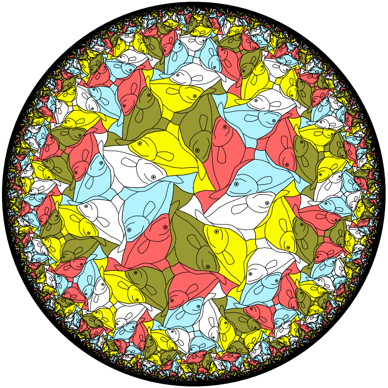
M & D 2010

CREATING FAMILIES OF REPEATING PATTERNS
CREANDO FAMILIAS DE ESQUEMAS REPETITIVOS

Douglas Dunham
University of Minnesota Duluth
Duluth, Minnesota
USA



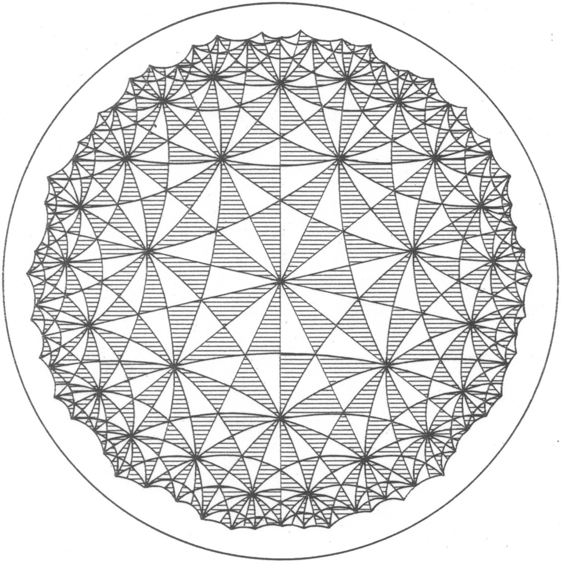
Outline

- Brief history of repeating patterns
- Review of hyperbolic geometry
- Repeating patterns and regular tessellations
- A family of fish patterns
- The families of Escher's "Circle Limit" patterns
- A family of lizard patterns
- A family of butterfly patterns
- Future work

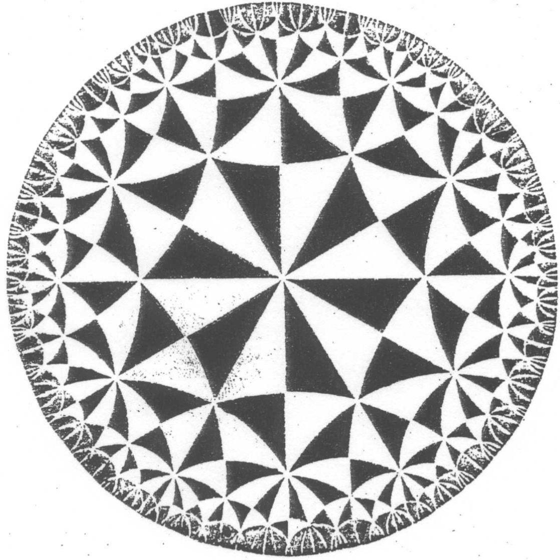
History

- People have created repeating Euclidean and spherical patterns for thousands of years.
- Hyperbolic geometry, the third "classical geometry", was discovered by Bolyai, Gauss, and Lobachevsky in the 1820's.
- In the late 1800's mathematicians started creating repeating hyperbolic patterns.
- In 1957 the Canadian mathematician H.S.M. Coxeter sent M.C. Escher a hyperbolic triangle pattern.
- With that inspiration, Escher became the first artist to create hyperbolic patterns — his 4 "Circle Limit" patterns from 1958 to 1960.
- In the late 1970's and early 1980's the first computer programs were written to draw repeating hyperbolic patterns.

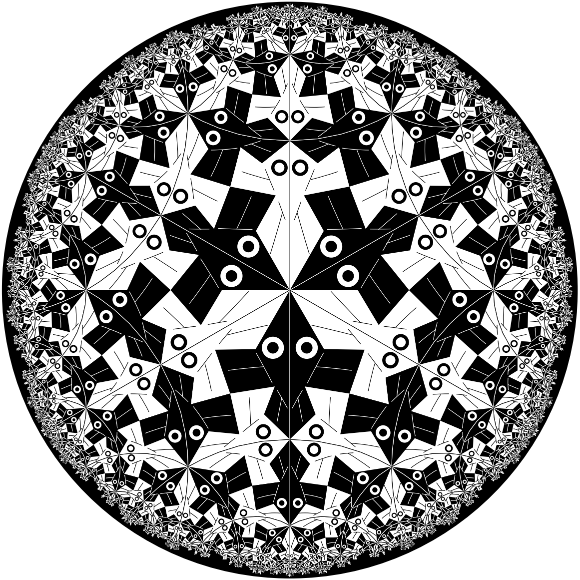
Figure: F. Klein, R. Fricke, 1890.



H.S.M. Coxeter's Figure 7
in: Crystal Symmetry and Its Generalizations
Trans. Royal Soc. of Canada, 1957.

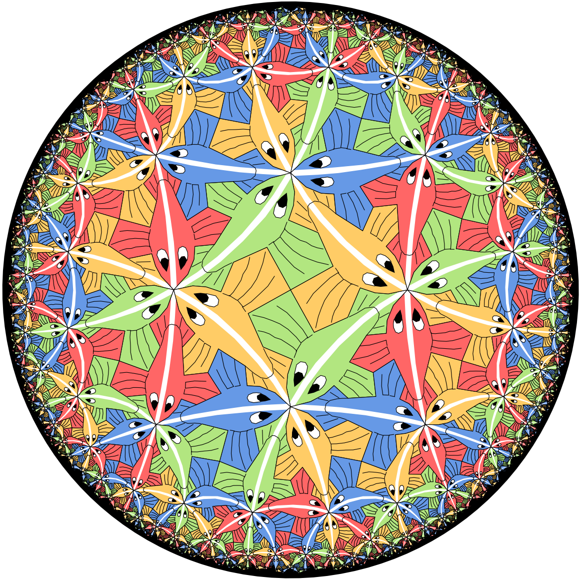


M.C. Escher's "Circle Limit" Patterns
Circle Limit I, 1958.

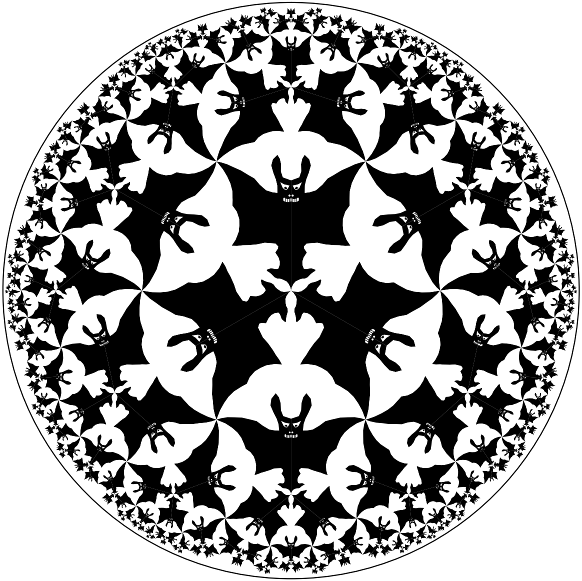


Circle Limit II, 1959.

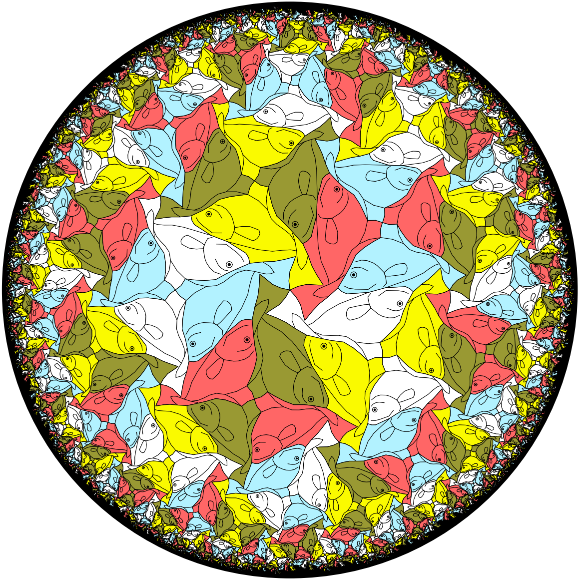
Circle Limit III, 1959.



Circle Limit IV, 1960.



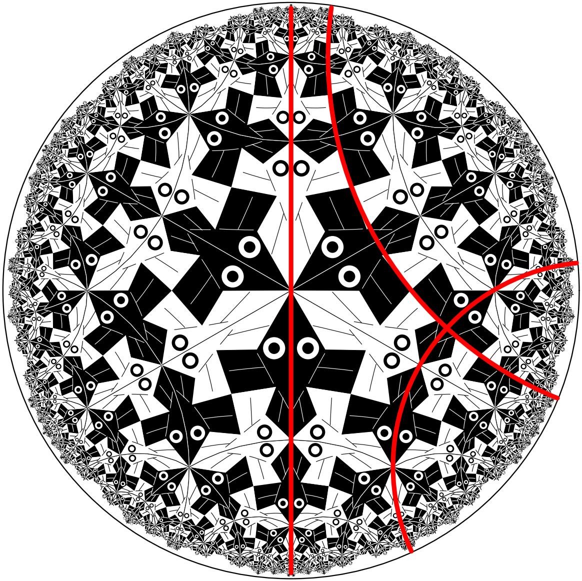
Computer Generated Fish Pattern, 1980's
Inspired by Escher's Notebook Drawing 20



Hyperbolic Geometry

- In 1901, David Hilbert proved that, unlike the sphere, there was no isometric (distance-preserving) embedding of the hyperbolic plane into ordinary Euclidean 3-space.
- This is probably the reason for its late discovery.
- Thus we must use *models* of hyperbolic geometry in which Euclidean objects have hyperbolic meaning, and which must distort distance.
- One such model, used by Escher, is the *Poincaré disk model*.
- The hyperbolic points in this model are represented by interior point of a Euclidean circle — the *bounding circle*.
- The hyperbolic lines are represented by (internal) circular arcs that are perpendicular to the bounding circle (including diameters as special cases).
- This model was preferred by Escher since (1) angles have their Euclidean measure (i.e. it is conformal), so that motifs of a repeating pattern retain their approximate shape as they get smaller toward the edge of the bounding circle, and (2) it could display an entire pattern in a finite area.

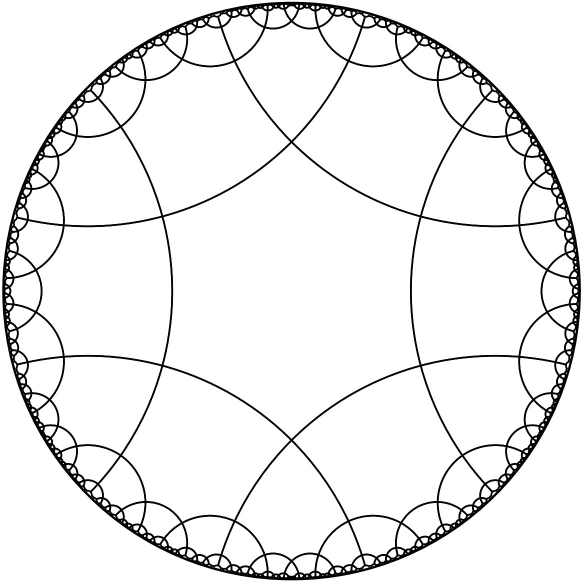
Figure: Escher's *Circle Limit I* showing hyperbolic lines.



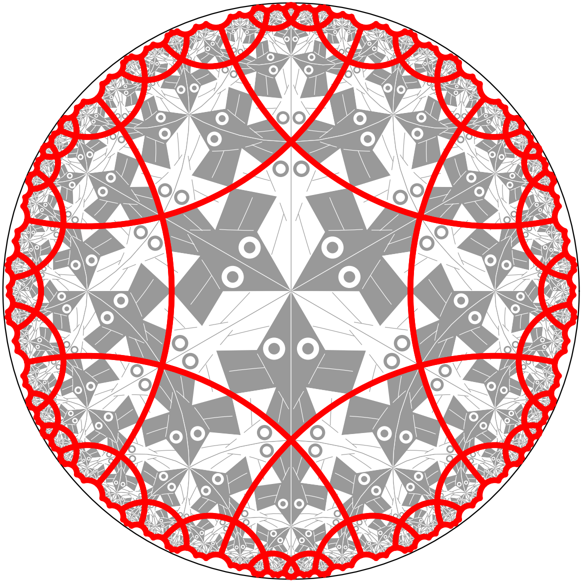
Repeating Patterns and Regular Tessellations

- A *repeating pattern* in any of the 3 "classical geometries" is composed of congruent copies of a basic subpattern or *motif*.
- For example if we ignore color, one fish is a motif for *Circle Limit III* above.
- The *regular tessellation*, $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{6, 4\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

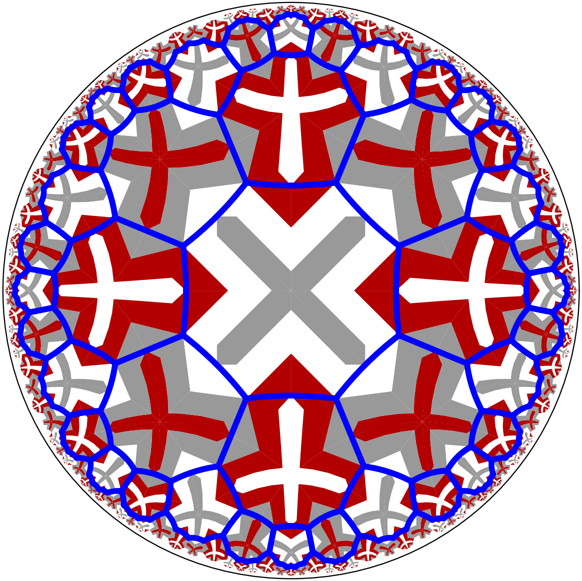
The $\{6, 4\}$ tessellation.



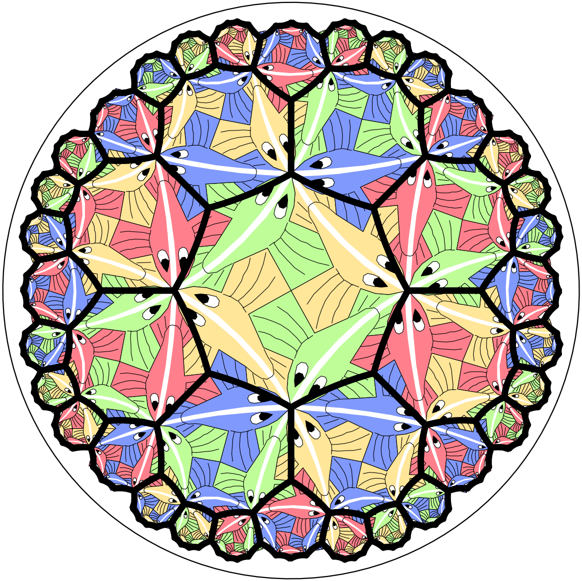
The $\{6, 4\}$ tessellation underlying *Circle Limit I*



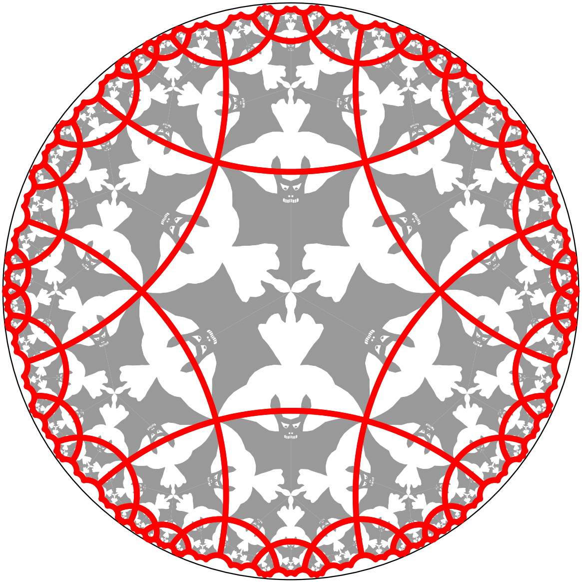
The $\{8, 3\}$ tessellation underlying *Circle Limit II*



The $\{8, 3\}$ tessellation underlying *Circle Limit III*



The $\{6, 4\}$ tessellation underlying *Circle Limit IV*



A Table of the Regular Tessellations

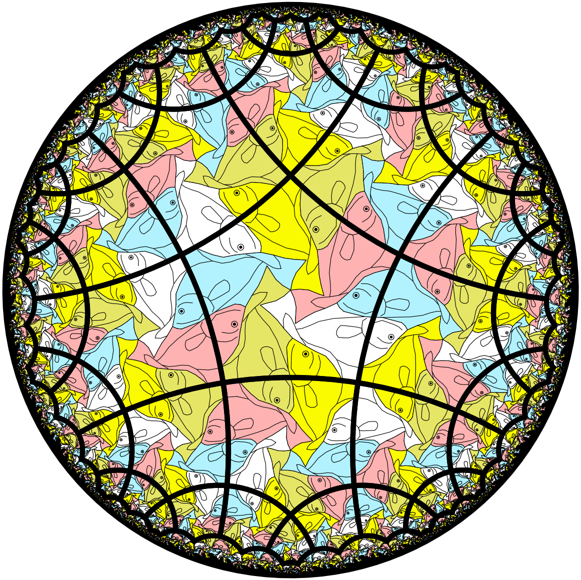
q							
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
8	$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$
7	$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$
6	□	$*$	$*$	$*$	$*$	$*$	$\cdots$
5	○	$*$	$*$	$*$	$*$	$*$	$\cdots$
4	○	□	$*$	$*$	$*$	$*$	$\cdots$
3	○	○	○	□	$*$	$*$	$\cdots$
	3	4	5	6	7	8	$\cdots$
				p			

□ - Euclidean tessellations

○ - spherical tessellations

$*$ - hyperbolic tessellations

The $\{5,4\}$ tessellation underlying the fish pattern



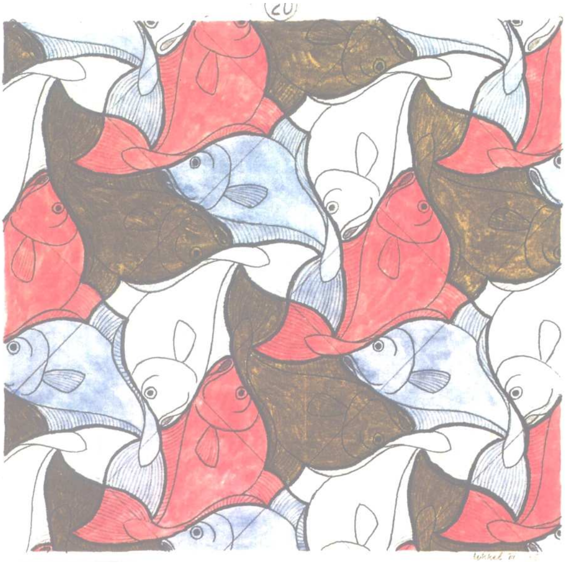
Families of Patterns

- If a pattern is based on an underlying $\{p, q\}$ tessellation, we can conceive of other patterns with the same motif (actually slightly distorted) based on a different tessellation $\{p', q'\}$.
- This observation leads us to consider an whole *family* of such patterns indexed by p and q.
- We use (p, q) to denote the pattern of the family that is based on $\{p, q\}$.
- For example, the previous fish pattern would be denoted $(5, 4)$.

A Family of Fish Patterns

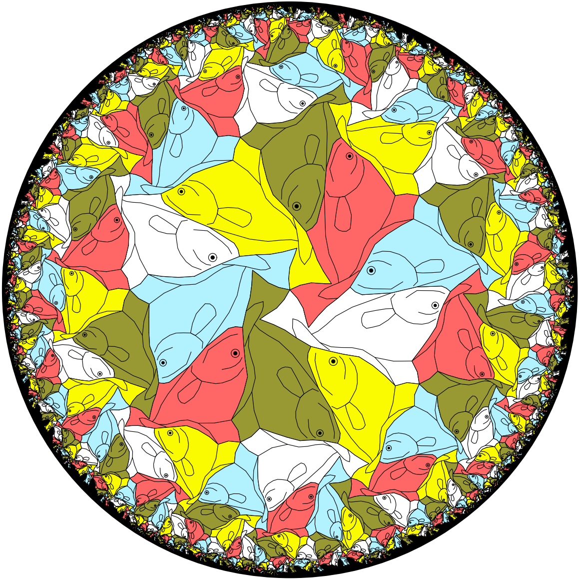
- Theoretically, we can create a fish pattern (p, q) like the one above for any values of p and q provided $p \geq 3$ and $q \geq 3$.
- For these patterns, p is the number of fish that meet their tails and q is the number of fish that meet at their dorsal fins.
- This family of fish patterns is based on Escher's Notebook Drawing Number 20 (1938) which is based on the Euclidean "square" tessellation $\{4, 4\}$, and is shown in the following slide.
- Escher also created a spherical version of this pattern based on the tessellation $\{4, 3\}$, and is also shown below.
- Unfortunately large values of p or q or both do not produce aesthetically appealing patterns, since such values lead to distortion of the motif and/or push most of the pattern outward near the bounding circle.

Escher's Notebook Drawing Number 20 — a $(4,4)$ pattern

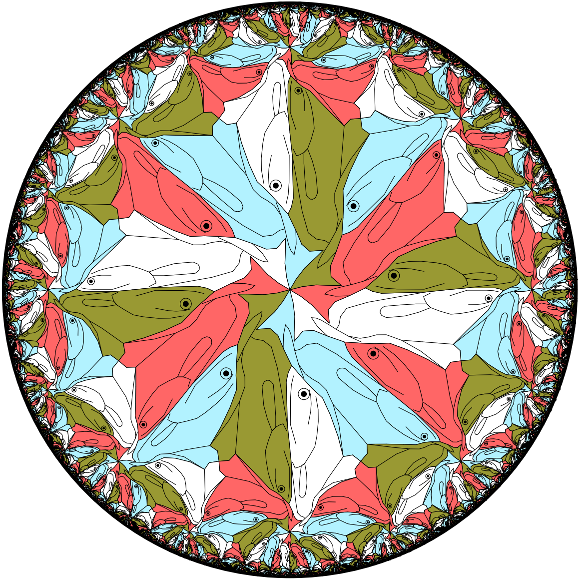


Escher's $(3,3)$ fish pattern on a sphere

A $(5, 5)$ fish pattern.



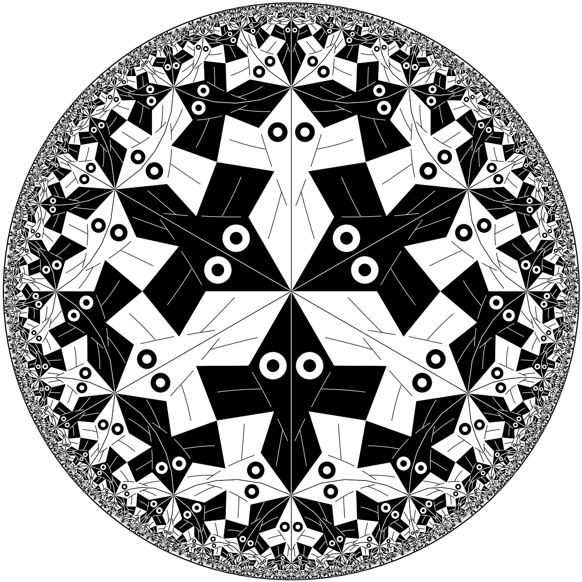
An $(8, 4)$ pattern of distorted fish



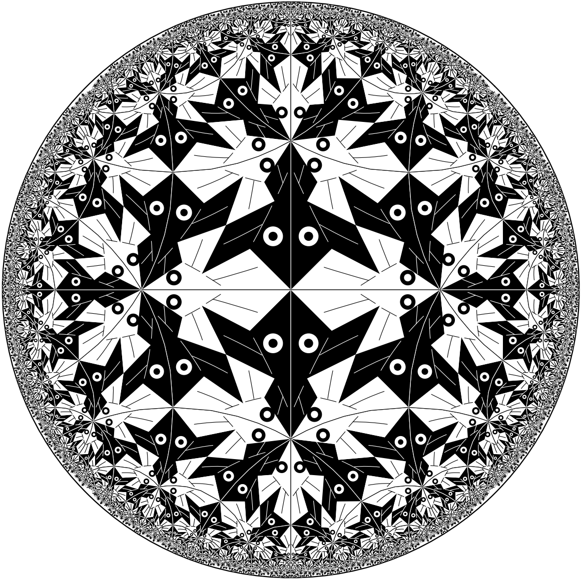
The *Circle Limit I* Family of Patterns

- Unlike the preceding family of fish patterns, for a *Circle Limit I* pattern based on a $\{p, q\}$ tessellation, both p and q must be even.
- For these patterns, $p/2$ is the number of black fish meeting at their noses and $q/2$ is the number of white fish that meeting at noses.
- For this family, we let $(p/2, q/2)$ denote the pattern based on the $\{p, q\}$ tessellation.
- *Circle Limit I* would be $(3, 2)$ in this notation.

A $(3,3)$ *Circle Limit I* pattern.



A $(2,3)$ *Circle Limit I* pattern.

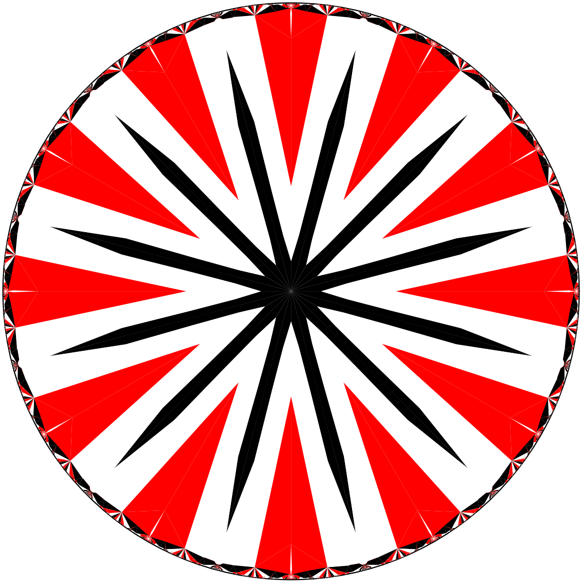


The *Circle Limit II* Family

- For a pattern based on a $\{p, q\}$ tessellation in the *Circle Limit II* family p must be even but there is no restriction on q.
- For these patterns, $p/2$ is the number of arms of the crosses, and q is the number of background crosses that meet near their ends.
- For this family, we let $(p/2, q)$ denote the pattern based on the $\{p, q\}$ tessellation.
- *Circle Limit II* would be $(4, 3)$ in this notation.
- Since the motif is simple for this family, large values of p or q can produce interesting patterns.

A $(5, 3)$ *Circle Limit II* pattern.

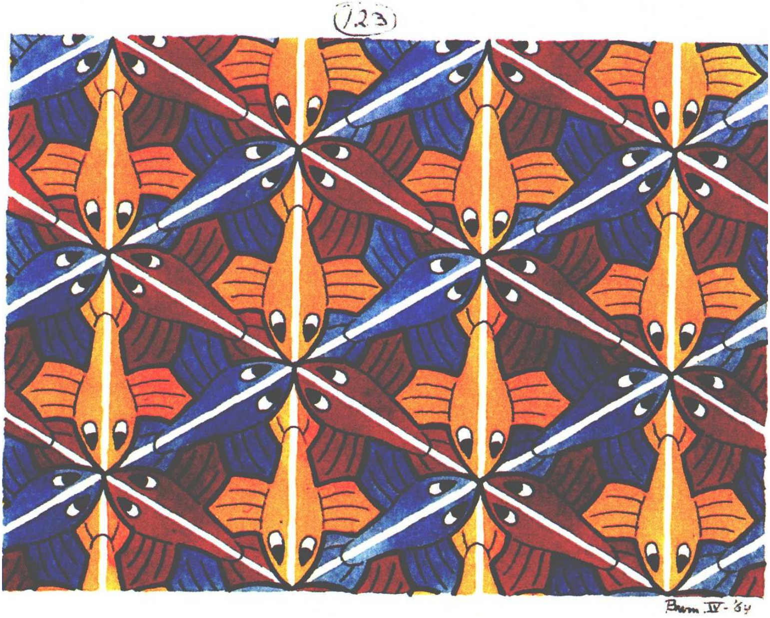
A $(12,12)$ *Circle Limit II* pattern.



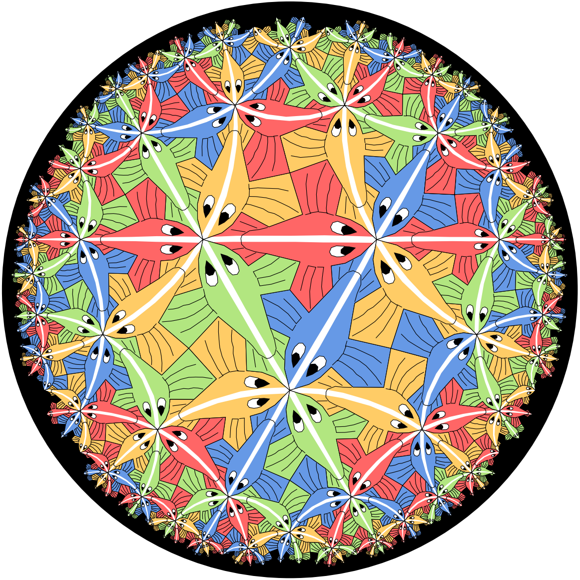
The *Circle Limit III* Family

- The *Circle Limit III* family of patterns depends on 3 numbers, p, q, and r.
- For these patterns, p and q are the numbers of fish meeting at right and left fin tips, respectively, and r is the number of fish meeting at noses.
- There is no restriction on p or q, but r must be odd so that the fish swim head-to-tail. Of course p, q, and r must be at least 3. We let (p, q, r) denote such a pattern.
- If $1/p + 1/q + 1/r < 1$, the pattern will be hyperbolic.
- If $<$ is replaced by $=$ or $>$ we could could theoretically obtain a Euclidean or spherical pattern, respectively. There are no spherical patterns in this family, and only one Euclidean pattern, $(3, 3, 3)$, which Escher realized as Notebook Drawing 123.
- *Circle Limit III* would be $(4, 3, 3)$ in this notation.

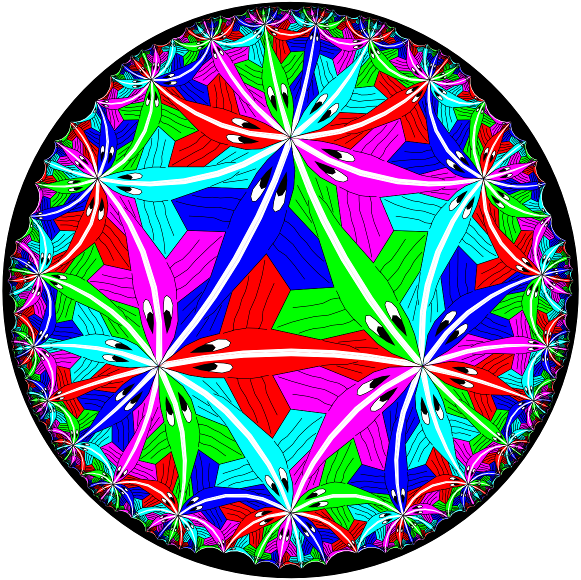
Escher's $(3, 3, 3)$ pattern — Notebook Drawing 123.



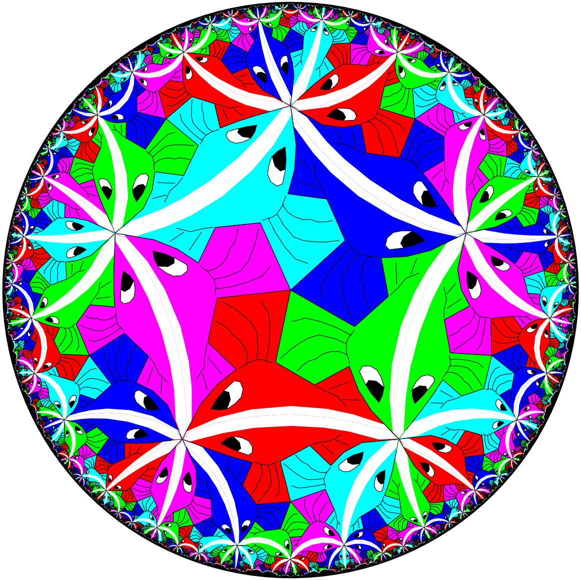
A $(3,4,3)$ *Circle Limit III* pattern.



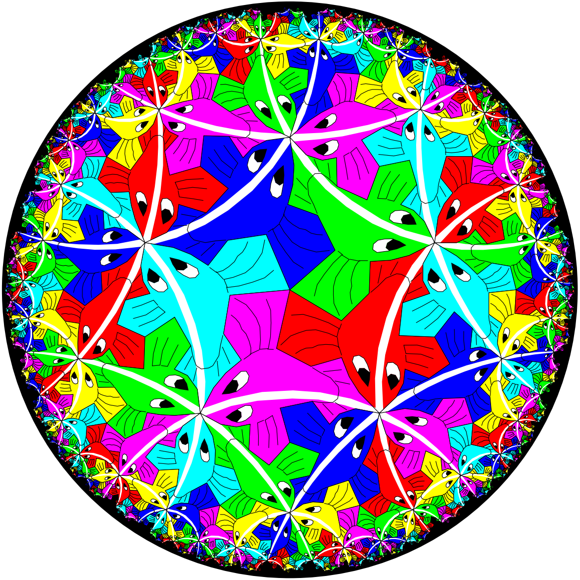
A $(3, 3, 5)$ *Circle Limit III* pattern.



A $(5, 5, 3)$ *Circle Limit III* pattern.



A $(5, 3, 3)$ *Circle Limit III* pattern.

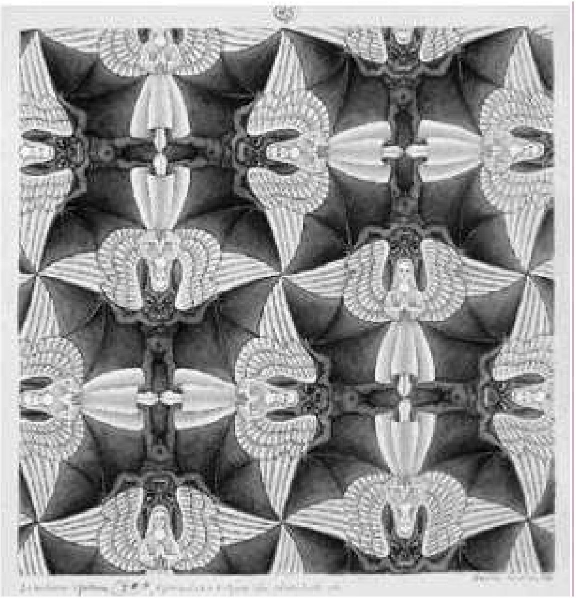


The *Circle Limit IV* Family

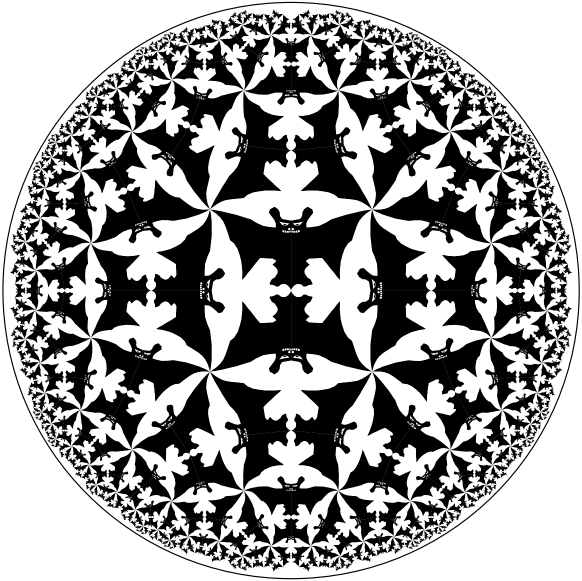
- For a pattern based on a $\{p, q\}$ tessellation in the *Circle Limit IV* family, p must be even but there is no restriction on q (the same as for the *Circle Limit II* family).
- For these patterns, $p/2$ is the number of of angels or devils meeting at their feet, and q is the number angels or devils that meet at their wingtips.
- For this family, we let $(p/2, q)$ denote the pattern based on the $\{p, q\}$ tessellation.
- Thus *Circle Limit IV* would be denoted $(3, 4)$.
- This is the only family for which Escher provided 3 examples, one in each of the 3 "classical geometries".

Escher's spherical $(2, 3)$ *Circle Limit IV* pattern.

Escher's Euclidean Notebook Drawing 25 —
A $(2, 4)$ *Circle Limit IV* pattern.



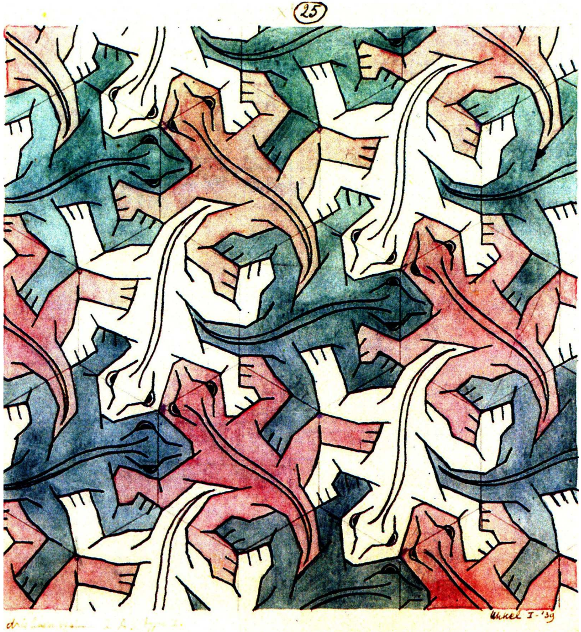
A $(2,5)$ *Circle Limit IV* pattern.



A Lizard Pattern Family

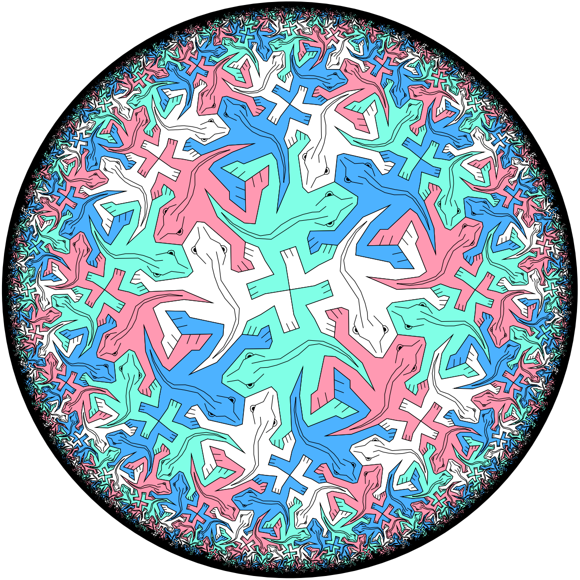
- Similarly to the *Circle Limit III* family, the family of lizard patterns depends on 3 numbers, p, q, and r, the number of lizards meeting at their heads, right knees, and left rear feet, respectively.
- There are no restrictions on p, q, or r (except that p, q, and r must be at least 3). We let (p, q, r) denote such a pattern.
- If $1/p + 1/q + 1/r < 1$, the pattern will be hyperbolic.
- If $<$ is replaced by $=$ or $>$ we could could theoretically obtain a Euclidean or spherical pattern, respectively. There is only one possible Euclidean pattern, $(3, 3, 3)$, which Escher realized as Notebook Drawing 25.
- However, Escher bent the rules slightly to create a spherical $(3, 2, 3)$ pattern in which the the right rear shins meet at a 2-fold rotation point.

Escher's Lizard $(3,3,3)$ pattern — Notebook Drawing 25.



Escher's Lizard $(3, 2, 3)$ Pattern on a Sphere

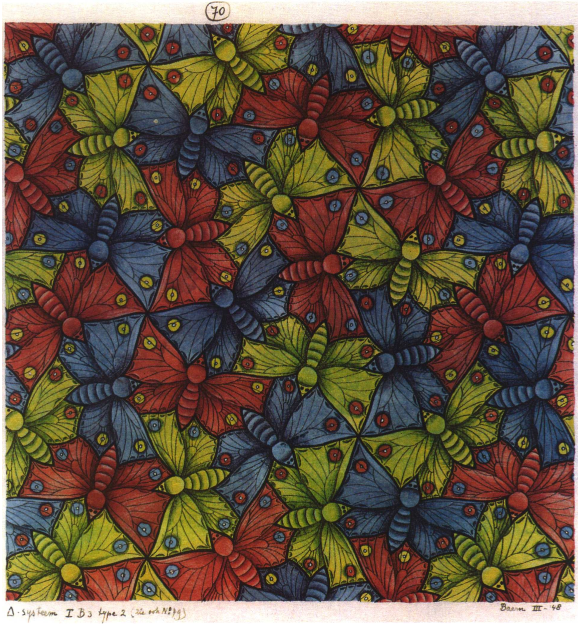
A Hyperbolic Lizard $(3, 4, 3)$ Pattern



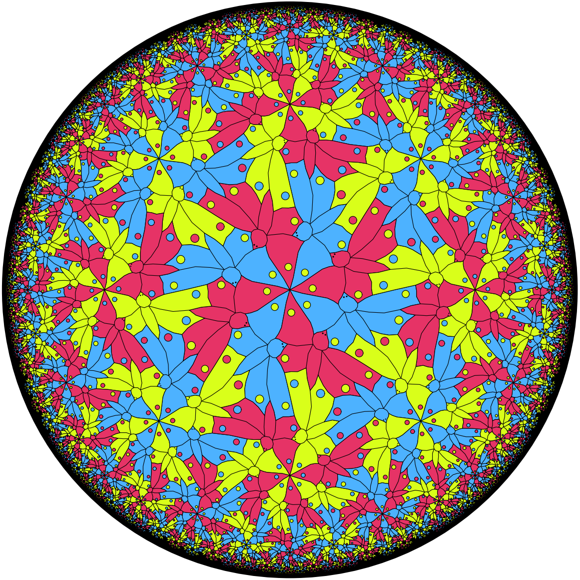
A Butterfly Pattern Family

- The patterns in the butterfly family are based on the $\{p, q\}$ tessellations and just depend on p and q, and there is no restriction except that they must be greater than or equal to 3.
- For these patterns, p is the number of butterflies meeting at left front wingtips, and q is the number of butterflies meeting at their left rear wings. We let (p, q) denote such a pattern.
- Escher only created one pattern in this family, his Euclidean Notebook Drawing 70, which would be called $(6, 3)$ in this notation.

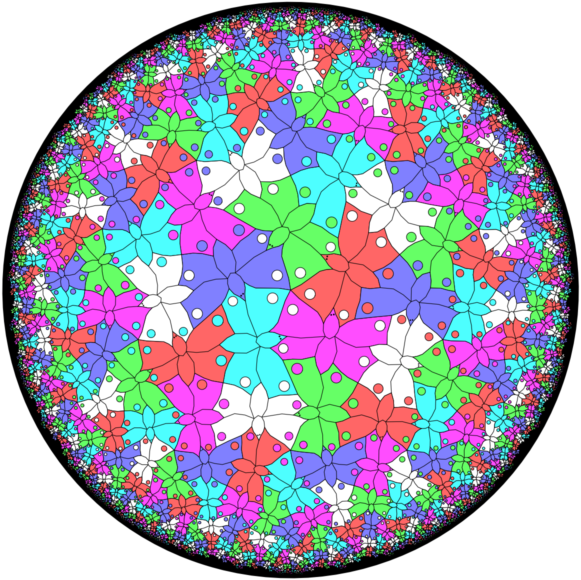
Escher's Butterfly Pattern $(6,3)$ — Notebook Drawing 70



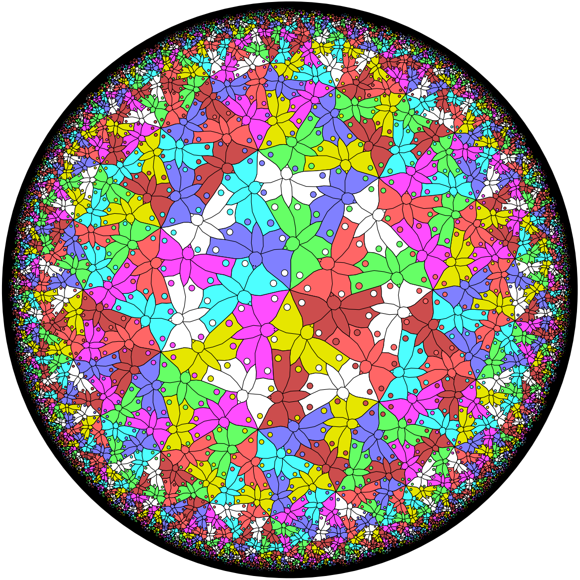
An $(8, 3)$ Butterfly Pattern



A (5, 4) Butterfly Pattern



A $(7,3)$ Butterfly Pattern



Future Work

- Extend the repeating pattern program so that it can also draw Euclidean and spherical patterns.
- Automatically generate the colors so that the pattern is symmetrically colored. Currently this must be done manually for each pattern in a family.
- Make more patterns!

Thank you!
¡Gracias!

Vera,

Organizers, / Organizadores
and/y
All who worked on M & D 2010
Todos los que trabajaron en M & D 2010