On regular handicap graphs of order $n \equiv 0 \pmod 8$

Aaron Shepanik [1]

Department of Mathematics and Statistics
University of Minnesota Duluth
Duluth, USA

Dalibor Froncek [2]

Department of Mathematics and Statistics
University of Minnesota Duluth
Duluth, USA

Abstract

A *handicap distance antimagic labeling* of a graph $G = (V, E)$ with n vertices is a bijection $\acute{f} : V \to \{1, 2, ...n\}$ with the property that $\acute{f}(x_i) = i$ and the sequence of the weights $w(x_1), w(x_2), ..., w(x_n)$ forms an increasing arithmetic progression. A graph G is a *handicap distance antimagic graph* if it allows a distance antimagic labeling. We construct r-regular handicap distance antimagic graphs of order $n \equiv 0 \pmod 8$ for all feasible values of r. An overview of this and other related results can be found in [5].

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[1] Email: `shepa107@d.umn.edu`
[2] Email: `dfroncek@d.umn.edu`

1 Introduction

The study of handicap distance antimagic graphs was motivated by incomplete round-robin type tournaments that posess various properties.

A *complete round robin tournament of n teams* is a tournament in which every team plays each of the remaining $n-1$ teams. Since each team plays every other team, complete round robin tournaments are sometimes considered fair tournaments. When the teams are ranked $1, 2, \ldots, n$ according to their standings, then the sum of rankings of all opponents of the i-th ranked team, denoted $w(i)$, is $w(i) = n(n+1)/2 - i$, and the sequence $w(1), w(2), \ldots, w(n)$ is a decreasing arithmetic progression with difference one. A tournament of n teams in which every team plays precisely r opponents, where $r < n-1$ and the sequence $w(1), w(2), \ldots, w(n)$ is a decreasing arithmetic progression with difference one is called a *fair incomplete round robin tournament*. Since a team doesn't play itself, a natural property of a such a tournament is that strong teams get to play weaker teams, and weak teams play stronger teams. This property is removed in *equalized incomplete round robin tournaments* where the sum of rankings of all openents played is the same for each team. Some results on fair incomplete round robin tournaments can be found in [2] and [4].

Still, we can design a tournament that may allow weak teams an even better chance at winning than an equalized incomplete tournament. By allowing weak teams to play other weak teams, and having strong teams play other strong teams, the sequence $w(1), w(2), \ldots, w(n)$ should be an increasing arithmetic progression. A tournament in which this condition is satisfied, and every team plays $r < n-1$ games is called a *handicap incomplete round robin tournament*. A summary of results of handicap tournaments obtained by the authors et al. is found in [5]. In this paper we provide the details of the construction for $n \equiv 0 \pmod 8$ for all feasible regularities.

2 Basic Notions

By $G = (V, E)$ we mean a simple graph of order n. We will identify vertex names with their labels, thus by stating i we refer to the vertex labeled i.

As previously mentioned, the study of handicap distance antimagic graphs was motivated by other tournament types, each of which had their own properties and associated graph labelings. The graph of a fair incomplete round robin tournament admits a *distance antimagic labeling*, while the graph of an equalized incomplete round robin tournament admits a *distance magic label-*

ing.

The term distance magic labeling has evolved throughout the years. The concept was originally coined as a sigma labeling by Vilfred [10] in 1994, and then by Miller et. al. [11] using the name 1-vertex magic vertex labeling. The definition of distance antimagic labeling nicely follows after the definition of distance magic labeling.

Definition 2.1 A distance magic labeling of a graph G of order n is a bijection $f : V \to \{1, 2, ..., n\}$ with the property that there is a positive integer μ such that

$$\sum_{y \in N(x)} f(y) = \mu \qquad \forall x \in V.$$

The constant μ is called the magic constant of the labeling f, and $N(x)$ denotes the set of all vertices adjacent to v. The sum $\sum_{y \in N(x)} f(y)$ is called the weight of vertex x and is denoted $w(x)$. A graph that admits a distance magic labeling is called a distance magic graph. [10]

Definition 2.2 A distance d-antimagic labeling of a graph G with n vertices is a bijection $\bar{f} : V \to \{1, 2, ..., n\}$ with the property that there exists an ordering of the vertices of G such that the weights $w(x_1), w(x_2), ..., w(x_n)$ forms an arithmetic progression with difference d. When $d = 1$, then $\bar{f}$ is called just distance antimagic labeling. A graph G is a distance d-antimagic graph if it allows a distance d-antimagic labeling, and a distance antimagic graph when $d = 1$. [3]

A survey on distance magic graphs can be found in [1], while an often updated overview of results of all types of labelings can be found in [8].

The term *handicap labeling* was originally introduced by Petr Kovář and Tereza Kovářová and previously referred to as *ordered distance antimagic labeling* by Froncek in [3].

Definition 2.3 A handicap distance d-antimagic labeling of a graph G with n vertices is a bijection $\hat{f} : V \to \{1, 2, ...n\}$ with the property that $\hat{f}(x_i) = i$ and the sequence of the weights $w(x_1), w(x_2), ..., w(x_n)$ forms an increasing arithmetic progression with difference d. A graph G is a handicap distance d-antimagic graph if it allows a distance d-antimagic labeling, and a handicap distance antimagic graph when $d = 1$.

Note that in a handicap labeling a vertex with a lower label has a lower weight than a vertex with higher label. Thus, if we think of the vertices as teams and label them according to their strength, an r-regular handicap distance antimagic graph is in fact representative of a handicap incomplete round robin tournament.

3 Preliminary and Related Results

We often seek to know for which pairs (n, r) does a fair, equalized, or handicap incomplete tournament exist. The following results for fair and equalized incomplete tournaments with an even number of teams and can be found in [4].

Theorem 3.1 *Let* $\mathrm{EIT}(n, r)$ *be an equalized incomplete complete tournament. Then* r *is even.*

Theorem 3.2 *For* n *even an* EIT *(n, r) exists if and only if* $2 \leq r \leq n-2$, $r \equiv 0 \pmod 2$ *and either* $n \equiv 0 \pmod 4$ *or* $n \equiv r+2 \equiv 2 \pmod 4$.

Theorem 3.3 *For* n *even a fair incomplete tournament* FIT *(n, k) exists if and only if* $1 \leq k \leq n-1$, $k \equiv 1 \pmod 2$ *and either* $n \equiv 0 \pmod 4$ *or* $n \equiv k+1 \equiv 2 \pmod 4$.

For odd n, the following is known and obtained in [2].

Theorem 3.4 *Let* n *be an odd number and* $r = 2^s q$ *with* $s \geq 1$ *and* q *odd. Then an* EIT*(n , r) exists whenever* $r \leq \frac{2}{7}(n-2)$.

Theorem 3.5 *Let* n *be an odd number and* k *be an even number such that* $k < n$ *and* $n - k - 1 \neq 2^z$ *for any* $z > 0$. *Then a fair incomplete tournament* FIT*(n, k) exists whenever* $k > \frac{5}{7}(n-2)$.

Recently, some results on handicap distance d-antimagic graphs where $d = 2$ have been obtained, including a full characterization for $n \equiv 0 \pmod{16}$.

Theorem 3.6 *If* G *is a* k*-regular 2 handicap graph, then* k *is even.* [7]

Theorem 3.7 *There exists a* k*-regular 2-handicap graph of order* n *for every positive* $n \equiv 8 \pmod{16}$, $n \geq 56$ *and every even* k *satisfying* $6 \leq k \leq n-50$. [7]

Theorem 3.8 *There exists a* k*-regular 2-handicap graph of order* $n \equiv 0 \pmod{16}$ *if and only if* $n \geq 16$ *and* $4 \leq k \leq n-6$. [6]

Even some results have been obtained for more general d-handicap tournaments by Freyberg in [13]. These include a variety of results for even d, a partial characterization of order n that permits d odd, and multiples restrictions on the feasible regularities based on n and d. More details are known about handicap tournaments when $d = 1$, for the remainder of this paper this is the case.

For any graph with given regularity r and order n a simple counting argument shows the weight of each vertex i is already known as in the following lemma (see [12]).

Lemma 3.1 *In an r-regular handicap graph with n vertices the weight of every vertex is $w(i) = (r-1)(n+1)/2 + i$.*

Each vertex weight is an integer value obtained as a sum of integers. The previous lemma is used in a number of non-existance results. The following can be found amongst other non-existance results, see e.g [12] or [14].

Lemma 3.2 *There exists no r-regular handicap graph with n vertices if both r and n are even.*

Lemma 3.3 *No nontrivial r-regular handicap graph with n vertices exists if $r = 1$ and $r = n - 1$.*

Lemma 3.4 *There is no $(n-3)$-regular handicap graph of order n.*

We now proceed to the primary focus of this paper.

4 Construction for $n \equiv 0 \pmod 8$

If i is joined to k by an edge, we will use the notation $[i|k]$.Further, $[a, b|c, d]$ will denote the complete bipartite graph where a and b are both adjacent to c and d and vice-versa. The construction aims to prove the following proposition.

Proposition 4.1 *For $n \equiv 0 \pmod 8$ and $r \equiv 1, 3 \pmod 4$, there exists an r-regular handicap graph G on n vertices for all feasible values of r, that is, $3 \le r \le n - 5$.*

First note that Lemmas 3.2, 3.3, and 3.4 provide non-existance for all other r values than those claimed above. Since r is odd and at least 3, we can partition the edges at each vertex as follows: $2s$ black edges, 2 blue edges, and 1 red edge, for some nonnegative integer s. In other words we will have $2s$ 1-factors with edges colored black, a pair of 1-factors that are colored blue, and a single 1-factor colored red. The construction is complete in a three step process.

Step 1: The red edges will be used specifically to create the arithmetic progression required in the labeling by connecting $[1|4k+1]$, $[2|4k+2]$, $[3|4k+3]\ldots$, and $[4k|8k]$. This naturally partitions the vertex set into "lower" and "upper" sets.

Let $w_r(i)$ denote the weight of vertex i obtained from the red edges. We have that

$$w_r(i) = 4k + i \text{ for } i \in [1, 4k]$$

and

$$w_r(i) = -4k + i \text{ for } i \in [4k + 1, 8k]\,.$$

Step 2: Now we construct the two blue edges to each vertex. For the lower vertices, the blue edges will be copies of $K_{2,2}$ as: $[1, 4k|2, 4k-1], [3, 4k-2|4, 4k-3], \ldots, [2k-1, 2k+2|2k, 2k+1]$, and the upper vertices will be done in a similar manner: $[4k+1, 8k|4k+2, 8k-1], [4k+3, 8k-2|4k+4, 8k-3], \ldots, [6k-1, 6k+2|6k, 6k+1]$. See Figures 1 and 2.

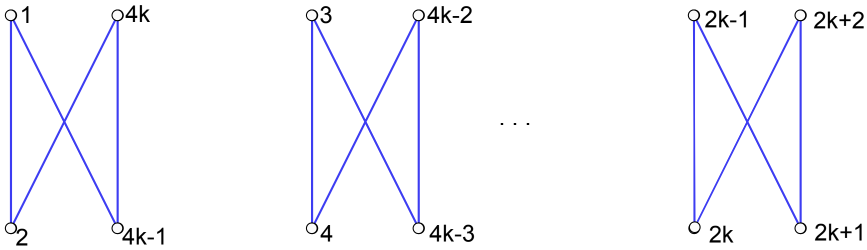

Fig. 1. Lower Blue Edges

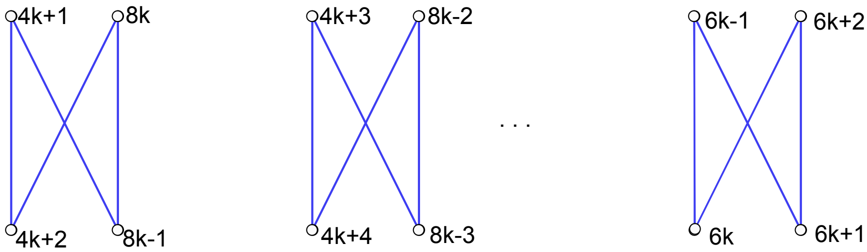

Fig. 2. Upper Blue Edges

Let $w_b(i)$ denote the weight of vertex i obtained from the blue edges. Then

$$w_b(i) = 4k + 1 \text{ for } i \in [1, 4k]$$

and

$$w_b(i) = 12k + 1 \text{ for } i \in [4k + 1, 8k]$$

so we have that

$$w_b(i) + w_r(i) = 4k + 1 + 4k + i = 8k + 1 + i \text{ for } i \in [1, 4k]$$

and

$$w_b(i) + w_r(i) = 12k + 1 - 4k + i = 8k + 1 + i \text{ for } i \in [4k+1, 8k]\,.$$

Thus the weight of each vertex with the red and blue edges is $8k + 1 + i$ for each i, which is exactly what we want. The graph of red and blue edges is currently 3-regular and handicap. All that is left is to show we can increase the regularityfor any $r \equiv 1, 3 \pmod 4$ up to $n - 5$ as claimed in proposition 4.1.

Step 3: Our goal is to add $2s$ black edges such that the subgraph induced by the black edges is distance magic. In doing so, we will not be effecting the arithmetic progression of our weights, and therefore, still have a handicap graph with higher regularities. We need to be careful, though, to make sure that we are not trying to reuse any of the red or blue edges that are used in Steps 1 and 2. To do this, we pair the vertices 1 with $8k$, 2 with $8k - 1$, ..., and $4k$ with $4k + 1$, so that the sum of these pairs is $8k + 1$. Each of these pairs can be thought of as a graph H with with no edges. Each pair becomes a vertex in our bubble graph B. In B, there will be an edge between two bubbles $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ if and only if there would be a red or blue edge (or both) between either x_1 or x_2 and y_1 or y_2. For clarity, we will color an edge red in B if it comes from Step 1. Once all edges from Step 1 are accounted for, we then add the edges from Step 2 and of course color those blue. While the colors in the bubble graph are not important, it helps to see where the edges came from. What happens here is the red and blue edges create separate components of B, each of which is K_4.

To see this, take any bubble $J = (a, 8k + 1 - a)$. Since there is a red edge $[a|4k+a]$, we have $[J|K]$ where $K = (4k+1-a, 4k+a)$. We know the other half of the bubble K must have weight $4k+1-a$ since the sum inside each bubble is $8k+1$. We also have the blue $K_{2,2}$ involving a, namely $[a, 4k+1-a|a+1, 4k-a]$. Specifically, since there exists a blue edge $[a|a + 1]$, we have $[J|L]$ where $L = (a + 1, 8k - a)$. Similarly, $[J|M]$ where $M = (4k - 1, 4k + 1 + a)$. Checking all other existing red and blue edges, we have a red edge $[4k - a|8k - a]$, and the blue $K_{2,2} = [4k + a, 8k + 1 - a|4k + 1 + a, 8k - a]$. Observe that any red or blue edges that would emerge from the four bubbles J, K, L, and M only result in edges between these four bubbles. See Figure 3.

Since we have $\frac{n}{2}$ bubbles, $\overline{B} = K_{\frac{n}{2}} - \frac{n}{8}K_4$. This is in fact isomorphic to the complete multipartite graph $K_{\frac{n}{8}[4]}$, that is, a graph with $\frac{n}{8}$ partite sets of size 4. Observe the bubble graph B is 3-regular, and the complement $\overline{B}$ will be $(\frac{n}{2} - 4)$-regular.

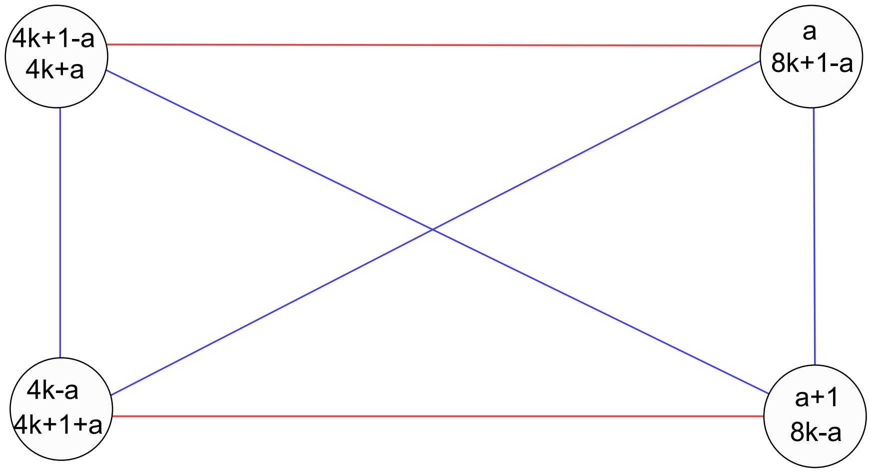

Fig. 3. Bubble Structure

$\overline{B}$ is the graph where we will pull our black edges from. It is a well known result that the complete multipartite graph on an even number of vertices can be 1-factored [9]. Each black edge in $\overline{B}$ will equate to a $K_{2,2}$ in the blown up graph $\overline{B}[H]$, where $\overline{B}[H]$ is the lexicographic product of $\overline{B}$ and H. Therefore, each 1-factor induced on $\overline{B}$ will consist of a 2-regular distance magic graph we can add to the red and blue edges, as desired. If we use all available black edges, we can add $2(\frac{n}{2} - 4) = n - 8$ black edges to increase regularity, for a max regularity of $n - 8 + 1 + 2 = n - 5$.

Our constuction is now complete, and we can state proposition 4.1 as a theorem.

Theorem 4.1 *For $n \equiv 0 \pmod 8$ and $r \equiv 1, 3 \pmod 4$, there exists an r-regular handicap graph G on n vertices for all feasible values of r, that is, $3 \leq r \leq n - 5$.*

Detailed visual examples of the construction can be found in [14].

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