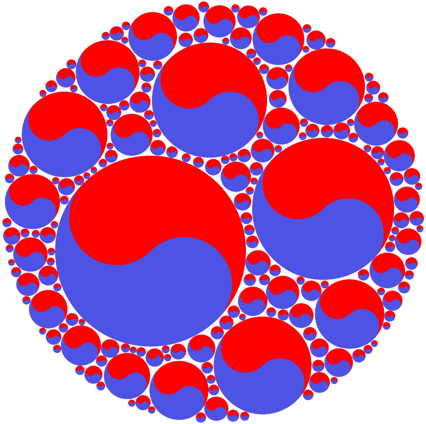
UMD Undergraduate Mathematics Colloquium Oct. 16, 2014

Creating Random Fractal Patterns

Douglas Dunham
Dept. of Computer Science
Univ. of Minnesota, Duluth
Duluth, MN 55812, USA

John Shier
6935 133rd Court
Apple Valley, MN 55124 USA



Outline

- Background and the "Area Rule"
- The algorithm
- A conjecture
- Dependence on parameters c and N
- Sample patterns
- A 3D pattern
- Conclusions and future work
- Contact information

Background

Our original goal was to create patterns by randomly filling a region R with successively smaller copies of a motif, creating a fractal pattern.

This goal can be achieved if the motifs follow an "area rule" which we describe in the next slide.

The resulting algorithm is quite robust in that it has been found to work for hundreds of patterns in (combinations of) the following situations:

- The region R is connected or not.
- The region R has holes — i.e. is not simply connected.
- The motif is not connected or simply connected.
- The motifs have multiple (even random) orientations.
- The pattern has multiple (even all different) motifs.
- If R is a rectangle, the pattern can be **periodic** — it can repeat horizontally and vertically, and thus tile the plane. The code is different and more complicated in this case.

The Area Rule

If we wish to fill a region R of area A with successively smaller copies of a motif (or motifs), it has been found experimentally that this can be done for $i = 0, 1, 2, \ldots$, with the area A_i of the i-th motif obeying an inverse power law:

$$A_i = \frac{A}{\zeta(c, N)(N + i)^c}$$

where where $c > 1$ and $N > 0$ are parameters, and $\zeta(c, N)$ is the Hurwitz zeta function: $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$ (and thus $\sum_{k=0}^{\infty} A_i = A$).

We call this the **Area Rule**

The Algorithm

The algorithm works by successively placing copies m_i of the motif at locations inside the bounding region R.

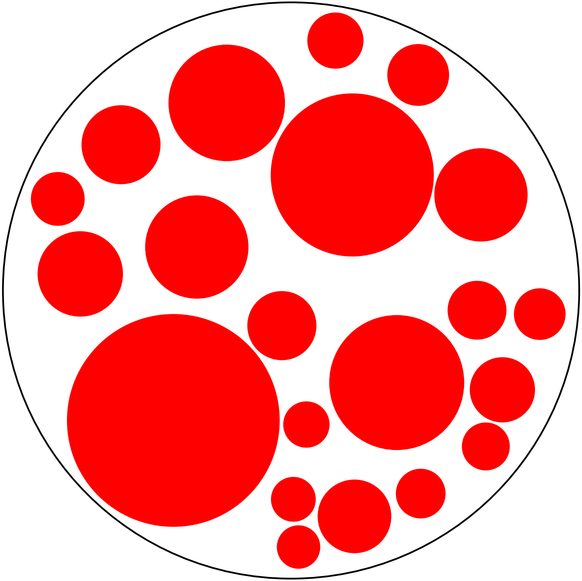
This is done by repeatedly picking a random **trial** location (x, y) inside R until the motif m_i placed at that location doesn't intersect any previously placed motifs.

We call such a successful location a **placement**. We store that location in an array so that we can find successful locations for subsequent motifs.

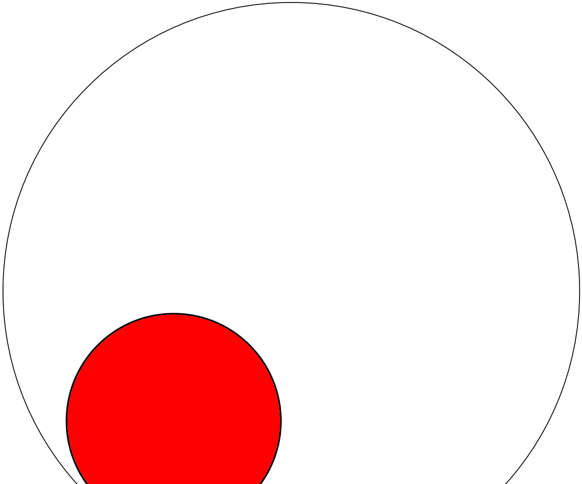
We show an example of how this works in the following slides.

A pattern of 21 circles partly filling a circle

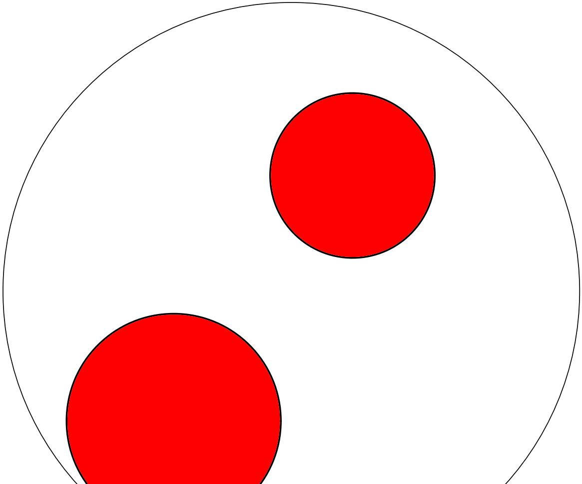
(Note: $c = 1.30$ and $N = 2$ in this example)



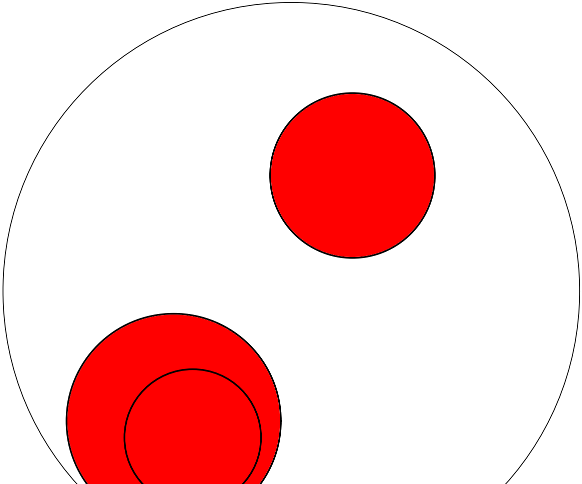
Placement of the first motif



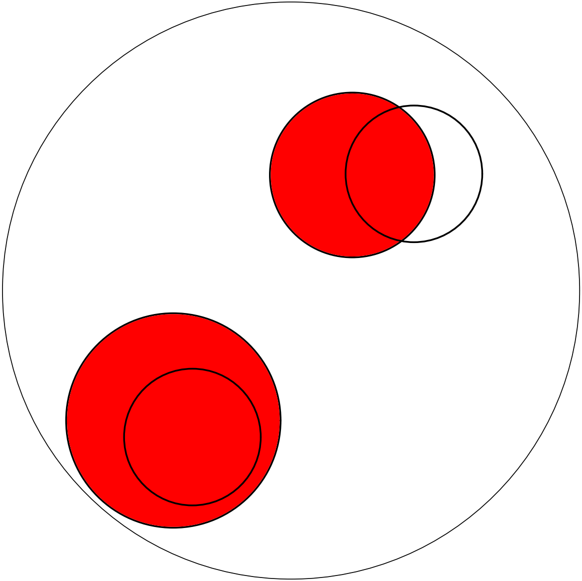
Placement of the second motif



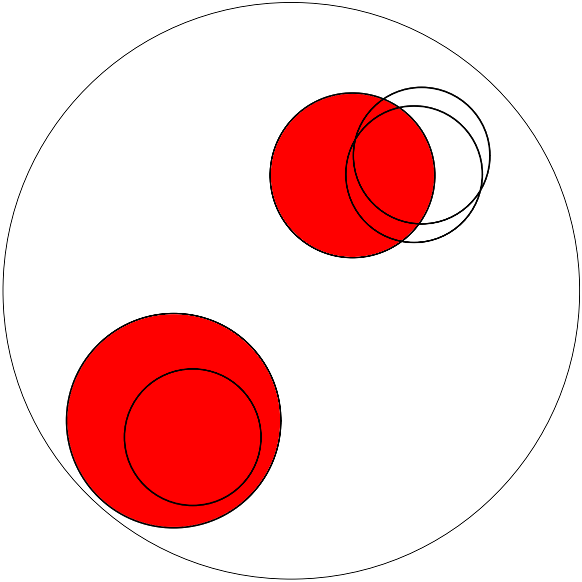
First trial for the third motif



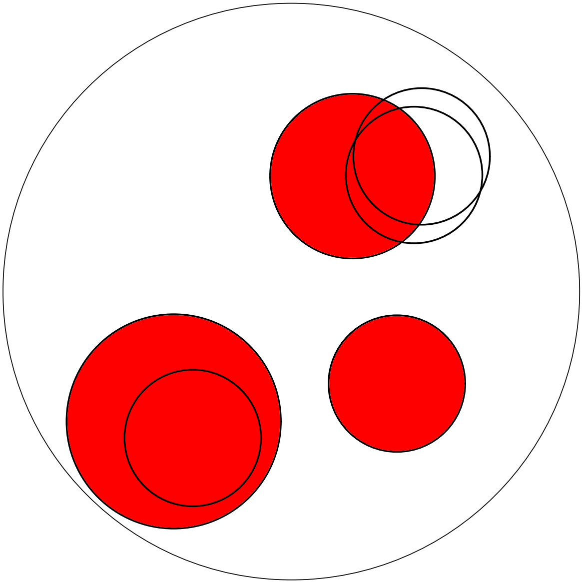
Second trial for the third motif



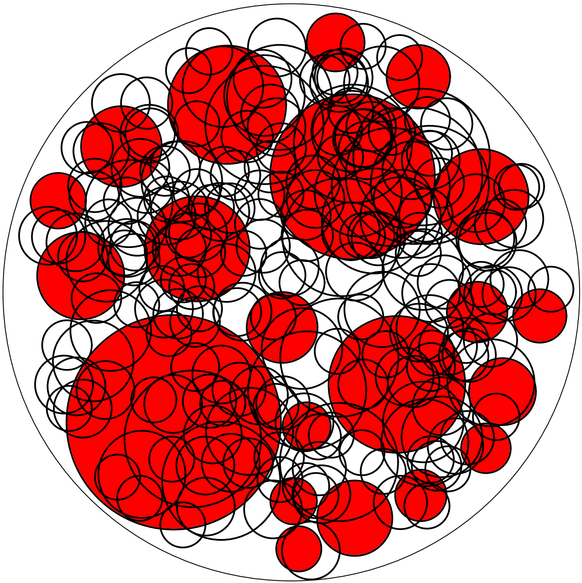
Third trial for the third motif



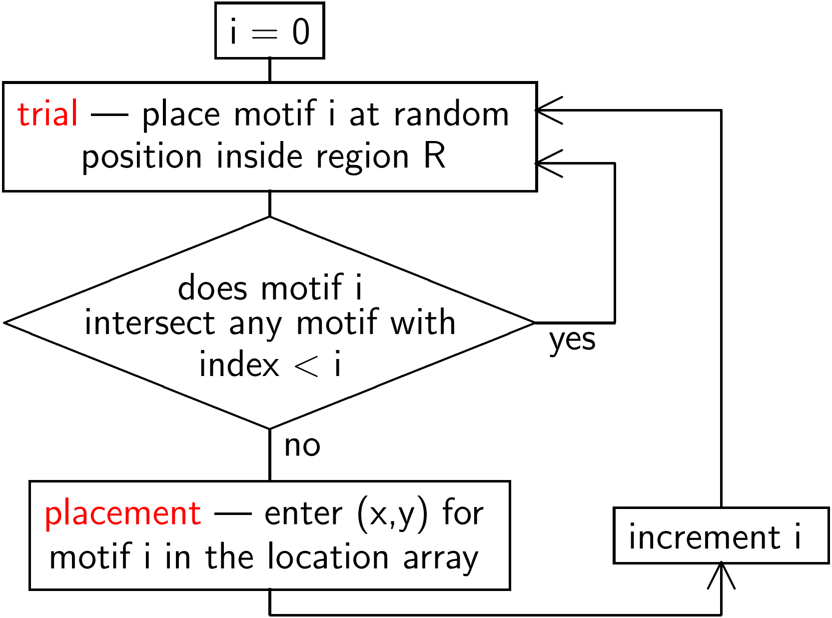
Successful placement of the third motif



All 245 trials for placement of the 21 circles



A Flowchart for the Algorithm



A Conjecture

Conjecture: The algorithm will randomly fill *any* reasonably defined region R with *any* reasonably defined motif(s), and it will not halt for $1 < c < c\_max$ and $N > N\_min > 0$, for appropriate values of c_max and N_min (which depend on the shapes of R and the motifs).

Typically values of c_max seem to be somewhat less than 1.5; often the values of N that were used were 2 or greater (not necessarily integer).

This algorithm has been implemented in dimensions 1, 2, 3, and 4, though we note that 1D patterns are not very interesting, and the "front" motifs in 3D and 4D obscure the motifs behind them.

In 1D, in which the motifs are line segments, it has been proved that the algorithm never halts for any c with $1 < c < 2$.

Also, the fractal dimensions of the patterns (not the unused portion of R) can be calculated to be $1/c, 2/c$, and $3/c$ in the 1D, 2D, and 3D cases respectively, which leads to the conjecture that the fractal dimension is d/c in d-dimensional space.

Dependence of patterns on **c** and **N**

By examining the formula that gives the Area Rule:

$$A_i = \frac{A}{\zeta(c,N)(N+i)^c}$$

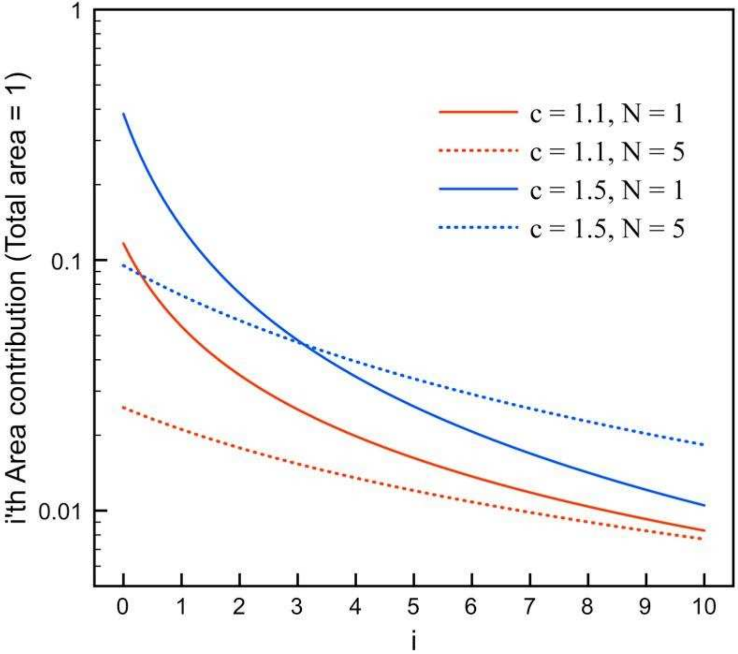
one can see that as c increases or N decreases, there is a larger difference in the sizes of the first few motifs.

Conversely, as c decreases or N increases, the first few motifs are closer in size.

The next slide shows a graph of how the sizes of the i-th motif decrease for different values of c and N.

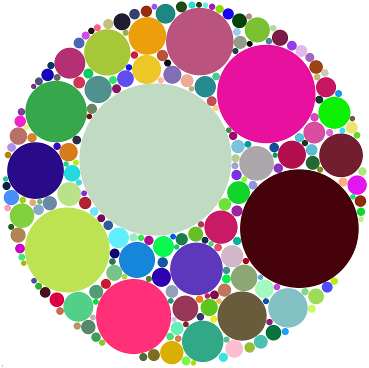
Following that, we show how patterns depend on c and N.

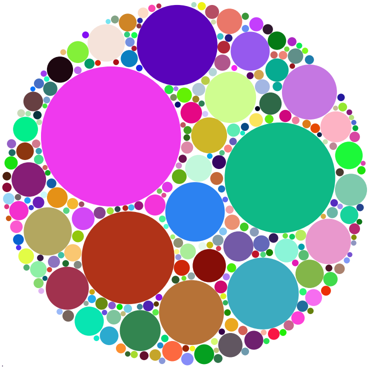
Graph of areas A_i for different values of c and N



Circle patterns with c = 1.48 and 1.40

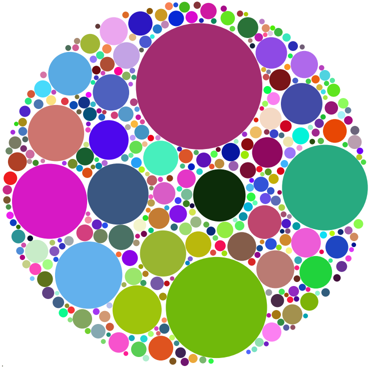
The value of N is 2.5 for each of these patterns.





Circle patterns with c = 1.32 and 1.24

The value of N is 2.5 for each of these patterns.



The dependence of the algorithm on N

(The value of c is 1.4 in the figures below.)

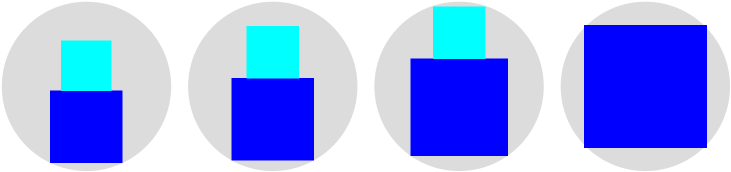
In each case the largest square is blue, and the second-largest is cyan. The blue square is placed with its lower corners on the bounding circle, and the cyan and blue squares are touching.

With N=1.50 there is plenty of room for squares with $i > 1$.

With N=1.10 if the blue square is placed near the center of the circle, the algorithm halts; it continues if it gets past the first few placements.

With N=0.70 the bounding circle can barely hold squares 0 and 1 and the algorithm halts because square 2 can't be placed.

With N=0.30 even square 0 doesn't fit.



N=1.50 N=1.10 N=0.70 N=0.30

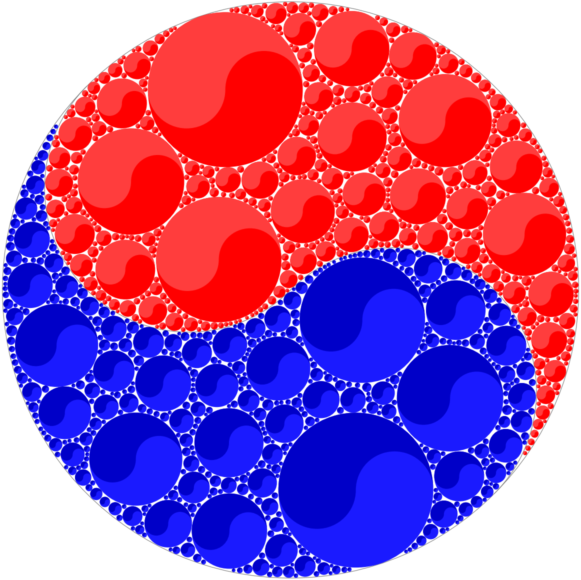
Sample Patterns

In the following slides, we exhibit the robustness of the algorithm by showing combinations of:

- Connectivity of the bounding region R.
- Non simply connected regions R.
- Non connected or non simply connected motifs.
- The motifs with multiple or even random orientations.
- Multiple, even all different, motifs.
- Periodicity for rectangular regions R.

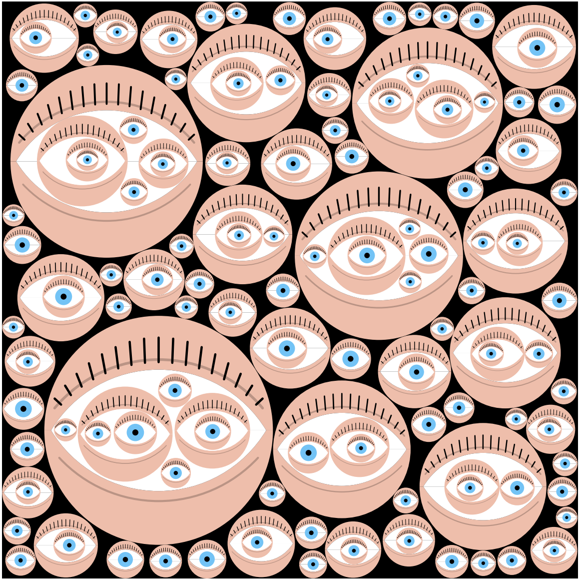
Two regions forming a yin and yang

In this pattern, $c = 1.47$ and $N = 3$, with 92% fill; it has 180° rotational color symmetry.



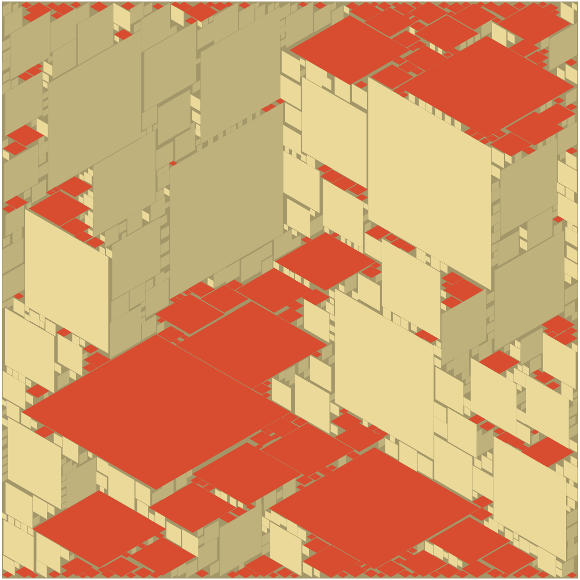
A pattern non-simply connected eye motifs

In this pattern, $c = 1.20$ and $N = 3$, with 56% fill; only eyes with no contained eyes have pupils.



Rhombi in three orientations and colors

In this pattern, $c = 1.52$ and $N = 8$ with 91% fill.

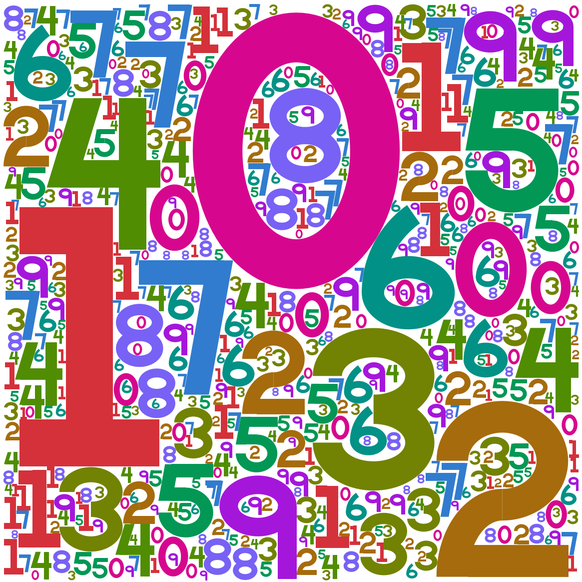


A periodic pattern of randomly oriented peppers

In this pattern, $c = 1.26$ and $N = 3$ with 80% fill.

A pattern of the 10 digit motifs

In this pattern, $c = 1.19$ and $N = 2$ with 68% fill.

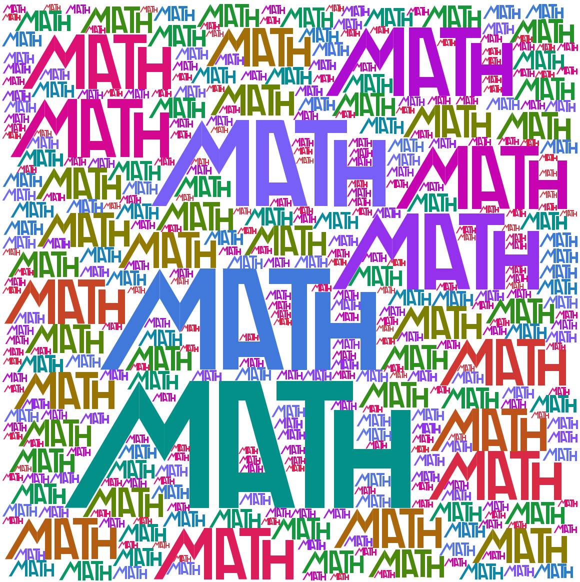


A pattern with the word ART as a motif

In this pattern, $c = 1.15$ and $N = 3$ with 53% fill.

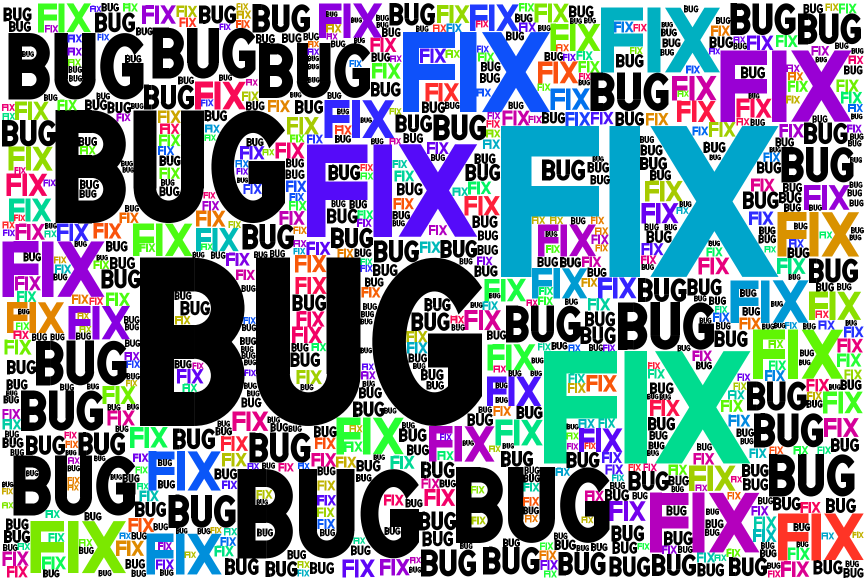
A pattern with the word MATH as a motif

In this pattern, $c = 1.26$ and $N = 2$ with 50% fill.

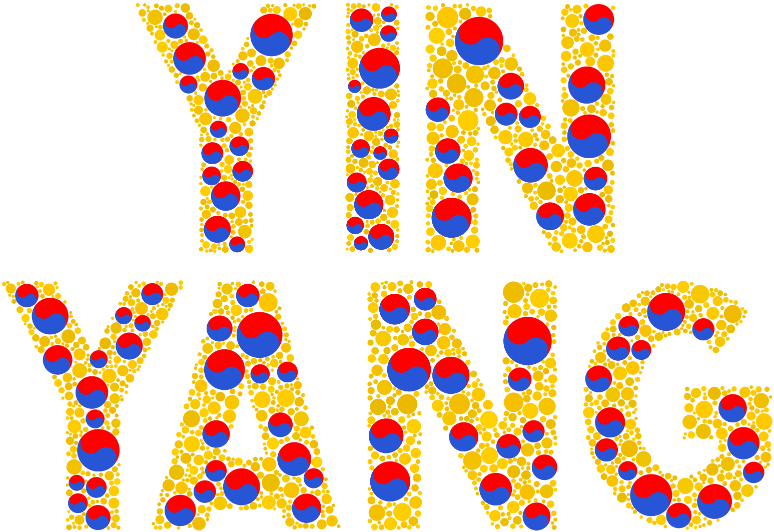


A pattern with the words BUG and FIX as motifs

In this pattern, $c = 1.155$ and $N = 2$ with 62% fill.

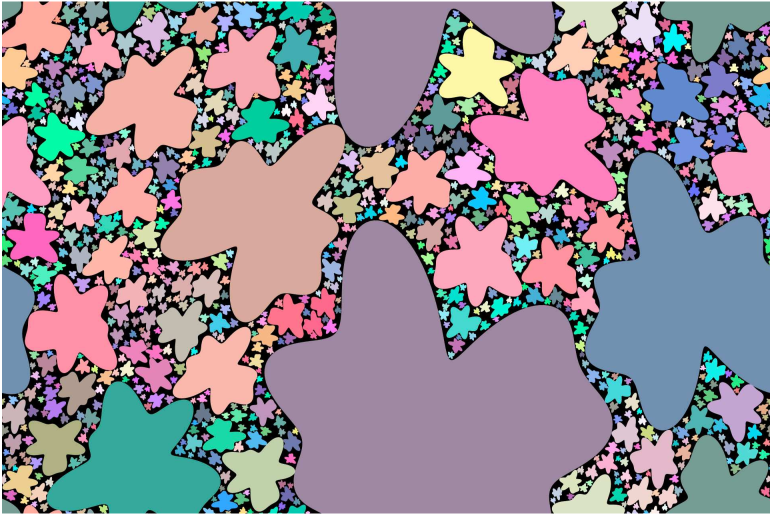


YIN YANG latin letters filled with two motifs



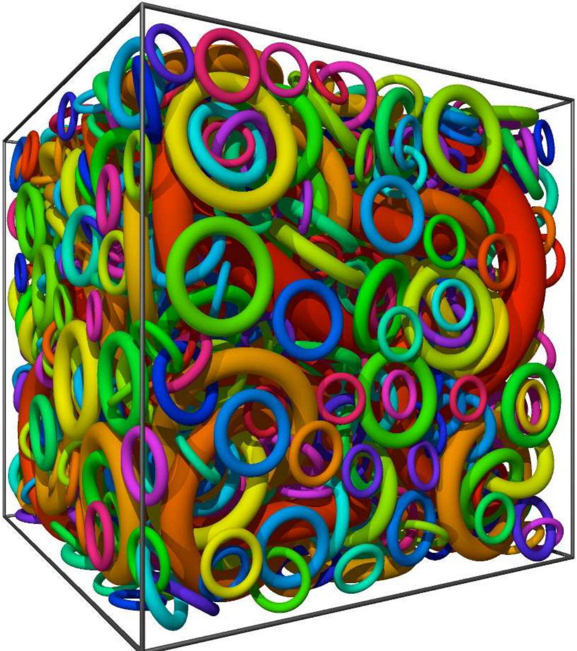
A periodic pattern of different random blobs

In this pattern, $c = 1.23$ and $N = 1$ with 82% fill.



A 3D pattern of tori by Paul Bourke

Note that some tori are linked.



Future Work

- Since the algorithm seems to be so robust, it would be reasonable to test it with new combinations of 2D regions and motifs.
- Though 1D patterns are not very interesting, and the motifs of 3D patterns block views of the interior, still there may be interesting 3D patterns to be discovered.
- We have displayed two patterns that are periodic, and thus tile the plane. Such a tiling would have the simplest plane symmetry group $p1$ (or **o** in Conway notation). It would also seem possible to create locally fractal patterns having global symmetries of the other 16 plane symmetry groups using our techniques.
- There are a few things that can be proved mathematically about these patterns, but there are a number of conjectures that have yet to be proved — such as the non-halting of the 2D algorithm for reasonable values of c and N.

Acknowledgements and Contact

We would like to thank Paul Bourke for his contributions to fractal geometry and for his 3D work in particular. His web page is `http://paulbourke.net/`

We would also like to thank Reza and *all* the other Bridges organizers.

Contact Information:
Doug Dunham
Email: `ddunham@d.umn.edu`
Web: `http://www.d.umn.edu/~ddunham`

John Shier
Email: `johnpf99@frontiernet.net`
Web: `http://www.john-art.com/stat_geom_linkpage.html`