Distance Magic Graphs—A Survey

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Abstract

Let $G = (V, E)$ be a graph of order n. A bijection $f : V \to \{1, 2, \ldots, n\}$ is called a *distance magic labeling* of G if there exists a positive integer k such that $\sum_{u \in N(v)} f(u) = k$ for all $v \in V$, where $N(v)$ is the open neighborhood of v. The constant k is called the magic constant of the labeling f. Any graph which admits a distance magic labeling is called a *distance magic graph.* In this paper we present a survey of existing results on distance magic graphs along with our recent results, open problems and conjectures.

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1 Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [3].

Several practical problems in real life situations have motivated the study of labelings of the vertices(edges) of graphs with real numbers or subsets of sets, which are required to obey variety of conditions. There is an enormous literature built up on several kinds of labelings of graphs over the past three decades or so. For a survey of various graph labeling problems one may refer to Gallian [8].

As pointed out by Gallian in his dynamic survey [8], finding out what has been done for any particular kind of labeling and keeping up with new discoveries is difficult because of the sheer number of papers and because many of the papers have appeared in journals that are not widely available. As a consequence for any particular type of graph labeling, the same classes of graphs have been done by several authors and in some cases the same terminology is used for different concepts. Again the same concept has been investigated by different authors with different terminology. One such concept is distance magic labeling which has been investigated under different names such as sigma labeling and 1-vertex magic vertex labeling. In this paper we present a survey of existing results on distance magic graphs along with our recent results, open problems and conjectures.

2 Basic Results

The concept of distance magic labeling of a graph has been motivated by the construction of magic squares. A magic square of side n is an $n \times n$ array whose entries are an arrangement of the integers $\{1, 2, \ldots, n^2\}$, in which all elements in any row, any column, or either the main diagonal or main back-diagonal, add to the same sum r. Now if we take a complete n partite graph with parts $V_1, V_2, \ldots, V_n$ with $|V_i| = n, 1 \leq i \leq n$ and label the vertices of V_i with the integers in the i^{th} row of the magic square, we find that the sum of the labels of all the vertices in the neighborhood of each vertex is the same and is equal to $r(n-1)$. Motivated by this observation in 1994 Vilfred [21] in his doctoral thesis introduced the concept of sigma labelings. The same concept was introduced by Miller et al. [15] under the name 1-vertex magic vertex labeling. Sugeng et al. [20] introduced the term distance magic labeling for this concept. In this paper we use the term distance magic labeling.

Definition 2.1. [21] *Distance magic labeling of a graph G of order n is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer k such that $\sum_{y \in N(x)} f(y) = k$ for every $x \in V$. The constant k is called the magic constant of the labeling f. The sum $\sum_{y \in N(x)} f(y)$ is called the weight of the vertex x and is denoted by $w(x)$.*

A natural generalization of the concept of magic square is the concept of magic rectangle.

Definition 2.2. *A magic rectangle $A = (a_{ij})$ of size $m \times n$ is an $m \times n$ array whose entries are $\{1, 2, \ldots, mn\}$, each appearing once, with all its row sums equal and with all its column sums equal.*

The sum of all entries in the array is $\frac{1}{2}mn(mn+1)$; it follows that

$$\sum_{i=1}^{m} a_{ij} = \frac{1}{2}n(mn+1) \text{ for all } j \quad \text{and} \tag{1}$$

$$\sum_{j=1}^{n} a_{ij} = \frac{1}{2}m(mn+1) \text{ for all } i. \tag{2}$$

Hence m and n must either both be even or both odd. It has been proved in [11, 12] that such an array exists whenever m and n have the same parity, except for the impossible cases where exactly one of m and n is 1, and for $m = n = 2$. We state the result formally here.

Theorem 2.3. *An $m \times n$ magic rectangle exists if and only if $m, n > 1, mn > 4$, and $m \equiv n \pmod 2$.*

A simpler construction is given in [10]. As in the case of magic squares, we can construct a distance magic complete m partite graph with each part size equal to n

by labeling the vertices of each part by the columns of the magic rectangle. Hence we have the following theorem. While there is no 2×2 magic rectangle, notice that the partite sets of $K_{2,2}$ can be labeled $\{1, 4\}$ and $\{2, 3\}$, respectively, to obtain a distance magic labeling.

Theorem 2.4. *Let $m, n > 1$. The complete m partite graph with each part of size n is distance magic if and only if n is even or both n and m are odd.*

The above result has been independently proved in [15].

We now present some basic results on distance magic graphs, which have been independently discovered.

The following lemma gives a necessary condition for the existence of distance magic labeling.

Lemma 2.5. [13, 15, 17, 21] *A necessary condition for the existence of a distance magic labeling f of a graph $G = (V, E)$ is $\sum\limits_{x \in V(G)} deg(x)f(x) = kn$, where n is the number of vertices of G and k is the magic constant.*

Corollary 2.6. [13, 15, 17, 21] *Let G be a r-regular distance magic graph on n vertices. Then $k = \frac{r(n+1)}{2}$.*

Corollary 2.7. [13, 15, 17, 21] *No r-regular graph with r-odd can be a distance magic graph.*

Theorem 2.8. [15]

(i) The path P_n of order n is a distance magic graph if and only if $n = 1$ or $n = 3$.

(ii) The cycle C_n of length n is a distance magic graph if and only if $n = 4$.

(iii) The complete graph K_n is a distance magic graph if and only if $n = 1$.

(iv) The Wheel $W_n = C_n + K_1$ is a distance magic graph if and only if $n = 4$.

(v) A tree T is a distance magic graph if and only if $T = P_1$ or $T = P_3$.

Theorem 2.9. [20] *If G is distance magic and $\delta(G) = 1$, then either G is isomorphic to P_3 or G contains exactly one component isomorphic to P_3 and all other components are isomorphic to $K_{2,2}$.*

Let G be a distance magic graph with distance magic labeling f and magic constant k. Obviously $k \geq n$ and a characterization of graphs for which $k = n$ is given in [13, 21]. We give an alternative proof of this result.

Theorem 2.10. *Let G be a distance magic graph with labeling f and magic constant k. Then the following are equivalent.*

(i) $k = n$.

(ii) $\delta = 1$.

(iii) Either G is isomorphic to P_3 or G contains exactly one component isomorphic to P_3 and all other components are isomorphic to C_4.

Proof. Suppose $k = n$. Then any vertex which is adjacent to n has degree 1 and hence (i) implies (ii). It follows from Theorem 2.9 that (ii) implies (iii). We now prove that (iii) implies (i).

If $G = P_3$ then G is a distance magic graph with $k = n = 3$, as shown in Figure 1.

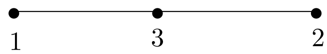

Figure 1:

Suppose $G = P_3 \cup tC_4$.

Let $P_3 = (v_1, v_2, v_3)$ and let the i^{th} copy of C_4 be $(v_{i1}, v_{i2}, v_{i3}, v_{i4}, v_{i1})$. Now define $f : V(G) \to \{1, 2, \ldots, 4t+3\}$ as follows:

$f(v_1) = 1$,
$f(v_2) = 4t + 3$,
$f(v_3) = 4t + 2$,
$f(v_{i1}) = 4t - 2(i-1) + 1$,
$f(v_{i2}) = 4t - 2(i-1)$,
$f(v_{i3}) = 2i$ and
$f(v_{i4}) = 2i + 1$, where $1 \leq i \leq t$.

Clearly f is a distance magic labeling of G with magic constant $k = 4t+3 = n$. □

The following theorem gives a characterization of distance magic graphs of order n with magic constant $k = n + 1$. This has been stated without proof in [13].

Theorem 2.11. *A graph G of order n is a distance magic graph with magic constant $k = n + 1$ if and only if $G = tC_4$.*

Theorem 2.12. [20] *If G is distance magic, $d(x) = d(y) = d(z) = 2$ and y is adjacent to x and z then either G is isomorphic to C_4 or G contains a component isomorphic to C_4.*

Theorem 2.13. [20] *If G is a complete multipartite graph and G has a distance magic labeling then $G = K_{s_1,s_2,\ldots,s_r}$ with $1 \leq s_1 \leq s_2 \leq \cdots \leq s_r$ and $s_i \geq 2, i = 2, 3, \ldots, r$.*

Acharya et al. [1] obtained a characterization of all complete bipartite graphs which are distance magic by proving the following two results.

Let $\theta(n)$ be the largest value of s such that $1 + 2 + \cdots + s = \frac{s(s+1)}{2} \leq \frac{n(n+1)}{4}$.

Lemma 2.14. [1] *If the complete bipartite graph K_{n_1,n_2} is distance magic graph, then $n \equiv 0$ or $3 (mod\ 4)$ and $\left\lceil \frac{n}{\sqrt{2}} \right\rceil - 2 \leq \theta(n) < \left\lceil \frac{n}{\sqrt{2}} \right\rceil$, where $n = n_1 + n_2$.*

Theorem 2.15. [1] *The complete bipartite graph K_{n_1,n_2} is a distance magic graph if and only if $n \equiv 0$ or $3 (mod\ 4)$ where $n_1 + n_2 = n$ and $\frac{n}{2} \leq n_1 \leq \theta(n)$.*

The same results were obtained in an alternative form in [2].

Theorem 2.16. [2] *Let m and n be two positive integers such that $m \leq n$. The complete bipartite graph $K_{m,n}$ is a distance magic graph if and only if*

(i) $m + n \equiv 0$ or $3 (mod\ 4)$ and

(ii) either $n \leq \lfloor (1 + \sqrt{2})m - \frac{1}{2} \rfloor$ or $2(2n+1)^2 - (2m + 2n + 1)^2 = 1$.

Definition 2.17. [19] *Let G and H be two graphs where $\{x_1, x_2, \ldots, x_p\}$ are vertices of G. Based upon the graph G, an isomorphic copy H_j of H replaces every vertex x_j, for $j = 1, 2, \ldots, p$ in such a way that a vertex in H_j is adjacent to a vertex in H_i if and only if $x_j x_i$ was an edge in G. Let $G[H]$ denote the resulting graph.*

Theorem 2.18. [15, 19] *Let $r \geq 1$, $n \geq 3$, G be an r-regular graph and C_n the cycle of length n. Then $G[C_n]$ admits a distance magic labeling if and only if $n = 4$.*

Miller at al. [15] proved the following result.

Theorem 2.19. [15] *Let G be an arbitrary regular graph. Then $G[\overline{K}_n]$ is distance magic for any even n.*

Shafiq et al. [19] considered distance magic labeling for disconnected graphs and obtained the following theorems.

Theorem 2.20. [19] *Let $m \geq 1, n > 1$ and $p \geq 3$. $mC_p[\overline{K}_n]$ has a distance magic labeling if and only if either n is even or mnp is odd or n is odd and $p \equiv 0 (mod\ 4)$.*

Theorem 2.21. [19]

(i) If n is even or mnp is odd, $m \geq 1, n > 1$ and $p > 1$, then $mK_{p[n]}$ has a distance magic labeling.

(ii) If np is odd, $p \equiv 3(mod\ 4)$ and m is even, then $mK_{p[n]}$ does not have a distance magic labeling.

Froncek at al. [7] strengthened the existence part of Theorem 2.20, part (i) of Theorem 2.21 and complemented Theorem 2.19 by proving the following.

Theorem 2.22. [7] *Let G be an arbitrary r-regular graph with k vertices, where k is an odd number, and n be an odd positive integer. Then r is even and the graph $G[\overline{K}_n]$ is distance magic.*

For $k \equiv r \equiv 2 \pmod 4$ they proved the non-existence of distance labeling for such graphs (see Theorem 4.9). The remaining cases are still open.

Froncek at al. [7] also solved the case of $p \equiv 1 \pmod 4$ which was not covered by part (ii) of Theorem 2.21 by proving the non-existence of such labeling. We list the result separately later as Theorem 4.8. Their result along with Theorem 2.21 then gives a necessary and sufficient condition.

Theorem 2.23. [7, 19] *The graph $mK_p[\overline{K}_n] = mK_{p[n]}$, where np is odd and m is even, $p > 1, m \geq 2$, is distance magic if and only if $p \equiv 3 \pmod 4$.*

For complete multipartite graphs that are not necessarily regular, Miller at al. [15] proved the following result.

Theorem 2.24. [15] *Let $1 \leq a_1 \leq \cdots \leq a_p$ where $2 \leq p \leq 3$. Let $s_i = \sum_{j=1}^{i} a_j$. There exists a distance magic labeling of the complete multipartite graph $K_{a_1,a_2,\ldots,a_p}$ if and only if the following conditions hold.*

(i) $a_2 \geq 2$

(ii) $n(n+1) \equiv 0(mod\ 2p)$, where $n = s_p = |V(K_{a_1,a_2,\ldots,a_p})|$ and

(iii) $\sum_{j=1}^{s_i}(n+1-j) \geq \frac{in(n+1)}{2p}$ for $1 \leq i \leq p$.

3 Embedding Theorems

Vilfred [21] proved the following theorem.

Theorem 3.1. [13, 21] *Every graph is a subgraph of a distance magic graph.*

Proof. If k is the chromatic number of G and r is the independence number of G, then G is to isomorphic to subgraph of the complete k partitie graph where each part has cardinality $2r$. □

Acharya [1] proved the following stronger theorem.

Theorem 3.2. [1] *Every graph H is an induced subgraph of a regular distance magic graph.*

Proof. The given graph H can be embedded as an induced subgraph of a connected r-regular graph $G = \{V, E\}$ with $r > 0$. Let $V = \{v_1, v_2, \ldots, v_n\}$. By the neighborhood expansion of G denoted by $D_2(G)$ we mean the graph obtained by taking another copy of G denoted $G' = (V', E')$ such that $V \cap V' = \emptyset$ and then by joining each vertex $u \in V$ to each vertex in the neighborhood of u' in G' by a new edge. Note that the new graph $D_2(G)$ is a $2r$-regular graph of order $2n$. Now the function f defined on $V(D_2(G))$ defined by $f(v_i) = i$ and $f(v_i') = 2n+1-i$, $1 \leq i \leq n$, gives a distance magic labeling of $D_2(G)$ with magic constant $k = r(2n+1)$. □

Corollary 3.3. [1] *There is no forbidden subgraph characterization for distance magic graph.*

In the following theorem we obtain a stronger version of Theorem 3.2. The result and its proof were communicated to the first author by S.B. Rao [18].

Theorem 3.4. *Given any graph H there is an Eulerian distance magic graph G with chromatic number same as that of H such that H is an induced subgraph of G.*

Proof. First we prove that H can be embedded as an induced subgraph of an r-regular graph G of degree $r = \Delta(H)$ such that $\chi(G) = \chi(H)$. Let H_1 be the graph obtained from H by attaching at each vertex u of H, $\Delta(H) - deg\ u$ pendant edges. In H_1 the degree of each vertex of H is $\Delta(H)$. Clearly $\chi(H) = \chi(H_1)$.

Now take $\Delta(H)$ copies of this H_1 and identify the corresponding $\Delta(H)$ pendant vertices of the same new pendant vertex of these copies to a single vertex to get the graph G which is $\Delta(H)$ regular. Clearly H is an induced subgraph of G. Since $\chi(H)$ coloring of each of these $\Delta(H)$ copies of H_1 gives a $\chi(H)$ coloring of G, we have $\chi(H) = \chi(H_1) = \chi(G)$. Now proceeding as in Theorem 3.2 we embed G as an induced subgraph of the $2r$-regular graph $D_2(G)$, which is Eulerian and distance magic. Clearly $\chi(G) = \chi(D_2(G))$ and hence the result follows. □

Corollary 3.5. *The problem of deciding whether the chromatic number of an Eulerian distance magic graph is at least 3 is NP-complete.*

Proof. Since the problem of deciding whether the chromatic number $\chi(G)$ is less than or equal to k, where $k \geq 3$ is NP-complete, ([9], Page 190), the result follows from Theorem 3.4. □

4 Graphs which are not Distance Magic

In this section we present several families of graphs which are not distance magic. We start with following simple observation given in [13, 21] which is very useful in this regard.

Theorem 4.1. [13, 21] *Let u and v be vertices of a distance magic graph G. Then $|N(u) \oplus N(v)| = 0$ or ≥ 3 (Here $A \oplus B$ denotes the symmetric difference of the two sets A and B).*

Corollary 4.2. [13, 21] *Let G be a graph of order n which has two vertices of degree $n-1$. Then G is not a distance magic graph.*

Corollary 4.3. [13, 21] *Any complete multipartite graph with two partite sets of cardinality 1 is not a distance magic graph.*

Corollary 4.4. [13, 21] *If a graph G has a path* (u, v, w, t, p) *with* $deg(v) = deg(t) = 2$*, then G is not a distance magic graph.*

Corollary 4.5. [13, 21] *If C is a cycle component of a distance magic graph G, then C is a* 4*-cycle.*

Lemma 4.6. [15] *If G contains two vertices u and v such that* $|N(u) \cap N(v)| = d(v) - 1 = d(u) - 1$*, then G has no distance magic labeling.*

Lemma 4.7. [15] *Let G be a graph on n vertices with maximum degree* Δ *and minimum degree* δ*. If* $\Delta(\Delta + 1) > \delta(2n - \delta + 1)$ *then G does not have a distance magic labeling.*

Theorem 4.8. [7] *The graph* $mK_p[n]$*, where np is odd, m is even,* $p \equiv 1 \pmod 4$*, and* $p > 1$*, is not distance magic.*

Theorem 4.9. [6] *Let n be odd,* $k \equiv r \equiv 2 \pmod 4$*, and G be an r-regular graph with k vertices. Then* $G[\overline{K}_n]$ *is not distance magic.*

5 Distance Magic Labeling and Cartesian Product

M.I. Jinnah [13] has proved that $P_n \square C_3$ and $P_n \square C_4$ are not distance magic graphs by using Theorem 4.1. Hence the following problem arises naturally.

Problem 5.1. *If G and H are two graphs, is the Cartesian product* $G \square H$ *distance magic?*

S.B. Rao [16] answered the above problem when G and H are both cycles or both complete graphs by proving the following sequence of results.

Theorem 5.2. [16] $C_n \square C_k$*,* $n, k \geq 3$ *is a distance magic graph if and only if* $n = k \equiv 2(mod\ 4)$.

Theorem 5.3. [16] $K_m \square K_n, m, n \geq 3$ *is not a distance magic graph.*

Beena [2] proved the following theorem regarding Cartesian products of graphs with minimum degree 1.

Theorem 5.4. [2] *The product of paths* $P_n \square P_k$ *is not a distance magic graph.*

Theorem 5.5. [2] *Let* G_1 *and* G_2 *be connected graphs with* $\delta(G_i) = 1, |V(G_i)| \geq 3$ *for* $i = 1, 2$*. Then* $G_1 \square G_2$ *is not a distance magic graph.*

6 Distance Magic Labelings of Bi-regular Graphs

A generalization of magic rectangles is useful in constructions of distance magic graphs with vertices of two different degrees. The results below were introduced by Sugeng et al. [20] with the use of *Kotzig arrays* (see [14]) and *lifted Kotzig arrays*, which are a generalization of magic rectangles. We use their idea to introduce a *lifted magic rectangle*, which in turn is a special case of the *lifted Kotzig array.*

Definition 6.1. *A lifted magic rectangle* $LMR(a, b; l)$ *is an* $a \times b$ *matrix whose entries are elements of* $\{l + 1, l + 2, \ldots, l + ab\}$*, each appearing once, such that the sum of each column is* $\sigma(a, b; l) = \frac{1}{2}a(ab + 2l + 1)$ *and the sum of each row is* $\tau(a, b; l) = \frac{1}{2}b(ab + 2l + 1)$*.*

Consider $LMR(a, b; 0)$ and a p-regular graph H which has n' vertices with $n' = b + d$, where $b = |V_1(H)|$ and $d = |V_2(H)|$ for $V_1, V_2 \subset V(H)$ such that $V_1 = \{x_1, x_2, \dots, x_b\}$ and $V_2 = \{y_1, y_2, \dots, y_d\}$ form a partition of $V(H)$.

Denote by $G = H[b \times a, d \times c]$ a graph arising from H by expanding each vertex $x_i \in V_1(H)$ into a set X_i of a independent vertices $\{x_{i1}, x_{i2}, \dots, x_{ia}\}$ and similarly expanding each $y_j \in V_2(H)$ into a set of c independent vertices $\{y_{j1}, y_{j2}, \dots, y_{jc}\}$. Further every edge $x_i x_j$ between two vertices of $V_1(H)$ will be replaced by a^2 edges of $K_{a,a}$ while every edge $y_i y_j$ between two vertices of $V_2(H)$ will be replaced by c^2 edges of $K_{c,c}$. Also any edge $x_i y_j$ between a vertex in $V_1(H)$ and a vertex in $V_2(H)$ will be replaced by ac edges of $K_{a,c}$. Denote $V_1(G) = X_1 \cup X_2 \cdots \cup X_b$ and $V_2(G) = Y_1 \cup Y_2 \cup \cdots \cup Y_d$.

Lemma 6.2. [20] *Let a, b, c, d be positive integers such that $a > c$ and both $LMR(a, b; 0)$ and $LMR(c, d; ab)$ exist. Then $\sigma(a, b; 0) = \sigma(c, d; ab)$ if and only if $d = \frac{(a^2b - 2abc + a - c)}{c^2}$.*

Lemma 6.3. [20] *Let a, b, c, d be positive integers such that both $LMR(a, b; 0)$ and $LMR(2, d; ab)$ exist and $\sigma(a, b; 0) = \sigma(2, d; ab)$. Then either $a \equiv 2 (mod\ 4)$ or a is odd and $a \equiv b (mod\ 4)$ and $a \geq 5$.*

Theorem 6.4. [20] *Let H be a p-regular graph on $b + d$ vertices and $G = H[b \times a, d \times c]$ be a graph with a, b, c, d satisfying conditions*

(i) $a > c$,

(ii) both $LMR(a, b; 0)$ and $LMR(c, d; ab)$ exist, and

(iii) $d = \frac{a^2b - 2abc + a - c}{c^2}$.

Then G is a distance magic graph.

7 Fair and handicap incomplete tournaments

So far we were looking at problems that can be in general stated as follows: For a given class of graphs $\Gamma(n, r, \dots)$, find all values of parameters $(n, r, \dots)$ for which a graph $G \in \Gamma$ allows distance magic labeling f. Most typical parameters were the number of vertices n and regularity r. We have seen above that for any given family $\Gamma(n, r)$ the spectrum of values of (n, r), for which the graphs are distance magic, can be very sparse. Sometimes, however, we do not need a particular class of graphs, but for a given pair (n, r) can settle for *any* graph with a distance magic vertex labeling.

Although the problem seems to be too random, there is a real life motivation in sports tournament scheduling. Suppose we want to schedule a one-divisional tournament, but do not have enough time to play the complete round robin tournament. What format should we select? We want to schedule a *fair incomplete round robin tournament* with the following properties:

1. Every team plays the same number of opponents.
2. The difficulty of the tournament for each team mimics the difficulty of the complete round robin tournament.

Condition 2 can be justified as follows. If we know the strength of each team based on team standings in the previous year, the teams can be ranked from 1 to n. Based on their rankings, we can define the *strength* of the i-th ranked team (or just team i for short) in a tournament with n teams as $s_n(i) = n + 1 - i$. The *total strength of opponents* of team i in a complete round robin tournament is then defined as $S_{n,n-1}(i) = n(n+1)/2 - s_n(i) = (n+1)(n-2)/2 + i$. We observe that the total strengths form an arithmetic progression with difference one. Therefore, we want the total strengths of opponents for respective teams in our incomplete tournament to form such a progression as well. In general, we want to find a tournament of n teams with each team playing k games in which the total strength of opponents of the i-th ranked team is $S_{n,k}(i) = (n+1)(n-2)/2 + i - m$ for some integer m.

Obviously, this is equivalent to finding the set of games that are left out of the complete tournament with the property that the total strength of opponents in the $n-k-1$ left out games, $S^*_{n,n-k-1}(i)$, is equal to some constant m for every team i.

A *fair incomplete tournament* of n teams with k rounds, FIT(n,k), is a tournament in which every team plays k other teams and the total strength of the opponents that team i plays is $S_{n,k}(i) = (n+1)(n-2)/2 + i - m$ for every i and some fixed constant m. The total strength of the opponents that each team misses is then equal to m. Hence, we can view the games that are *not* played as a complement of FIT(n,k), which is itself an incomplete tournament. In an *equalized incomplete tournament* of n teams with r rounds, EIT(n,r), every team plays exactly r other teams and the total strength of the opponents that team i plays is $S^*_{n,r}(i) = m$ for every i. Notice that EIT$(n, n-k-1)$ is the complement of FIT(n,k). Therefore, a FIT(n,k) exists if and only if an EIT$(n, n-k-1)$ exists.

One can notice that finding an EIT(n,r) is equivalent to finding a distance magic labeling of *any* r-regular graph on n vertices. We also observe that the complementary FIT$(n, n-r-1)$ is a *distance antimagic graph*.

Definition 7.1. *A* distance k-antimagic labeling *of a graph $G(V,E)$ with n vertices is a bijection $\bar{f} : V \to \{1,2,\dots,n\}$ with the property that there exists an ordering of the vertices of G such that the sequence of the weights $w(x_1), w(x_2), \dots, w(x_n)$ forms an arithmetic progression with difference k. When $k=1$, then $\bar{f}$ is called just* distance antimagic labeling. *A graph G is a* distance k-antimagic graph *if it allows a distance k-antimagic labeling, and* distance antimagic graph *when $k=1$.*

The weight $w(x)$ of a vertex x in a FIT(n,k) or EIT(n,r) is equal to $S_{n,k}(x)$ or $S^*_{n,r}(x)$, respectively.

In the language of distance magic graphs, our observation can be stated as follows.

Observation 7.2. *Graph G is distance magic if and only if its complement $\overline{G}$ is distance antimagic.*

It follows from Corollary 2.7 that if G is an r-regular distance magic graph, then r is even. The remaining feasible values of r for r-regular distance magic graphs with an even number of vertices were found in [6].

Theorem 7.3. [6] *For n even an r-regular distance magic graph with n vertices exists if and only if $2 \le r \le n-2, r \equiv 0 \pmod 2$ and either $n \equiv 0 \pmod 4$ or $r \equiv 0 \pmod 4$.*

For graphs with an odd number of vertices, the existence question of regular distance magic graphs was partially answered in [4].

Theorem 7.4. [4] *Let n, q be odd integers and s an integer, $q \ge 3$, $s \ge 1$. Let $r = 2^s q$, $q \mid n$ and $n \ge r + q$. Then an r-regular distance magic graph of order n exists.*

When the maximum odd divisor of r does not divide n, somewhat weaker result can be proved.

Theorem 7.5. [4] *Let n, q be odd integers and s an integer, $q \geq 3$, $s \geq 1$. Let $r = 2^s q$, $q \nmid n$ and $n \geq \frac{7r+4}{2}$. Then an r-regular distance magic graph of order n exists.*

The proofs are based on an application of magic rectangles.

Although the fair incomplete tournaments mimic the structure of the complete round-robin tournaments, they in fact favor the highest ranked team, because the total strength of its opponents, $S_{n,k}(1)$, is the lowest. Even in an equalized tournament the highest ranked team has the best chance of winning, because all teams face opponents with the same total strength. If we want to give all teams roughly the same chance of winning, we need to schedule a tournament with handicaps.

A *handicap incomplete tournament* of n teams with k rounds, HIT(n, k), is a tournament in which every team plays k other teams and the total strength of the opponents that team i plays is $S^h_{n,k}(i) = t - i$ for every i and some fixed constant t. This means that the strongest team plays strongest opponents, and the lowest ranked team plays weakest opponents. In terms of distance magic graphs this restriction corresponds to finding a distance antimagic graph with the additional property that the sequence $w(1), w(2), \ldots, w(n)$ (where team i is again the i-th ranked team) is an increasing arithmetic progression with difference one. We call this special case *ordered distance antimagic graphs*. The notions were introduced by D. Froncek in [5].

Definition 7.6. *An* ordered distance antimagic labeling *of a graph $G(V, E)$ with n vertices is a bijection $\vec{f} : V \to \{1, 2, \ldots, n\}$ with the property that $\vec{f}(x_i) = i$ and the sequence of the weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an increasing arithmetic progression with difference one. A graph G is an* ordered distance antimagic graph *if it allows an ordered distance antimagic labeling.*

Notice that this is an inverse ordering compared with the ordering of labeled vertices in a complete distance antimagic graph, or any distance magic graph which is a complement of a regular magic graph. There we have $w(1) > w(2) > \cdots > w(n)$, while in a graph with an ordered distance antimagic labeling we have $w(1) < w(2) < \cdots < w(n)$.

So far, only a sparse class of graphs is known to allow an ordered distance antimagic labeling.

Theorem 7.7. [5] *Let a, b be positive integers such that $a, b > 1, ab > 4$, and $a \equiv b \pmod 2$. Let $n = ab$ and $d = n - a - b + 1$. Then there exists a d-regular ordered distance antimagic graph with n vertices.*

A proof of Theorem 7.7 is based on magic rectangles. Recall that by Theorem 2.3 an $a \times b$ magic rectangle exists when the assumptions of Theorem 7.7 on a and b are satisfied. Let $G = K_a \square K_b$ with $V(G) = \{v_{ij} | 1 \leq i \leq a, 1 \leq j \leq b\}$ and $E(G) = \{v_{ij}v_{il} | 1 \leq i \leq a, 1 \leq j < l \leq b\} \cup \{v_{ij}v_{lj} | 1 \leq i < l \leq a, 1 \leq j \leq b\}$ and R_{ij} be an $a \times b$ magic rectangle with row sums s and column sums t. The labeling $\bar{f}(v_{ij}) = r_{ij}$ is obviously a distance 2-antimagic labeling, for when $\bar{f}(v_{ij}) = r_{ij} = q$, then $w_G(v_{ij}) = s + t - 2q$. Hence, the following observation holds.

Observation 7.8. [5] *The graph $G = K_a \square K_b$ is distance 2-antimagic when $a, b > 1, ab > 4$, and $a \equiv b \pmod 2$.*

The proof of Theorem 7.7 then follows easily. We show that $\overline{G}$, the complement of G, has and ordered antimagic labeling f. We define $\vec{f}(v_{ij}) = \bar{f}(v_{ij})$. For v_{ij} with $\vec{f}(v_{ij}) = q$ we have $w_G(v_{ij}) + w_{\bar{G}}(v_{ij}) = n(n + 1)/2 - q$ and because $w_G(v_{ij}) = s + t - 2q$, we have $w_{\bar{G}}(v_{ij}) = n(n + 1)/2 - s - t + q$. The values of q are $1, 2, \ldots, n$ and $\overline{G}$ has an ordered antimagic labeling.

8 Matrix Representation

Definition 8.1. *Let $G = (V, E)$ be a graph of order n with $V = \{v_1, v_2, \ldots, v_n\}$. Let $A = (a_{ij})$ be the adjacency matrix of G. Let $f : V \to \{1, 2, \ldots, n\}$ be a bijection, which gives a labeling of the vertices of G. The matrix $A_f = (b_{ij})$ of the labeling f is defined as follows.*

$$b_{ij} = \begin{cases} a_{ij} & \text{if } a_{ij} = 0 \\ f(v_j) & \text{if } a_{ij} = 1. \end{cases}$$

We observe that the matrix A_f is not symmetric. Also the matrix A_f is obtained from the adjacency matrix A by multiplying the i^{th} column of A by $f(v_i)$.

Further, if f is a distance magic labeling of G with magic constant k, then k is an eigenvalue of the matrix A_f. It is worth investigating whether the matrix A_f has any further property when f is a distance magic labeling of G.

9 Some Variants of Distance Magic Labelings

Acharya et al. [1] studied a variant of distance magic labeling in more general way, which they called neighborhood magic graph.

Definition 9.1. *A graph $G = (V, E)$ is said to be a neighborhood magic graph if there exists an injection $f : V \to R$ satisfying the condition $\sum\limits_{v \in N(u)} f(v) = Q(f)$, for all $u \in V(G)$. The constant $Q(f)$ is called the neighborhood magic index of f and the function f is called neighborhood magic lableing.*

Remark 9.2. *If f is a bijection from $V(G)$ to $N = \{1, 2, \ldots, |V|\}$, then the above definition coincides with the defintion of distance magic graphs.*

Jinnah [13] considered another variant of distance magic labelings which he called Σ^c-labeling.

Definition 9.3. *Let G be a graph on n vertices. Then a labeling $f : V(G) \to \{1, 2, \ldots, n\}$ is said to be a Σ^c-labeling if $\sum\limits_{u \in N[v]} f(u)$ is constant for each vertex v of G. S^c denotes the constant sum. We allow isolated vertices.*

Some basic results on this labeling are given in [13]. Beena [2] has given some more results on Σ^c-labeling.

10 Conjectures and Open Problems

We present several open problems and conjectures on distance magic graphs.

Conjecture 10.1. [15] *Let $1 \le a_1 \le \cdots \le a_p, p > 1$. Let $s_i = \sum\limits_{j=1}^{i} a_j$ and $n = s_p$. There exists a distance magic labeling of the complete multipartite graph $K_{a_1, a_2, \ldots, a_p}$ if and only if the following conditions hold.*

(i) $a_2 \ge 2$

(ii) $n(n+1) \equiv 0 (mod\ 2p)$ *and*

(iii) $\sum\limits_{j=1}^{s_i} (n + 1 - j) \ge \frac{in(n+1)}{2p}$ *for* $1 \le i \le p$.

Conjecture 10.2. [20] *If G is a distance magic graph different from $K_{1,2,2,\ldots,2}$ then the vertex set V can be partitioned into sets $V_1, V_2, \ldots, V_p$ such that for each i, has $|V_i| > 1$ and V_i is independent.*

Problem 10.3. [19] *If G is non-regular graph, determine if there is a distance magic labeling of $G[C_4]$.*

Problem 10.4. [17] *Characterize graphs G and H such that $G \square H$ is a distance magic graph.*

Problem 10.5. [17] *Characterize 4-regular distance magic graphs.*

Problem 10.6. *Does there exist a distance magic graph whose magic constant is a power of 2?*

Problem 10.7. *Does there exist an r-regular distance magic graph with n vertices where n is odd and r is a power of 2?*

Problem 10.8. *Does there exist a distance magic graph with two different distance magic labelings having different magic constants?*

Conjecture 10.9. [1] *For any even integer $n \geq 4$, the n-dimensional hypercube Q_n is not a distance magic graph.*

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