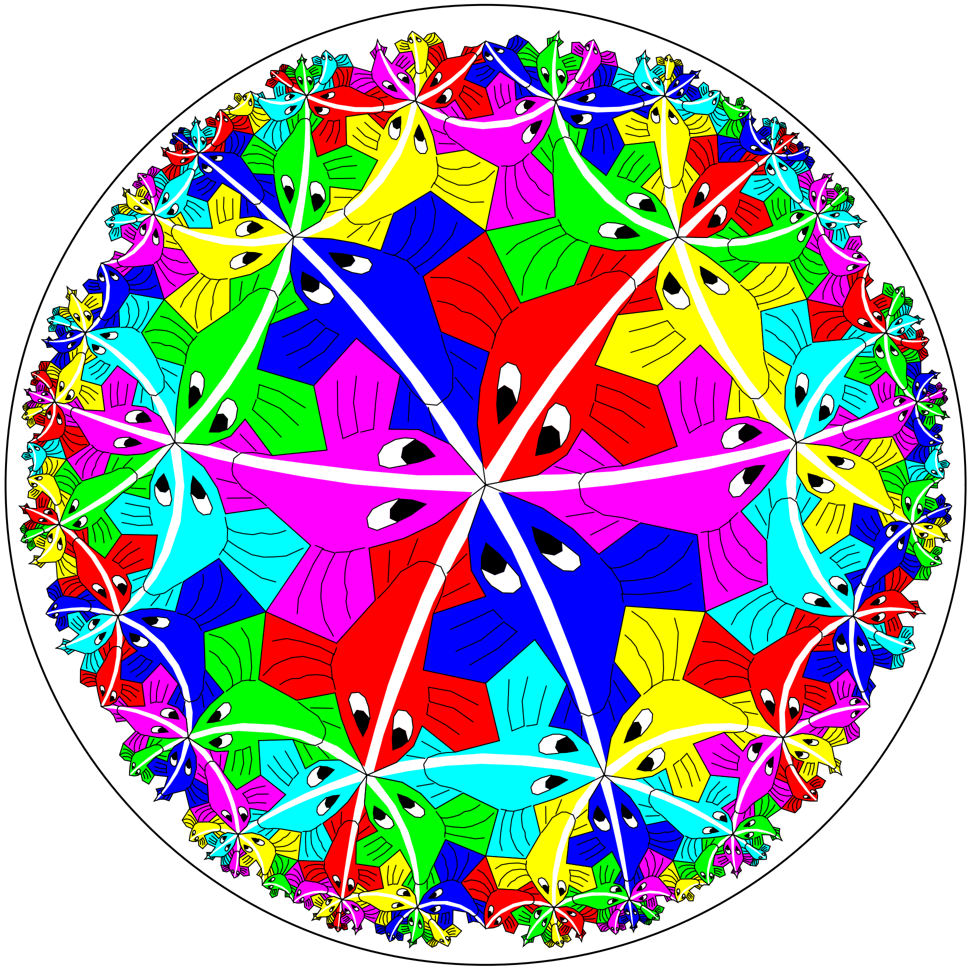
A (3,4,3) Pattern



A (3,5,3) Pattern.



A Nose-Centered (5,3,3) Pattern.



Future Work

- Write software to automatically convert the motif of a (p,q,r) pattern to a (p’,q’,r’) motif.
- Investigate patterns in which one of *q* or *r* (or both) is infinity. Also, extend the current program to draw such patterns.
- Find an algorithm for computing the minimum number of colors needed for a (p,q,r) pattern as in *Circle Limit III*: all fish along a backbone line are the same color, and adjacent fish are different colors (the “map-coloring principle”).