Joint Mathematics Meetings 2014

Patterns with Color Symmetry on Triply Periodic Polyhedra

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Outline

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- Hyperbolic geometry and regular tessellations
- Relation between periodic polyhedra and regular tessellations
- Color symmetry
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- Colored fish on the $\{6,4|4\}$ polyhedron
- A Pattern of fish on the $\{6,6|3\}$ polyhedron
- A Pattern of butterflies on a $\{3,8\}$ polyhedron
- Butterflies on another $\{3,8\}$ polyhedron
- A Pattern of fish on a $\{3,8\}$ polyhedron
- Future research

Background

- Over the past two years we have constructed several triply periodic polyhedra with patterns on them.
- Prior to that, we constructed many repeating patterns in the hyperbolic plane, and as we will see there is a relationship between such patterns and patterns on triply periodic polyhedra.
- So far we have not discussed the color symmetry of the patterns on our triply periodic polyhedra. We rectify that in this talk.

Triply Periodic Polyhedra

- A *triply periodic polyhedron* is a (non-closed) polyhedron that repeats in three different directions in Euclidean 3-space.
- We will consider the special case of triply periodic polyhedra whose faces are all regular p-sided polygons and which are *uniform*: there is a symmetry of the polyhedron that takes any vertex to any other vertex.
- We will also discuss a speciallization of uniform triply periodic polyhedra: *regular* triply periodic polyhedra which are "flag-transitive" — there is a symmetry of the polyhedron that takes any vertex, edge containing that vertex, and face containing that edge to any other such (vertex, edge, face) combination.
- In 1926 John Petrie and H.S.M. Coxeter proved that there are exactly three regular triply periodic polyhedra, which Coxeter denoted $\{4, 6|4\}$, $\{6, 4|4\}$, and $\{6, 6|3\}$, where $\{p, q|r\}$ denotes a polyhedron made up of p-sided regular polygons meeting q at a vertex, and with regular r-sided holes.

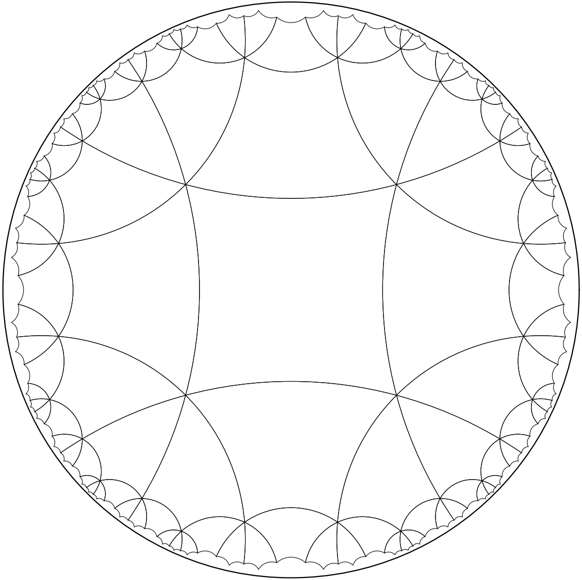
Hyperbolic Geometry and Regular Tessellations

- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.
- Thus we must use *models* of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.
- One such model is the *Poincaré disk model*. The hyperbolic points in this model are represented by interior point of a Euclidean circle — the *bounding circle*. The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (with diameters as special cases).
- This model is appealing to artests since (1) angles have their Euclidean measure (i.e. it is conformal), so that motifs of a repeating pattern retain their approximate shape as they get smaller toward the edge of the bounding circle, and (2) it can display an entire pattern in a finite area.

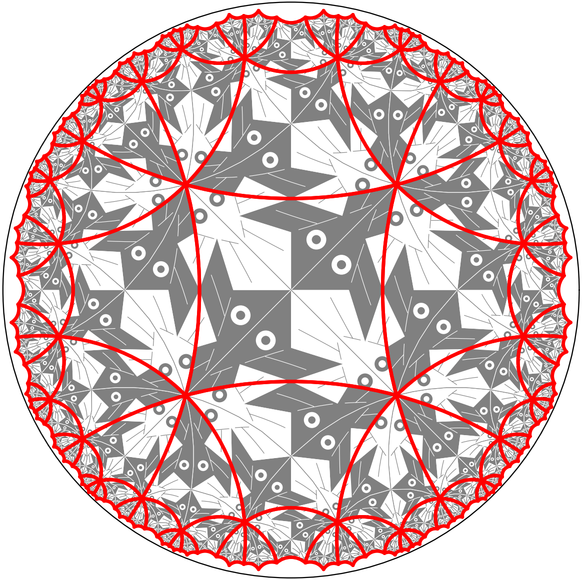
Repeating Patterns and Regular Tessellations

- A *repeating pattern* in any of the 3 "classical geometries" (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or *motif*.
- The *regular tessellation*, $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{6, 4\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

The Regular Tessellation $\{4, 6\}$



The tessellation $\{4, 6\}$ superimposed on a pattern of angular fish used to decorate the $\{4, 6|4\}$ polyhedron of the title slide



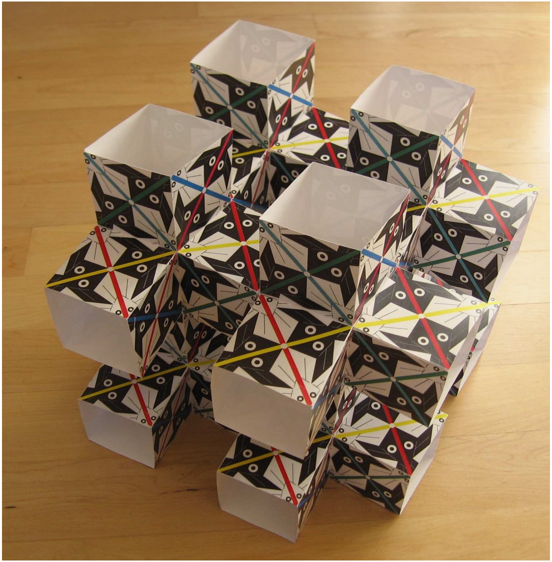
Relation between periodic polyhedra and regular tessellations — a 2-Step Process

- (1) Some triply periodic polyhedra approximate TPMS's.
- (2) As a minimal surface, a TPMS has negative curvature (except for isolated points of zero curvature), and so its universal covering surface also has negative curvature and thus has the same large-scale geometry as the hyperbolic plane.

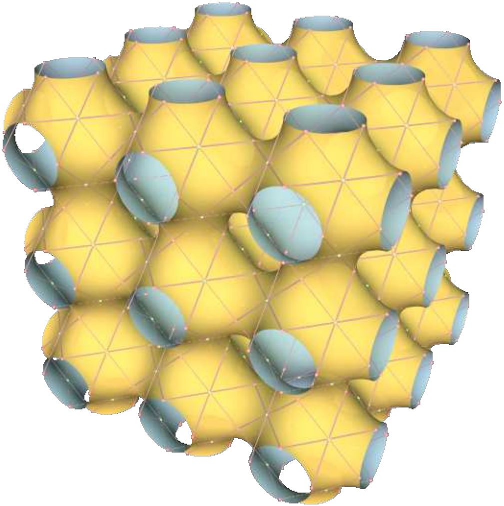
  So the polygons of the triply periodic polyhedron can be transferred to the polygons of a corresponding regular tessellation of the hyperbolic plane.

  Also, if the polyhedron has a pattern on it that lifts to a repeating pattern on the hyperbolic plane, we call that pattern the *universal covering pattern* of the patterned polyhedron.
- We show this relationship in the next slides.

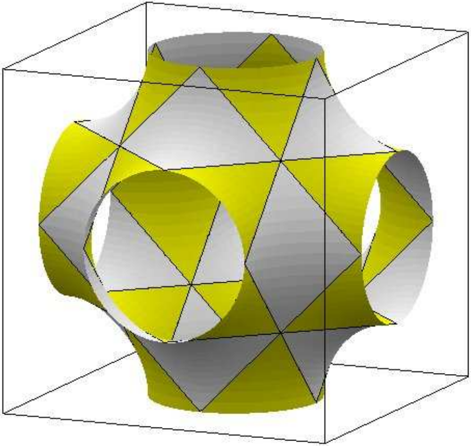
Angular fish on the triply periodic polyhedron $\{4, 6|4\}$ — showing colored embedded lines



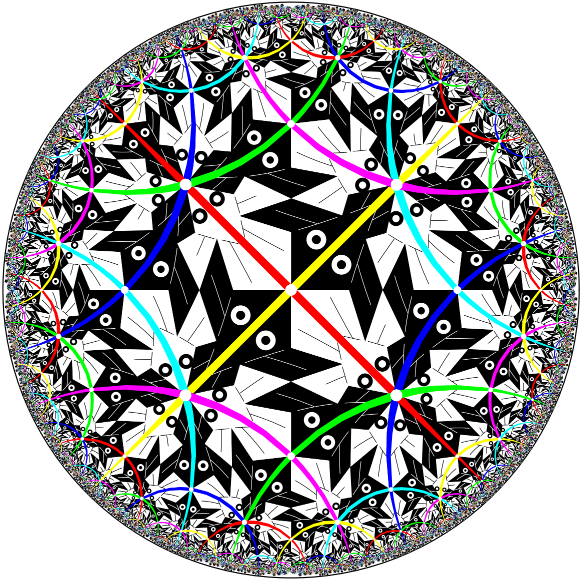
Schwarz's P-surface — approximated by the previous triply periodic polyhedron, and showing corresponding embedded lines



A close-up of Schwarz's P-surface showing corresponding embedded lines and "skew rhombi"



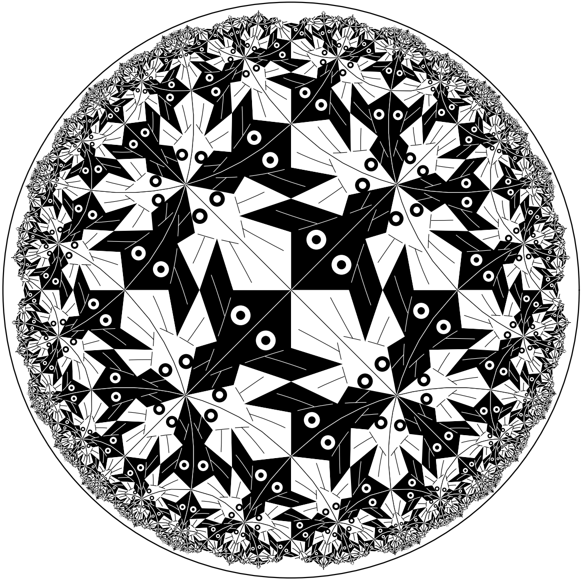
The angular fish polyhedron "unfolded" onto a repeating pattern of the hyperbolic plane — showing the embedded lines as hyperbolic lines, which bound the "skew rhombi".



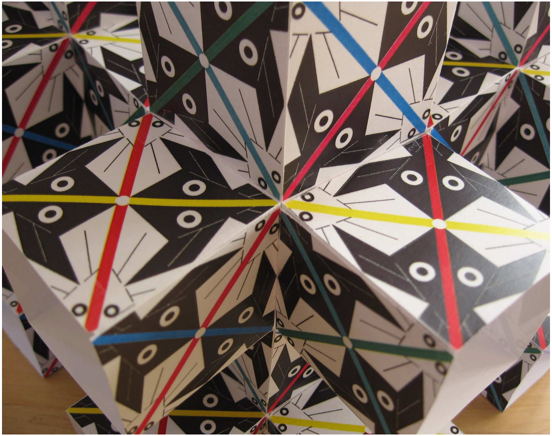
Color Symmetry

- A colored repeating plane pattern has (*perfect*) *color symmetry* if each symmetry of the pattern disregarding color causes a permutation of the colors of the motifs. The "plane" can either be Euclidean or hyperbolic.
- We can extend this notion to colored patterns on triply periodic polyhedra: a patterned polyhedron has color symmetry if each symmetry of the polyhedron in 3-space permutes the colors of the motifs comprising the pattern.

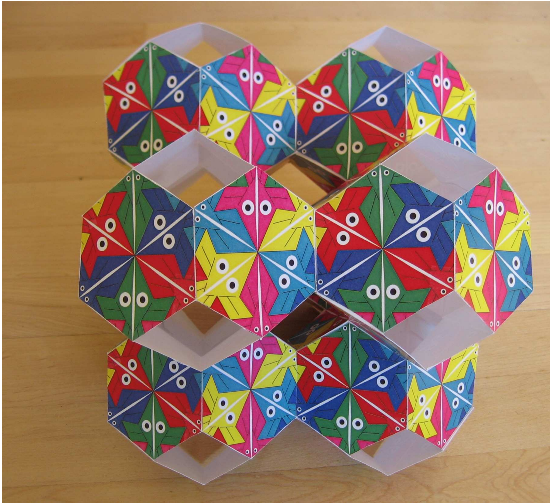
A pattern of angular fish with only trivial color symmetry



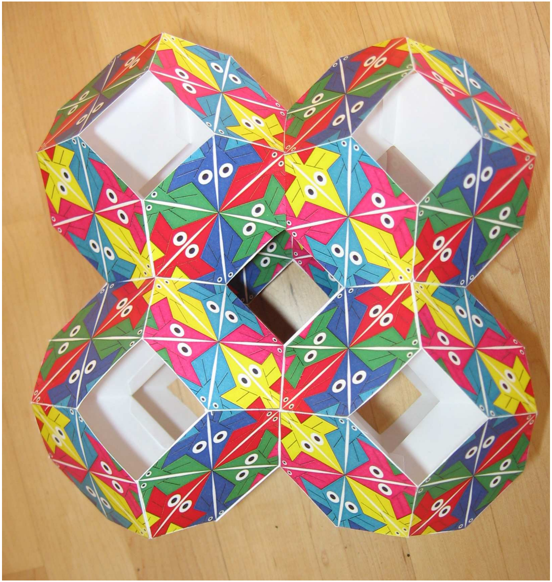
A close-up of a vertex of the angular fish polyhedron showing the color symmetry of the backbone lines



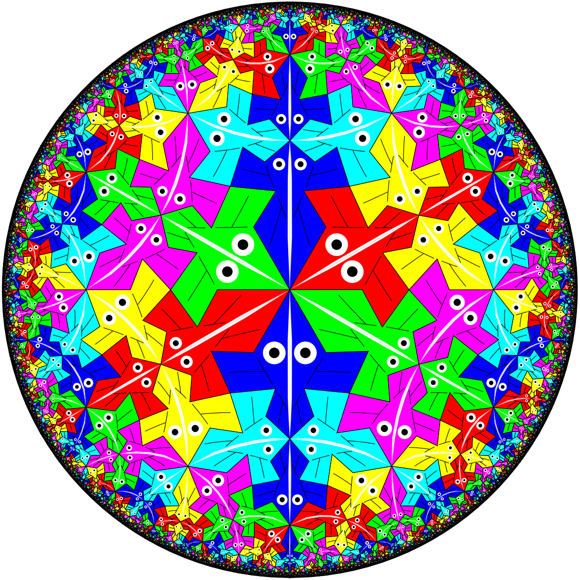
Colored Fish on the $\{6, 4|4\}$ Polyhedron



A top view of the $\{6, 4|4\}$ polyhedron with fish

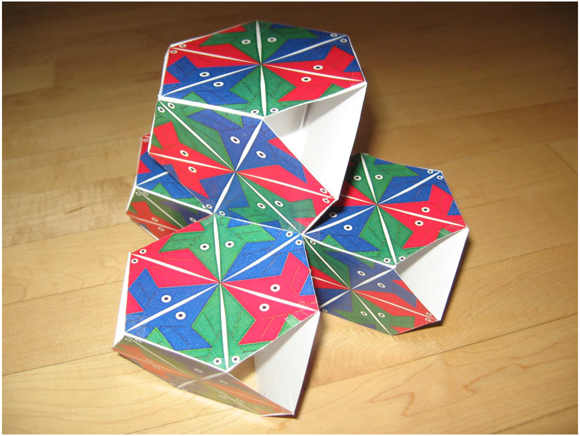


The corresponding hyperbolic pattern of fish — a version of Escher's Circle Limit I pattern with 6-color symmetry

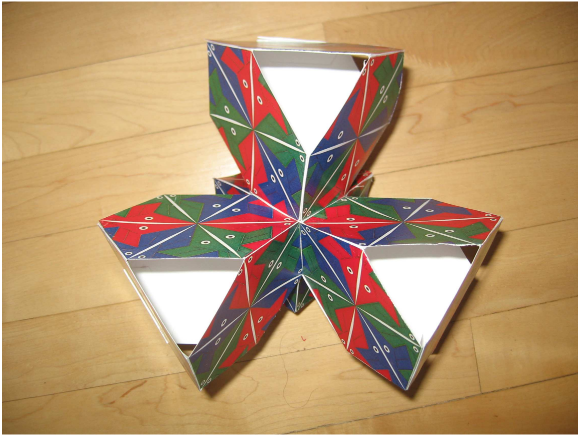


A Pattern of Fish on the $\{6, 6|3\}$ Polyhedron

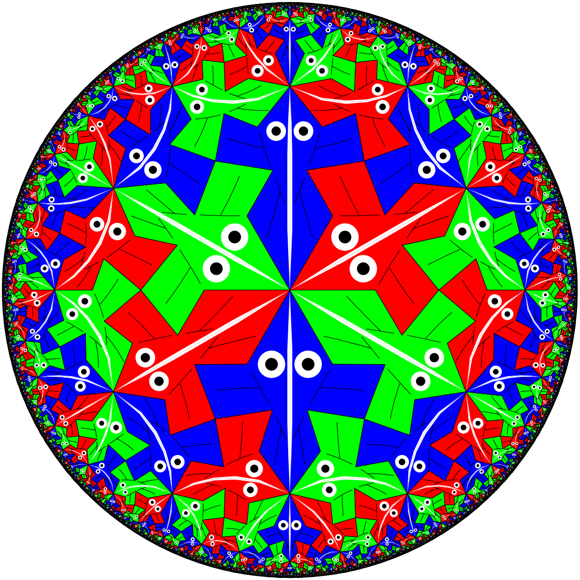
This patterned polyhedron clearly has 3-color symmetry.



A top view of the fish on the $\{6, 6|3\}$ polyhedron — showing how a 3-fold rotation about the vertex permutes the colors.



The corresponding hyperbolic pattern of fish — based on the $\{6,6\}$ tessellation



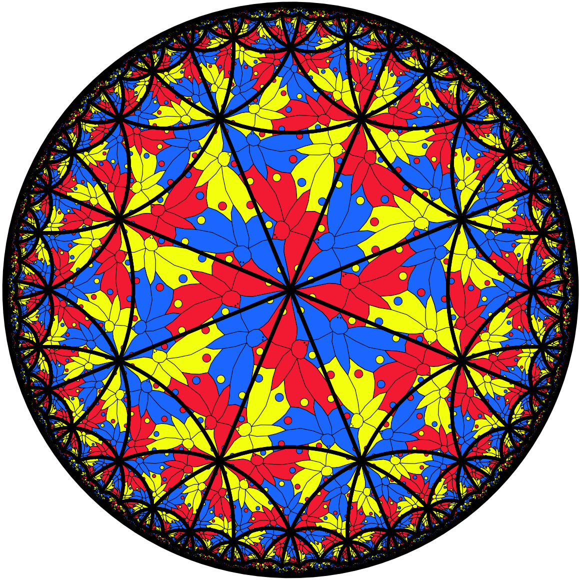
A Pattern of Butterflies on a $\{3, 8\}$ Polyhedron

This butterfly pattern was inspired by Escher's Regular Division Drawing # 70. This polyhedron is related to Schwarz's D-surface, a TPMS with the topology of a thickened diamond lattice. We show:

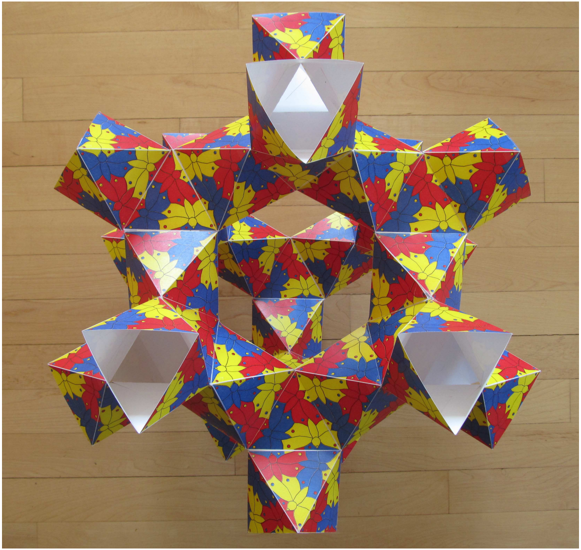
- Escher's Regular Division Drawing # 70.
- A hyperbolic pattern of butterflies based on the $\{3, 8\}$ tessellation — the "universal covering pattern" of the patterned polyhedron.
- A view of the patterned $\{3, 8\}$ polyhedron.
- A close-up of a vertex of the patterned polyhedron.

Escher's Regular Division Drawing # 70

A pattern of butterflies based on the $\{3,8\}$ tessellation — the "universal covering pattern" for the polyhedron.



The $\{3, 8\}$ polyhedron with butterflies, having 2-fold and 3-fold color symmetries.



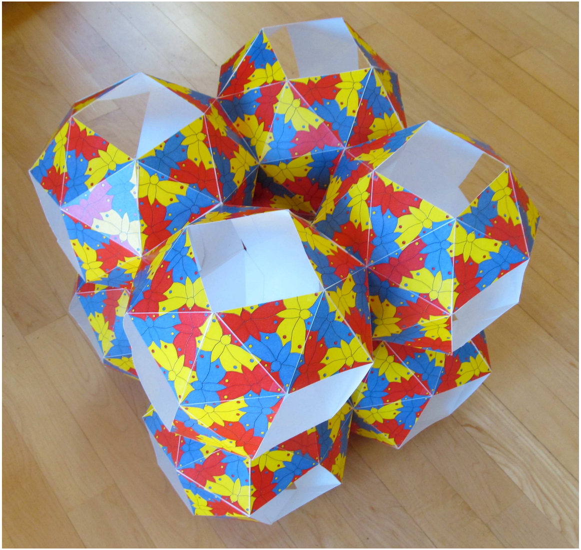
A close-up of a vertex of the patterned polyhedron, looking down a 2-fold axis of symmetry.

Butterflies on Another $\{3, 8\}$ Polyhedron

We show the pattern of butterflies on a different the triply periodic $\{3, 8\}$ polyhedron. This butterfly pattern was also inspired by Escher's Regular Division Drawing # 70. Thus the hyperbolic "covering pattern" is the same as for the previous polyhedron. This polyhedron has the same topology as Schwarz's P-surface, a TPMS with the topology of a thickened version of the 3-D coordinate lattice. We show:

- The $\{3, 8\}$ polyhedron, which is made up of snub cubes arranged in a cubic lattice, attached by their (missing) square faces, and alternating between left-handed and right-handed versions.
- A close-up of a vertex of the patterned polyhedron.

Another Patterned $\{3, 8\}$ Polyhedron —
with only trivial color symmetry.



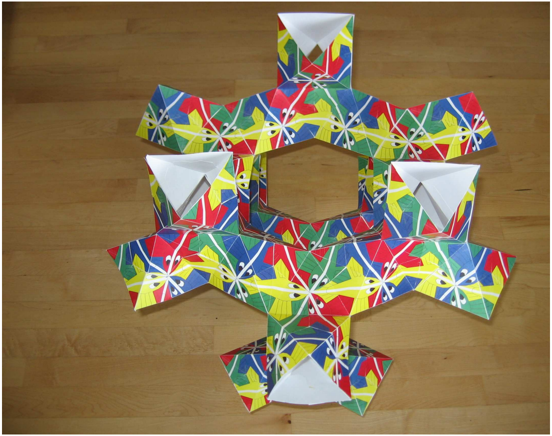
A close-up of a vertex of the patterned polyhedron.

A Pattern of Fish on a $\{3, 8\}$ Polyhedron

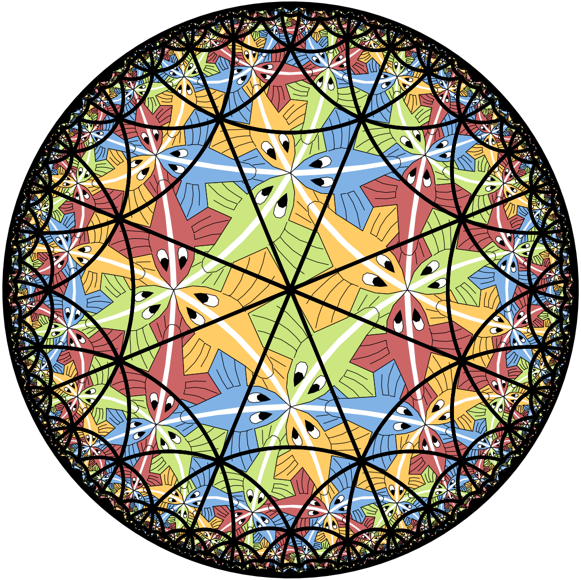
We show a pattern of fish on the first of the $\{3, 8\}$ polyhedra above. This pattern was inspired by Escher's hyperbolic print *Circle Limit III* (which is based on the regular $\{3, 8\}$ tessellation). The red, green, and yellow fish swim along those lines (the blue fish swim in loops around the "waists"). This polyhedron has only 3-color symmetry since the blue fish cannot be mapped to any other color by a symmetry of the polyhedron. We show:

- The patterned polyhedron.
- Escher's *Circle Limit III* with the equilateral triangle tessellation superimposed.
- Another view of the patterned polyhedron along a 3-fold axis.

The patterned polyhedron



Escher’s Circle Limit III with the equilateral triangle tessellation superimposed



A view down a 3-fold symmetry axis of the polyhedron

Future Work

- Put other patterns on the triply periodic polyhedra shown above.
- Place patterns on other triply periodic polyhedra.

Thank You!

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