Full spectrum of regular incomplete 2-handicap tournaments of order $n \equiv 0 \pmod{16}$

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Abstract

A *d-handicap distance antimagic labeling* of a graph $G = (V, E)$ with n vertices is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that $f(x_i) = i$ and the sequence of the weights $w(x_1), w(x_2), \ldots, w(x_n)$ (where $w(x_i) = \sum_{x_i x_j \in E} f(x_j)$) forms an increasing arithmetic progression with common difference d. A graph G is a *d-handicap distance antimagic graph* if it allows a d-handicap distance antimagic labeling.

We prove that a k-regular 2-handicap distance antimagic graph of order $n \equiv 0 \pmod{16}$ exists if and only if $n \geq 16$ and $4 \leq k \leq n - 6$.

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1 Motivation

The notion of handicap distance antimagic graphs has been motivated by tournament scheduling. Suppose we have a group of teams ranked according to their standings in the previous season, and we want to schedule a tournament in which every team plays the same number of games, say k, and we want that the weaker teams have good chance of winning. Then we need to schedule the tournament so that the strongest teams play the strongest sets of opponents, while the weakest teams have the weakest opponents.

When the strengths of schedules (that is, the sum of rankings of their opponents) of teams ranked i and $i + 1$ differ by d for any $i = 1, 2, \ldots, n - 1$, with team ranked 1 playing the most difficult schedule (sum of rankings is lowest) and team ranked n playing the easiest schedule (sum of rankings is highest), we speak about d-handicap graph or tournament.

Handicap tournaments with handicap $d = 1$ have been investigated in several papers. An overview of results on regular graphs of even order with more references has been published recently [9]. For even regular graphs of odd order, the results so far are sparse—see [6]. For $d = 2$, the author already studied graphs with $n \equiv 0 \pmod{16}$ vertices, but provided only a part of the possible regularities. For $n \equiv 8 \pmod{16}$, 2-handicap k-regular graphs were found by the author [8] for $n \geq 56$ and $6 \leq k \leq n-12$. Freyberg [1] recently stared investigating d-handicap graphs for $d \geq 3$.

We show that a k-regular 2-handicap graph of order $n \equiv 0 \pmod{16}$ exists if and only if $n \geq 16$ and $4 \leq k \leq n-6$.

2 Definitions, tools, and known results

The term "handicap distance antimagic labeling" was coined by Kovarova [14]. The author originally called the labeling *ordered distance antimagic* in [3].

Constructions of many classes of handicap graphs are based on properties of magic rectangles and magic rectangle sets, which are generalizations of the well-known notion of magic squares.

Definition 2.1. A *magic rectangle set* $MRS(a, b; c)$ is a collection of c arrays $\mathcal{R} = \{R^1, R^2, \ldots, R^c\}$, each of size $a \times b$ whose entries are elements of $\{1, 2, \ldots, abc\}$, each appearing once, with all row sums in every rectangle equal to a constant ρ and all column sums in every rectangle equal to a constant σ.

For our constructions, we need to modify the properties of magic rectangle sets as follows.

Definition 2.2. A *modified magic rectangle set* $MMRS(a, b; 2c)$ is a collection of $2c$ arrays $\mathcal{R} = \{R^1, R^2, \ldots, R^{2c}\}$, each of size $a \times b$ whose entries r^s_{ij} are elements of $\{1, 2, \ldots, 2abc\}$, each appearing once, where the set $\{R^1, R^2, \ldots, R^c\}$ is an $MRS(a, b; c)$ and for $s = c+1, c+2, \ldots, 2c$ the entries are defined as $r^{c+s}_{ij} = r^s_{ij} + abc$.

Constructions of magic rectangle sets and their applications can be found in [3] and [4], and more applications in [7] and [8].

Definition 2.3. A *handicap distance d-antimagic labeling* or shortly *d-handicap labeling* of a graph $G = (V, E)$ with n vertices is a bijection $f : V \to \{1, 2, \ldots, n\}$ with induced *weight function*

$$w(x_i) = \sum_{x_j : x_i x_j \in E} f(x_j)$$

such that $f(x_i) = i$ and the sequence of the weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an increasing arithmetic progression with difference d. When $d = 1$, the labeling is called just a *handicap labeling.*

A graph G is a *handicap distance d-antimagic graph* (or just *d-handicap graph*) if it allows a handicap distance d-antimagic labeling, and a *handicap distance antimagic graph* or a *handicap graph* when $d = 1$.

So, a graph G has a 2-handicap labeling defined by $f(x_i) = i$ if there exists a constant μ such that $w(x_i) = \mu + 2i$ for every $i = 1, 2, \ldots, n$.

It was proved in [8] that odd-regular 2-handicap graphs do not exist.

Theorem 2.4. *If G is a k-regular 2-handicap graph, then k is even.*

We denote by $H(n, k, d)$ a k-regular handicap distance d-antimagic graph of order n. The following existence results were proved by the author in [7] and [8], respectively.

Theorem 2.5. *There exists a k-regular 2-handicap graph $H(16m, k, 2)$ of order $16m$ for every positive m and every even k satisfying $4m + 2 \leq k \leq 12m - 2$.*

Theorem 2.6. *There exists a k-regular 2-handicap graph $H(n, k, 2)$ of order $n = 16m+8$ for every positive $m \geq 3$ and every even k satisfying $6 \leq k \leq n-50$.*

3 Necessary conditions

First we prove some necessary conditions.

Theorem 3.1. *If G is a k-regular 2-handicap graph, then k is even and $4 \leq k \leq n - 6$.*

Proof. Even parity of k follows from Theorem 2.4. Now we proceed by contradiction and assume that there is a 2-regular handicap graph on n vertices. We identify vertices with their labels, that is, we say "vertex l" rather than "vertex x_l with $f(x_l) = l$." It follows from the definition that $w(l) = w(1) + 2l - 2$. Hence,

$$\sum_{l=1}^{n} w(x_l) = \sum_{l=1}^{n}(w(1) + 2(l-1) = nw(1) + 2\sum_{l=1}^{n}(l-1) = nw(1) + n(n-1). \quad (1)$$

On the other hand, each label is in the above sum counted twice, so we also have

$$\sum_{l=1}^{n} w(x_l) = 2\sum_{l=1}^{n} l = n(n+1). \quad (2)$$

Comparing right-hand sides of (1) and (2), we obtain

$$nw(1) + n(n-1) = n(n+1),$$

which implies

$$nw(1) = 2n$$

and it follows that

$$w(1) = 2.$$

However, this is impossible, since the sum of two available lowest labels is $2+3=5$, a contradiction.

Similarly, we now assume that $k = n-4$. Then we have

$$\sum_{l=1}^{n} w(x_l) = \sum_{l=1}^{n}(w(n)-2(l-1) = nw(n)-2\sum_{l=1}^{n}(l-1) = nw(n)-n(n-1). \quad (3)$$

On the other hand, each label is in the above sum counted $(n-4)$ times, so we also have

$$\sum_{l=1}^{n} w(x_l) = (n-4)\sum_{l=1}^{n} l = (n-4)n(n+1)/2. \quad (4)$$

Comparing right-hand sides of (3) and (4), we obtain

$$n(w(n)-n+1) = (n-4)n(n+1)/2,$$

which yields

$$2w(n)-2n+2 = (n-4)(n+1) = n^2-3n-4$$

and it follows that

$$w(n) = (n^2-n-6)/2.$$

However, this is impossible, since the sum of $n-4$ available highest labels is

$$(n-1)+(n-2)+\cdots+5+4 = (n-4)(n+3)/2 = (n^2-n-12)/2.$$

But obviously,

$$n^2-n-6 > n^2-n-12$$

and we can never obtain the desired value $w(n) = n^2-n-6$, a contradiction.

For $k = n-2$, we have

$$\sum_{l=1}^{n} w(x_l) = (n-2)n(n+1)/2. \quad (5)$$

We compare right-hand sides of (3) and (5) and obtain

$$n(w(n)-n+1) = (n-2)n(n+1)/2,$$

which yields

$$2w(n)-2n+2 = (n-2)(n+1) = n^2-n-2$$

and it follows that

$$w(n) = (n^2+n-4)/2.$$

This is again impossible, since the sum of $n-2$ available highest labels is

$$\sum_{l=2}^{n-1} l = (n-2)(n+1)/2 = (n^2-n-2)/2.$$

But obviously,

$$n^2+n-4 > n^2-n-2.$$

a contradiction once more. This completes the proof. $\square$

4 Construction

We base our construction on modified magic rectangle sets. For any $c \geq 1$, we construct a modified magic rectangle set $\mathcal{R} = \mathrm{MMRS}(2,4;2c)$ consisting of $2c$ arrays of size 2×4 as follows. The entries of R^s are r^s_{ij} for $1 \leq s \leq 2c, 1 \leq i \leq 2$, and $1 \leq j \leq 4$.

In R^s for $j = 1,4$ and $s = 1,2,\ldots,c$ we have

$$r^s_{ij} = \begin{cases} 4s-4+j & \text{for } i=1 \\ 8c-4s+5-j & \text{for } i=2 \end{cases}$$

and for $j = 2,3$

$$r^s_{ij} = \begin{cases} 8c-4s+5-j & \text{for } i=1 \\ 4s-4+j & \text{for } i=2. \end{cases}$$

We observe that in all cases we have

$$f(x^s_{ij}) + f(x^s_{i+1\ j}) = 8c+1. \tag{6}$$

In R^s for $j = 1,4$ and $s = c+1, c+2, \ldots, 2c$ we have

$$r^s_{ij} = \begin{cases} 8c+4s-4+j & \text{for } i=1 \\ 16c-4s+5-j & \text{for } i=2 \end{cases}$$

and for $j = 2,3$

$$r^s_{ij} = \begin{cases} 16c-4s+5-j & \text{for } i=1 \\ 8c+4s-4+j & \text{for } i=2. \end{cases}$$

Here, we have in all cases

$$f(x^s_{ij}) + f(x^s_{i+1\ j}) = 24c+1. \tag{7}$$

A small example is shown in Figure 1.

First we prove the existence of 4-regular 2-handicap graphs of order $n \equiv 0 \pmod{16}$.

1	15	14	4
16	2	3	13

9	7	6	12
8	10	11	5

17	31	30	20
32	18	19	29

25	23	22	28
24	26	27	21

Figure 1: MMRS$(2, 4; 4)$

Theorem 4.1. *There exists a* 4*-regular* 2*-handicap graph* $H(16c, 4, 2)$ *of order* $16c$ *for every* $c \geq 1$.

Proof. We construct a 4-regular graph $H(16c, 4, 2)$ using the modified magic rectangle set $\mathcal{R} = \mathrm{MMRS}(2, 4; 2c)$ defined above. We denote the vertices x_{ij}^s for $s = 1, 2, \dots, 2c,\ i = 1, 2$ and $j = 1, 2, 3, 4$ and define a 2-handicap labeling as $f(x_{ij}^s) = r_{ij}^s$, where r_{ij}^s is an entry of $\mathcal{R}$.

We construct $H(16c, 4, 2)$ in two steps. First, we create $2c$ copies of 3-regular graphs $K_{4,4} - M$, where M is a perfect matching $4K_2$. Let $K^s = (V^s, E^s)$ be the s-th copy with bipartition $V^s = \{x_{1j}^s \mid j = 1, 2, 3, 4\} \cup \{x_{2t}^s \mid t = 1, 2, 3, 4\}$ and edges $x_{1j}^s x_{2t}^s$ for $j, t \in \{1, 2, 3, 4\}$ and $j \neq t$.

Observe that the labels in each partite set in all copies of $K_{4,4} - M$ come from one row of one rectangle in $\mathcal{R}$. The sum of all labels in $R^1, R^2, \dots, R^c$ is $8c(8c + 1)/2$ and we have c copies of $K_{4,4} - M$ and hence $2c$ partite sets, each corresponding to one row in a rectangle in $\mathcal{R}$, which means that the sum of the labels in each partite set for $s = 1, 2, \dots, c$ is equal to $\rho = 2(8c + 1)$.

The temporary weight $w'(x_{ij}^s)$ for $s = 1, 2, \dots, c$ is now

$$w'(x_{ij}^s) = \rho - f(x_{i+1\ j}^s)$$

and for $s = c + 1, c + 2, \dots, 2c$ it is

$$w'(x_{ij}^s) = 24c + w'(x_{ij}^{s-c}) = 32c + \rho - f(x_{i+1\ j}^s)$$

In the second step, we add edges joining corresponding vertices in copies R^s and R^{c+s}, namely $x_{ij}^s x_{ij}^{c+s}$.

It follows from (6) that $f(x_{i+1\ j}^s) = 8c+1-f(x_{ij}^s)$ and hence for $s = 1, 2, \dots, c$ we obtain

$$\begin{aligned} w(x_{ij}^s) &= w'(x_{ij}^s) + f(x_{ij}^{s+c}) = \rho - f(x_{i+1\ j}^s) + f(x_{ij}^s) + 8c \\ &= \rho - (8c + 1 - f(x_{ij}^s)) + f(x_{ij}^s) + 8c = \rho + 2f(x_{ij}^s) - 1. \end{aligned}$$

Because we identified vertices with their labels, that is, we assumed that $f(l) = l$, we actually obtain

$$w(l) = \rho - 1 + 2l$$

for $l = 1, 2, \ldots, c$.

Similarly, for $s = c+1, c+2, \ldots, 2c$ from (7) we have $f(x^s_{i+1\ j}) = 24c + 1 - f(x^s_{ij})$ and hence

$$\begin{aligned} w(x^s_{ij}) &= w'(x^s_{ij}) + f(x^{s-c}_{ij}) = 32c + \rho - f(x^s_{i+1\ j}) + f(x^s_{ij}) - 8c \\ &= 24c + \rho - (24c + 1 - f(x^s_{ij})) + f(x^s_{ij}) + 8c = \rho + 2f(x^s_{ij}) - 1. \end{aligned}$$

This again after identifying vertices with their labels yields

$$w(l) = \rho - 1 + 2l$$

for $l = 1, 2, \ldots, c$.

Clearly the sequence $w(1), w(2), \ldots, w(16c)$ is an increasing arithmetic progression with common difference 2, and the proof is complete. □

Next we construct even-regular 2-handicap graphs of higher degree. This can be achieved by placing edges of graphs $K_{2,2}$ between appropriate pairs of vertices whose labels add up to $8c + 1$. We present a detailed proof just for the case when c is even. For c odd the proof is very similar, and thus we offer just a sketch with the main ideas.

Theorem 4.2. *There exists a k-regular 2-handicap graph of order n for every positive $n \equiv 0 \pmod{32}, n \geq 32$ and every even k satisfying $4 \leq k \leq n - 6$.*

Proof. We keep our previous notation. Hence, we need to assume here that c is even and set $c = 2c'$. We recall that $f(x^s_{ij}) + f(x^{s+c}_{i+1\ j}) = 16c + 1$. Adding unused edges of a 2-regular graph consisting of copies of $K_{2,2}$ with partite sets $\{x^s_{ij}, x^{s+c}_{i+1\ j}\}$ and $\{x^t_{ij}, x^{t+c}_{i+1\ j}\}$ for $t \notin \{s, s+c\}$ clearly increases the weight of every vertex by $16c+1$. We have so far in our construction in Theorem 4.1 used edges within the copies K^i, and edges $x^s_{ij}x^{s+c}_{ij}$.

First we construct a complete graph $K_{2c'}$ with vertices $z_1, z_2, \ldots, z_{2c'}$, where each z_i corresponds to the graph induced by vertices of K^i and K^{i+c} defined above.

First we look at the graph induced by the vertices of K^s and K^{s+c}. Edges $x^s_{ij}x^{s+c}_{ij}$ have already been used, and edges $x^s_{ij}x^{s+c}_{i+1\ j}$ will not be used at all. We have never used edges in graphs K_4 induced by $x^s_{i1}, x^s_{i2}, x^s_{i3}, x^s_{i4}$ for any $s = 1, 2, \ldots, 2c$ and $i = 1, 2$. Thus, we use these unused edges for s and $s+c$ along with the edges between K^s and K^{s+c} to create 4-cycles $x^s_{i1}, x^s_{i2}, x^{s+c}_{i+1\ 1}, x^{s+c}_{i+1\ 2}$ and $x^s_{i3}, x^s_{i4}, x^{s+c}_{i+1\ 3}, x^{s+c}_{i+1\ 4}$ to increase each degree by 2. Notice that the weight of each vertex increases by $16c+1$, since each vertex is joined to a pair of vertices $x^t_{ab}, x^{t+c}_{a+1\ b}$.

Next, we add cycles $x^s_{i1}, x^s_{i3}, x^{s+c}_{i+1\ 1}, x^{s+c}_{i+1\ 3}$ and $x^s_{i2}, x^s_{i4}, x^{s+c}_{i+1\ 2}, x^{s+c}_{i+1\ 4}$ to increase the degrees by 2 again, and finally cycles $x^s_{i1}, x^s_{i4}, x^{s+c}_{i+1\ 1}, x^{s+c}_{i+1\ 4}$ and $x^s_{i2}, x^s_{i3}, x^{s+c}_{i+1\ 2}, x^{s+c}_{i+1\ 3}$ with the same effect. This covers constructions of graphs $H(16c, k, 2)$ for $k = 6, 8, 10$.

It is well known that $K_{2c'}$ has a one-factorization. Now each edge in every one-factor joins vertices z_s and z_t, where z_s represents all vertices inducing K^s and K^{s+c} and similarly, z_t represents all vertices inducing K^t and K^{t+c}. Hence, the edge $z_s z_t$ of our $K_{2c'}$ represents all edges between these two sets of 16 vertices each, namely, $\{x^a_{ij} : a = s; s+c; i = 1, 2, 3, 4\}$ and $\{x^a_{ij} : a = t; t+c; i = 1, 2, 3, 4\}$. None of these edges has been previously used.

We now unify each pair x^s_{1j}, x^{s+c}_{2j} into a vertex u_{1j}, each pair x^s_{2j}, x^{s+c}_{1j} into a vertex u_{2j}, and similarly pairs x^t_{1j}, x^{t+c}_{2j} and x^t_{2j}, x^{t+c}_{1j} into v_{1j} and v_{2j}, respectively. Recall that the labels of each of the respective pairs add up to $16c + 1$. What we obtain is the complete bipartite graph $K_{8,8}$, which can obviously be factorized into one-factors. Now an edge in each one-factor, say $u_{ab}v_{gh}$ corresponds in the original graph to the 4-cycle in which the neighbors of each vertex have labels that add up to $16c + 1$. This step then increases the degree of every vertex by 2, while maintaining the 2-handicap property. Obviously, each one-factor of $K_{2c'}$ allows us to increase the vertex degrees by $2, 4, \dots, 16$. Hence, all one-factors together allow us to increase the degrees by up to $(c - 1)16 = 16c - 16 = n - 16$. Along with the previously added edges, we obtain maximum degree $n - 16 + 10 = n - 6$, as desired. $\square$

We notice that a similar construction can be used for $n \equiv 16 \pmod{32}$. Suppose $c = 2c'+1$. Then the condensed graph with K^s and K^{s+c} amalgamated into a vertex z_s is $K_{2c'+1}$. It is well known that it can be factorized into Hamiltonian cycles. We take one of the cycles, blow it up back to the graph $C_{2c'+1} \circ \overline{K_{16}}$ and decompose each $K_{16,16}$ into eight copies of $8C_4$, one at a time. Each such step increases the degree of our 2-handicap graph by 4. Now, if we want to have a graph $H(n, k, 2)$ where $k \equiv 0 \pmod 4$ and $k \le n - 12$, we skip the first step in our previous construction in the proof of Theorem 4.2 . That means we do not add any edges into the graph consisting of the copies K^s and K^{s+c}, just increase the degree by an appropriate multiple of four. If we want $k \equiv 2 \pmod 4$, we add just one of the three 2-factors consisting of the cycles described in the first step. Finally, for $k = n - 8$ and $n - 6$, we add one or two more of the remaining one-factors, respectively. While we have not provided a complete, rigorous proof, it should be obvious that the following result holds.

Theorem 4.3. *There exists a k-regular 2-handicap graph of order n for every positive $n \equiv 16 \pmod{32}, n \ge 16$ and every even k satisfying $4 \le k \le n - 6$.*

The complete result now follows from Theorems 2.4, 3.1, 4.2 and 4.3.

Theorem 4.4. *There exists a k-regular 2-handicap graph of order $n \equiv 0 \pmod{16}$ if and only if $n \ge 16$ and $4 \le k \le n - 6$.*

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