Symmetry: Culture and Science
Vol. 26, No. x, page_first-page_last, 2015

ARTISTIC PATTERNS: FROM RANDOMNESS TO SYMMETRY

Douglas Dunham[1,*], John Shier[2]

[1] Mathematician, (b. Pasadena, Calif., U.S.A., 1938).

Address: Department of Computer Science, University of Minnesota Duluth, 1114 Kirby Drive, Duluth Minnesota, 55812, U.S.A. E-mail: ddunham@d.umn.edu .

Fields of interest: Mathematical Art, Geometry, Algorithms.

Publications: with John Shier, The Art of Random Fractals, *Proceedings of Bridges Seoul*, (2014); with John Shier, Fractal Wallpaper Patterns, *Proceedings of Bridges Baltimore*, (2015).

[2] Physicist, (b. Columbus, Ohio, U.S.A., 1938).

Address: 6935 133rd Court, Apple Valley, Minnesota, 55124, U.S.A. E-mail: johnpf99@frontiernet.net. .

Fields of interest: Fractals, Randomness, Geometry, Butterflies.

Publications: with Paul Bourke, An Algorithm for Random Fractal Filling of Space, *Computer Graphics Forum*, (2013); with D. Dunham, The Art of Random Fractals, *Proceedings of Bridges Seoul*, (2014); with D. Dunham, Fractal Wallpaper Patterns, *Proceedings of Bridges Baltimore*, (2015).

Abstract: *We present an algorithm that can create a wide variety of artistic fractal patterns. The algorithm contains a parameter, c, that can be used to control the amount of perceived symmetry a patterns has, from completely random (for small c) to quite symmetric (for c near its upper limit). We describe the algorithm and present sequences of patterns in the spectrum from random to symmetric. As an example, we show patterns of triangles within a large triangle starting with a random arrangement and ending with a pattern approximating the symmetric Sierpinski triangle near the upper limit of c.*

Keywords: fractal, geometry, randomness, symmetry, algorithm

1. INTRODUCTION

In mathematics symmetry is often thought of as a precise attribute. A geometric object is symmetric if it is invariant under translation, rotation, mirroring, etc., and if it has no symmetry operations it is not symmetric. A general feature of such symmetric objects is a complete lack of randomness.

The authors have been exploring space-filling patterns involving randomness recently (Shier and Bourke, 2013; Dunham and Shier, 2014; Dunham and Shier 2015). In the "statistical geometry" algorithm (we use the term "statistical geometry" since the areas of the motifs are statistically distributed according to an inverse power law, and area is a geometric quantity) described below, a sequence of progressively smaller basic shapes or *motifs* (used in the sense of music or art) are placed by non-overlapping random search within a bounding region. The areas of the motifs obey a negative-exponent power law. The area sequence is defined in such a way that the areas of all the motifs sum to the area of the bounding region, i.e., the algorithm is space-filling in the limit of infinitely many shapes. It is observed that the space-filling patterns thus created have a progression from randomness to order and symmetry, controlled by the parameters of the process. Figure 1 shows a pattern of snowflake motifs within a rectangular region.

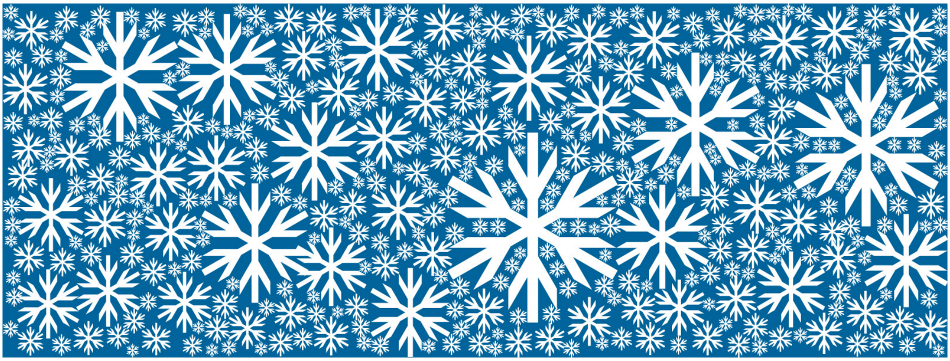

Figure 1: A fractal pattern of snowflake motifs within a bounding rectangle.

2. THE STATISTICAL GEOMETRY ALGORITHM

The statistical geometry algorithm (Shier and Bourke, 2013) places a sequence of shapes/motifs with areas A_0, A_1, … within a bounding region. The motifs are required to obey the *area rule*:

$$A_i = \frac{A}{\zeta(c,N)(i+N)^c} \qquad (1)$$

$$\zeta(c,N) = \sum_{j=0}^{\infty} \frac{1}{(j+N)^c} \qquad (2)$$

where A is the area of the bounding region, $\zeta(c,N)$ is the Hurwitz zeta function, and c and N are parameters. It follows from Equations (1) and (2) that the sum of all the A_i is A, i.e., the algorithm is space-filling in the limit $i \to \infty$ if it does not halt. When $N = 1$ the Hurwitz zeta function becomes the Riemann zeta function.

If the A_i are viewed as numbers, Equations (1) and (2) are just conventional mathematics and as such would be unremarkable. The connection with geometry arises from the interpretation of the A_i as *areas*.

The algorithm places of the shapes/motifs within the bounding region by starting with shape 0, and continuing with shapes 1, 2, … , as shown in the flow-chart of Figure 2.

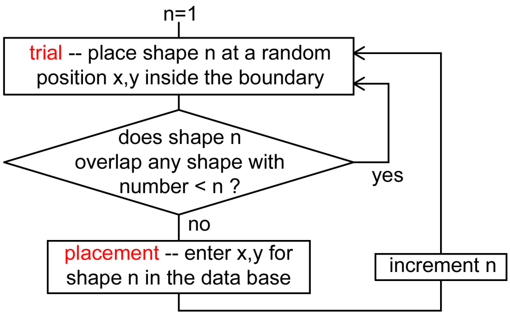

Figure 2: Flow-chart for non-overlapping random search, with definitions of what is meant by trial and placement.

The area rule (Equation (1)) and the search procedure of Figure 2 are a concise statement of the algorithm. Much computational evidence supports the idea that the algorithm does not halt over a wide range of the parameters c and N. It is conjectured that the statistical geometry algorithm is unconditionally non-halting for any motif or sequence of motifs which obey the area rule, over a substantial parameter range $1 < c < c_1$ and $N > N_{min}$. Here c_1 is the highest c value for which the algorithm is unconditionally non-halting. If $c > c_1$ some runs will halt, governed by a halting probability (Shier and Bourke, 2013). No exceptions to this conjecture have yet been found in extensive numerical studies with many motifs in one, two, and three dimensions. In fact it has recently been proven that the algorithm is unconditionally non-halting for circle motifs within a circular region for $1< c < 1.0965...$ (Ennis, 2016). Because the areas obey a power law, these constructions can be called fractal – in fact, by the form of the power-law, the fractal dimension can be computed to be 2/c (Shier and Bourke, 2013).

The particular shape of the motif (circle, square, etc.) is not specified because available data say that the algorithm applies to any shape. The value of c_1 is dependent upon just what shape is considered. It is highest for “compact” shapes such as circles and squares, and low for sparse shapes with a high perimeter-to-area ratio such as the snowflakes of Figure 1. The probability of any two shapes touching each other is vanishingly small.

The halting probability calculated for triangles fractalized within a triangle using the Monte Carlo method is shown in Figure 2. The corresponding fractal patterns are shown in Figure 8.

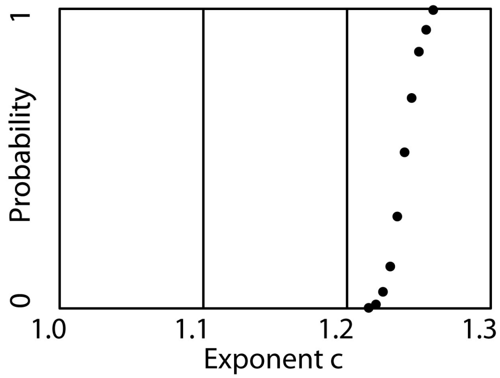

Figure 2: The halting probability versus exponent c for the statistical geometry algorithm where triangles are fractalized within a triangle with $N = 2$.

The data for Figure 2 was obtained by running the algorithm 1000 times for each of the c values. The run was assumed to halt if there were 6,000,000 trials without a placement; it was assumed to be non-halting if there were 65 placements. These limits are imposed by the requirements of computation but do not have large effects on the results. The data shows that runs which halt do so early in the run (see Figure 3).

The results form an s-shaped curve of halting probability versus c. There value of c_1 is around 1.21. Another parameter is c_2, the c value above which the algorithm always halts. Its value is around 1.26. It will be argued below that $c_2 < 1.26186$. As noted above, the algorithm is unconditionally non-halting for circles fractalized within a circle when $1 < c < 1.0965$ when $N = 1$ (Ennis, 2016). This proof leaves open the question of whether c_1 is greater than 1.0965.

Figure 3 gives a histogram of the halting placement number for the 1000 runs done with $c = 1.260$. This is the highest c value in the data set for Figure 2, and most runs halt.

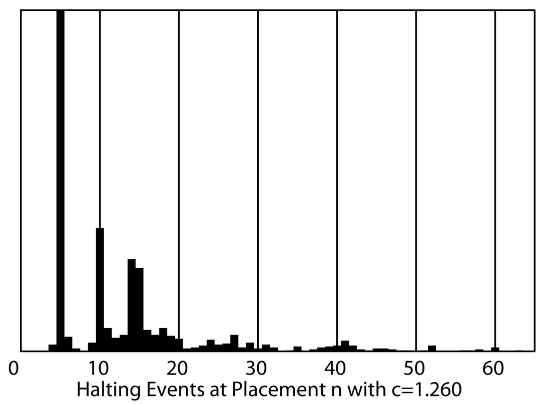

Figure 3: Histogram of the placement number at which the triangle-in-triangle algorithm halted for 1000 runs with c = 1.260 and N = 2.

There were very few halting events at fewer than 5 placements, but nearly 1/3 of the events occurred with n = 5 and many more runs halted at n = 10. The pattern of halting events shown in Figure 3 is interesting and largely unexplored. It is seen that the runs which halt do so early in the run.

As mentioned above, the fractal dimension D for the set of placed motifs is given by

$$D = 2/c \qquad (3)$$

and does not depend on N (Shier and Bourke, 2013).

In the limit $c \to 1$ the motif areas become infinitesimal and the algorithm reduces to a set of points randomly sprinkled within the bounding area. In this limit the fractal dimension $D = 2$, the same as the Euclidean dimension. The searches are fast, seldom requiring more than one trial to find a non-overlapping position. As c increases the motifs fit more closely and the searches become slower.

The fill factor depends only upon Equation (1) and not upon the random numbers used to place the motifs. Figure 4 shows universal curves for fill factor versus placement number when $N = 1$. The curves are similar for other N values.

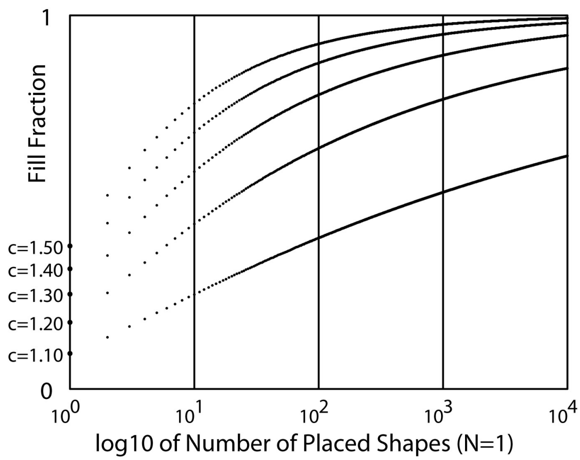

Figure 4: Semi-log plot of fill fraction versus the number of placed motifs for various c values with N = 1.

For high c values the bounding region fills quickly and the empty space falls rapidly. There is thus less average distance between motifs, but a far larger number of random trials is needed to place a given number of motifs as shown in Figure 5. It is always possible to find a particular c value for which the first two motifs exactly fit within the bounding region, and this c value is thus an upper bound on c_2.

If one could push the algorithm to a complete fill, given that it is space-filling, one would have a form of tiling of the plane. Of course a complete filling cannot be obtained with a finite number of placements, but by pushing c as high as possible one achieves patterns with only a tiny amount of space between motifs. In this limit the motifs are very close together and have a quite orderly arrangement.

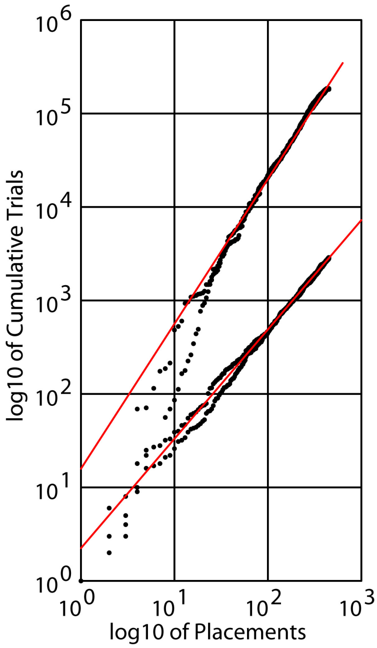

Figure 5: Log-log plot of the cumulative number of trials needed to place n triangle motifs fractalized within a triangle with N = 2. The upper data set has c = 1.2 and the lower one c = 1.1. Two runs are shown for each c value to give an idea of the amount of noise in the process..

Figure 5 shows the total number of trials needed to place a given number of triangles within a triangle. A straight line on a plot of this kind indicates that the y data follows a power law in x. The two straight lines are approximate regression lines for the data: $y = 1.17x + .35$ for $c = 1.1$, and $y = 1.55x + 1.2$ for c = 1.2 in log coordinates. Thus the trend in the data is a positive-exponent power law, with exponents approximately 1.17 for $c = 1.1$ and 1.55 for $c = 1.2$. If the data follows a power law, it is evidence that the algorithm does not halt; such a power law gives very high numbers of trials needed for many placements, but the average number of required trials never becomes infinite. The average number of trials needed to place n motifs can be read off from the graph. With $c = 1.2$ the average number of trials for 250 motifs is approximately 100,000.

3. SOME EXAMPLES

Because the algorithm is not widely known it is useful to present some examples. Most of them have relatively high c values, i.e., a high degree of filling and close spacing.

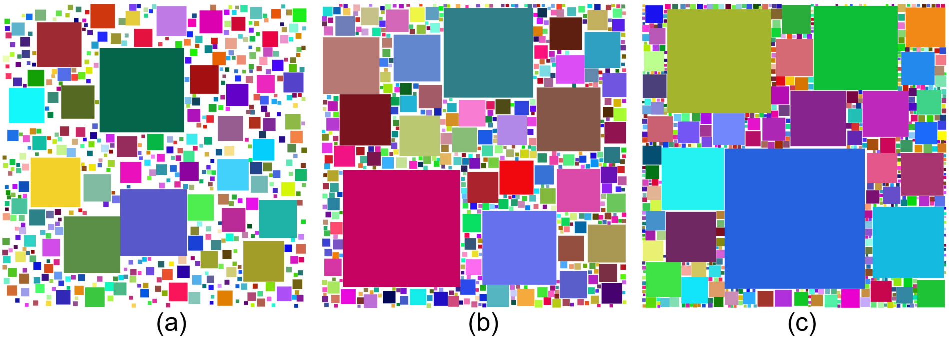

Figure 6: Five hundred randomly colored squares fractalized within a square for c values 1.16 (a), 1.32 (b) and 1.48 (c), with N = 2. Fill percentage: (a) fill = 60% with 2687 trials, (b) fill = 84% with 35,049 trials, (c) fill = 94% with 641,056 trials.

Figure 6 shows square motifs within a square region with three increasing values of *c*. Squares have the highest attainable *c* values of all motifs studied thus far. There is a steady progression from low fill and much randomness in (a) to high fill and much order in (c). The squares have random colors within a medium range which avoids very dark and very light colors. One of the problems in drawing these patterns is that one wishes the viewer to see all of the motifs with a very large range of sizes. Color variation is one way to accomplish this. If Figure 6(c) were drawn in a single color it would be difficult to distinguish the individual squares.

Figure 6(c) resembles results for a well-known mathematical problem called "squaring the square" (Wikipedia: Squaring the Square). The object is to completely fill a square with different sized smaller squares. Many solutions are known, ranging from just a few squares to very large numbers of them.

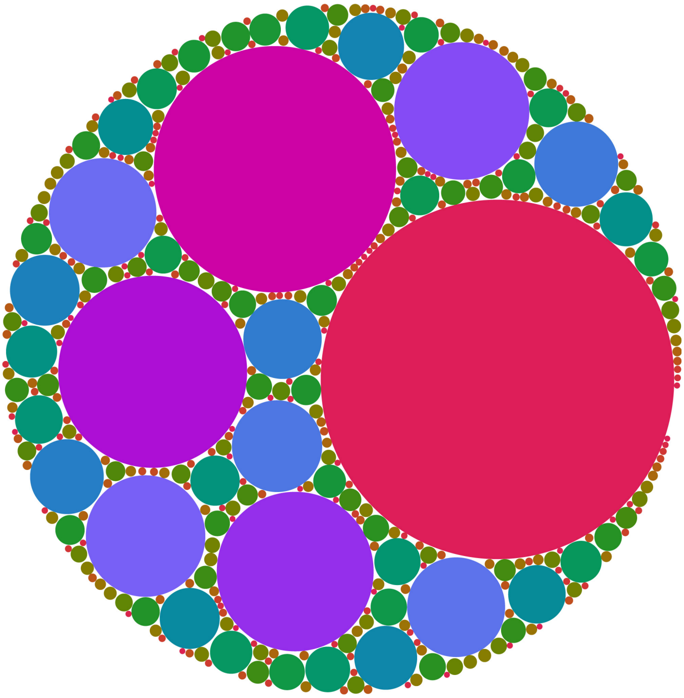

Figure 7: Circles fractalized within a circle. 400 circles, c = 1.48, N = 1.5, fill = 94.2%, 23,476,014 trials. Log-periodic color.

Figure 7 shows a highly ordered set of circles with *c* near its upper limit. The halting probability is high here, but this is a "survivor" run. The circles are very close together. It makes an interesting comparison with the Apollonian circles where the circles are mutually tangent (Wikipedia: Apollonian Gasket). In making such comparisons it needs to be kept in mind that there are many possible sets of Apollonian circles, some of which have symmetry, and some of which do not. The log-periodic color used here varies the color continuously along a closed orbit in RGB color space according to the logarithm of the circle area. This results in a pattern where circles that are close in area are also close in color.

Compact motifs such as circles and squares allow high *c* values. Sparse, lacy motifs such as the snowflakes of Figure 1 cannot go to such high c values. The snowflake pattern shown there has a maximum *c* value around 1.09. The percentage fill that can be achieved is low, but that is an advantage here since a too-dense fill would make it hard to see the individual snowflakes. For decorative uses it is important to choose the right number of motifs. If the motifs become too small they are indistinct to the viewer.

4. RANDOMNESS AND SYMMETRY

The power-law exponent *c* can be chosen over a wide range of values, but the particular choice made strongly influences the way the motifs fit. This is illustrated in Figure 8 for triangles fractalized within a triangle. There is a progression toward ever-greater order from (a) to (d) as c increases. In (d) there is a quite pronounced order in the pattern, where the smaller triangles have a triplet of near neighbors. Sierpinski's construction (e) is shown because of the interesting fact that the partial short-range order seen in (d) is complete for Sierpinski's triangle (e).

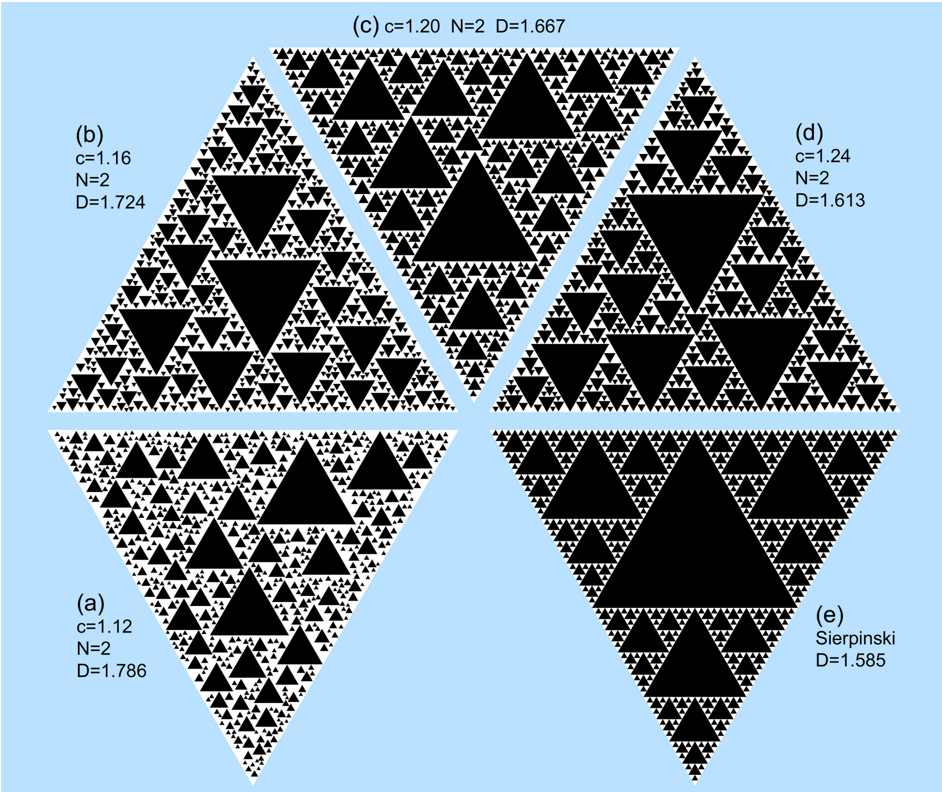

Figure 8: Fractalization of triangles within a triangle. The *c* values (and the degree of order) increase from (a) to (d). Sierpinski's construction (e) is seen to provide a symmetric end point for this sequence. The ratio of largest to smallest area is the same for (a)-(d).

The fractal dimension *D* shows an interesting progression, falling steadily as c increases. There is only a 1.7% drop of *D* between (d) and the Sierpinski structure (e), although the Sierpinski *D* is calculated from a different formula ($D = \log(3)/\log(2)$). We believe that Sierpinski's triangle is in a sense the end point of the progression from (a) to (d) in which degree of order steadily increases. One cannot get any more orderly than the complete order seen in the Sierpinski triangle. On this basis, using Equation (3), one can expect that $c = 2/D = 2/1.58549 = 1.26186$ is the limiting highest *c* value. It is thus conjectured that $c_2 \leq 1.26186$ for a triangle-in-triangle fractalization. There is good support for this in the data of Figure 2. This provides an alternative explanation for the existence of a maximum *c* value. Order increases with *c*, and it is not possible to have a higher degree of order than the Sierpinski triangle. The statistical geometry algorithm as defined cannot produce the Sierpinski triangle because it specifies that no two triangles can have the same area. It is conjectured that there is an algorithm that will allow all c values from 1+ to 1. 26186. Such an algorithm might begin with Sierpinski's triangle and add randomness, rather than beginning with randomness and progressing toward Sierpinski's triangle as shown here.

5. DISCUSSION

The statistical geometry algorithm offers a different way of filling space that is interesting for both mathematics and art. Careful reading of Section 2 gives an in-depth view of how the algorithm operates which is expanded with examples in Section 3.

The central point here is that symmetry is not necessarily an all-or-nothing property, but that it may be one end of a continuous gradation from complete randomness to complete order. Such order-disorder sequences are unusual in mathematics, but are well known in physics. University physics courses teach that there are three states of matter: solid, liquid, gas. It is said that only solids have symmetry properties, i.e., their crystals are built up of repeating units of a single cell. However it is found that in alloys such as the binary system Cu-Zn there is a continuous variation of the degree of order with temperature. At low temperatures with equal numbers of Cu and Zn atoms the unit cell has Cu atoms at the corners of the cubic unit cell and Zn atoms at the center. As temperature rises, the probability of a Zn at the corner increases from zero, and similarly for the Cu atoms at the center. At high temperatures (but still in the solid phase) the probability of a Zn at either cell position approaches 0.5, i.e., it is even random chance whether a Cu or a Zn atom occupies a particular cell site.

The fractal dimension D (or the corresponding power law exponent c) is seen to play the role of an order parameter. The degree of order can be specified a priori by a suitable choice of D or the equivalent c. This is one of several explanations for the existence of a maximum c value, and also provides a lower limit for D (and thus an upper limit for c) for triangles fractalized within a triangle.

If a list of interesting questions about the statistical geometry algorithm were made the great majority of the topics would be poorly explored or entirely unexplored. For example, Figure 2 gives a crude picture of the halting probability for triangles fractalized within a triangle. In principle this curve can be calculated by purely mathematical means, but how that could be done, or even begun, is unknown. Such a theory would also yield numerical values for c_1 and c_2. The power-law behavior of the cumulative trials versus placements (Figure 5) likewise has no explanation.

REFERENCES

Dunham, D., and Shier, J. (2014) The Art of Random Fractals, *Proceedings of Bridges Seoul*, 79-86.

Dunham, D., and Shier, J. (2015) Fractal Wallpaper Patterns, *Proceedings of Bridges Baltimore*, 183-190.

Ennis, C. (2016) Always Room for One More, *Math Horizons (*publication of the Mathematical Association of America), February, 2016, 8-12.

Shier, J., and Bourke, P.. (2013) An Algorithm for Random Fractal Filling of Space, *Computer Graphics Forum*, Vol. 32, Issue 8, 89-97.

Wikipedia: Apollonian gasket https://en.wikipedia.org/wiki/Apollonian_gasket
last accessed 29 May, 2016.

Wikipedia: Squaring the square https://en.wikipedia.org/wiki/Squaring_the_square last accessed 29 May, 2016.