Discussiones Mathematicae
Graph Theory xx (xxxx) 1–9

DISTANCE MAGIC CARTESIAN PRODUCTS OF GRAPHS

Christian Barrientos

Department of Mathematics,
Clayton State University

e-mail: ChristianBarrientos@clayton.edu

Sylwia Cichacz[1]

AGH University of Science and Technology

e-mail: cichacz@agh.edu.pl

Dalibor Froncek

Department of Mathematics and Statistics,
University of Minnesota-Duluth

e-mail: dfroncek@d.umn.edu

Elliot Krop

Department of Mathematics,
Clayton State University

e-mail: ElliotKrop@clayton.edu

and

Christopher Raridan

Department of Mathematics,
Clayton State University

e-mail: ChristopherRaridan@clayton.edu

Abstract

A distance magic labeling of a graph $G = (V, E)$ with $|V| = n$ is a bijection $\ell : V \to \{1, \dots, n\}$ such that the weight of every vertex v, computed as the sum of the labels on the vertices in the open neighborhood of v, is a constant.

In this paper, we show that hypercubes with dimension divisible by four are not distance magic. We also provide some positive results by providing necessary and sufficient conditions for the Cartesian product of certain complete multipartite graphs and the cycle on four vertices to be distance magic.

Keywords: Distance magic labeling, magic constant, sigma labeling, Cartesian product, hypercube, complete multipartite graph, cycle..

2010 Mathematics Subject Classification: 05C76, 05C78.

1. Introduction

1.1. Definitions

For standard graph theoretic definitions and notation, we refer to Diestel [9]. All graphs $G = (V, E)$ are finite undirected simple graphs with vertex set $V(G)$ and edge set $E(G)$. Given any vertex v, the set of all vertices adjacent to v is the *open neighborhood* of v, denoted $N(v)$ (or $N_G(v)$, if necessary), and the *degree* of v is $|N(v)|$. If every vertex in a graph G has the same degree r, the graph is called *r-regular*. The *closed neighborhood* of v is $N(v) \cup \{v\}$, denoted $N[v]$ (or, $N_G[v]$).

A *distance magic labeling* of a graph G of order n is a bijection $\ell : V(G) \to \{1, \ldots, n\}$ such that the weight of every vertex v, defined as $w(v) = \sum_{u \in N(v)} \ell(v)$, is a constant, which we call the *magic constant*, denoted simply as μ. Any graph which admits a distance magic labeling is called a *distance magic graph*. Distance magic graphs are analogue to *closed distance magic graphs*; see [3, 6].

We use the definition of Cartesian product given in [12]. Given two graphs G and H, the *Cartesian product* of G and H, denoted $G \Box H$, is the graph with vertex set $V(G) \times V(H)$, where two vertices (g, h) and (g', h') are adjacent if and only if $g = g'$ and h is adjacent to h' in H, or $h = h'$ and g is adjacent to g' in G.

The cycle on n vertices is denoted C_n. The complete graph on n vertices is denoted K_n. The complete bipartite graph with parts of cardinality m and n, respectively, is denoted $K_{m,n}$. The complete r-partite graph with n vertices in each part is denoted $K(n; r)$. The n-dimensional hypercube is denoted $\mathcal{Q}_n$. The vertices of $\mathcal{Q}_n$ are binary n-tuples and two vertices are adjacent if their corresponding tuples differ in exactly one position. For integers $0 \le k \le n$, we say that a vertex of $\mathcal{Q}_n$ belongs to *row* k, denoted r_k, if the corresponding n-tuple contains exactly k entries that are 1's. For a vertex $v \in r_k$, if $0 \le k \le n-1$, we say the *upper neighbors* of v, denoted $N_u(v)$, are those vertices in r_{k+1} that are adjacent to v, and if $1 \le k \le n$, we say the *lower neighbors*, denoted $N_l(v)$, are those in r_{k-1} that are adjacent to v. For a vertex $v \in V(\mathcal{Q}_n)$, let $\{v\}$ denote the label on v and let $N_u\{v\} = \sum_{x \in N_u(v)} \{x\}$ and $N_l\{v\} = \sum_{x \in N_l(v)} \{x\}$ denote the sum of the labels on the upper and lower neighbors of v, respectively. Note that $\mathcal{Q}_n$ also may be defined recursively in terms of the Cartesian product: $\mathcal{Q}_1 = K_2$ and $\mathcal{Q}_n = \mathcal{Q}_{n-1} \Box K_2$ for integers $n \ge 2$.

1.2. History and Motivation

Graph labelings have served as the focal point of considerable study for over forty years; see Gallian's survey [11] for a review of results in the field. For a detailed survey of previous work and open problems concerning distance magic labelings, see Arumugam, Froncek, and Kamatchi [5]. Some graph which are distance magic among (some) products can be seen in [2, 4, 6, 7, 8, 16, 18, 19]. The general question about characterizing graphs G and H such that $G \Box H$ is distance magic was posed in [5]. Some results along that line follow:

Theorem 1 [18]. *The Cartesian product $C_n \Box C_m$ is distance magic if and only if $n = m \equiv 2 \pmod 4$.*

Theorem 2 [19]. *(1) The Cartesian product $P_n \Box C_m$, where n is an odd integer greater than 1 or $n \equiv 2 \pmod 4$, has no distance magic labeling. (2) The Cartesian product $K_{1,n} \Box C_m$ has no distance magic labeling. (3) The Cartesian product*

$K_{n,n}\square C_m$, where $n \neq 2$ and m is odd, has no distance magic labeling. (4) The Cartesian product $K_{n,n+1}\square C_m$, where n is even and $m \equiv 1 \pmod 4$, has no distance magic labeling.

It was shown in [15, 16, 17, 20] that if G is an r-regular distance magic graph with n vertices, then the magic constant must be $\mu = r(n+1)/2$, implying that no graph with odd regularity can be a distance magic. That is, $\mathcal{Q}_n$ for odd n is not distance magic. The concept of distance magic labelings has been motivated by the construction of magic rectangles (see [10, 13, 14]) since we can construct a distance magic labeling of $K(n;r)$ by labeling the vertices in each part by the columns of the magic rectangle. Note, however, that lack of an $n \times r$ magic rectangle does not imply that $K(n;r)$ is not distance magic; for example, there is no 2×2 magic rectangle but $\mathcal{Q}_2 = K(2;2) = K_{2,2}$ is distance magic. In 2004, Acharya, Rao, Singh, and Parameswaran stated the following conjecture:

Conjecture 3 [1]**.** *For any even integer $n \geq 4$, the n-dimensional hypercube $\mathcal{Q}_n$ is not a distance magic graph.*

The following problem was given in [7]:

Problem 4. If G is a regular graph, determine if $G\square C_4$ is distance magic.

Notice that if G is an r-regular graph, then the necessary condition for $H = G\square C_4$ to be distance magic is that r is even (since H is $(r+2)$-regular).

In Section 2, we show that $\mathcal{Q}_n$, where $n \equiv 0 \pmod 4$, is not distance magic. In Section 3, we provide some positive results by giving necessary and sufficient conditions for which $K(n;r)\square C_4$, where $n \neq 2$, is distance magic.

2. Non-distance magic hypercubes

Theorem 5. *The hypercube $\mathcal{Q}_n$, where $n \equiv 0 \pmod 4$, is not distance magic.*

Proof. Assume that $\mathcal{Q}_n$, where $n \equiv 0 \pmod 4$, is distance magic with magic constant μ. Let $k = n/2$. By symmetry of the hypercube, we have that

$$\binom{n}{k-1}\mu = \sum_{v \in r_{k-1}} \left(N_l\{v\} + N_u\{v\}\right) = \sum_{v \in r_{k+1}} \left(N_l\{v\} + N_u\{v\}\right). \tag{2.1}$$

By considering the binary representation of the vertices of the hypercube, for $1 \leq j \leq k-1$ and every vertex $v \in r_j$, we have $|N_u(v)| = n - j$ and $|N_l(v)| = j$. Thus,

$$\begin{aligned}\binom{n}{k-1}\mu &= (k+1)\sum_{v \in r_k}\{v\} + (k-1)\sum_{v \in r_{k-2}}\{v\} \\ &= (k+1)\sum_{v \in r_k}\{v\} + (k-1)\sum_{v \in r_{k+2}}\{v\},\end{aligned}$$

which implies that

$$\sum_{v \in r_{k-2}}\{v\} = \sum_{v \in r_{k+2}}\{v\}. \tag{2.2}$$

Using (2.1) and (2.2) as the basis step, we perform induction on the hypercube rows. Assume that for some j, where $1 < j \leq k/2$, and all $i \leq j$,

$$\sum_{v\in r_{k-2(i-1)}} \{v\} = \sum_{v\in r_{k+2(i-1)}} \{v\}. \tag{2.3}$$

Now, by symmetry of the hypercube,

$$\binom{n}{k-2i+1}\mu = \sum_{v\in r_{k-2i+1}} (N_l\{v\} + N_u\{v\}) = \sum_{v\in r_{k+2i+1}} (N_l\{v\} + N_u\{v\}),$$

which implies

$$\begin{aligned}(k+2i-1)\sum_{v\in r_{k-2i}} \{v\} + (k-2i+1)\sum_{v\in r_{k-2(i-1)}} \{v\} \\ = (k+2i-1)\sum_{v\in r_{k+2i}} \{v\} + (k-2i+1)\sum_{v\in r_{k+2(i-1)}} \{v\}.\end{aligned}$$

Using (2.3) gives

$$\sum_{v\in r_{k-2i}} \{v\} = \sum_{v\in r_{k+2i}} \{v\};$$

in particular,

$$\sum_{v\in r_0} \{v\} = \sum_{v\in r_{2k}} \{v\}. \tag{2.4}$$

Since both r_0 and r_{2k} contain only one vertex, (2.4) implies that the labels on these vertices are the same, which contradicts that $\mathcal{Q}_n$ has a distance magic labeling. ■

3. Distance magic $K(n;r)\square C_4$

In this section the proof is based on an application of magic rectangles, which are a natural generalization of magic squares. A *magic rectangle* $\mathrm{MR}(a,b)$ is an $a \times b$ array with entries from the set $\{1, 2, \ldots, ab\}$, each appearing once, with all its row sums equal to a constant δ and with all its column sums equal to a constant η. Harmuth proved the following:

Theorem 6 [13, 14]. *A magic rectangle $MR(a,b)$ exists if and only if $a, b > 1$, $ab > 4$, and $a \equiv b \pmod 2$.*

To prove our main result in this section, we will need the following generalization of magic rectangles that was introduced in [10].

Definition 3.1. A magic rectangle set $\mathrm{MRS}(a,b;c)$ is a collection of c arrays $(a\times b)$ whose entries are elements of $\{1, 2, \ldots, abc\}$, each appearing once, with all row sums in every rectangle equal to a constant δ and all column sums in every rectangle equal to a constant η.

Moreover, Froncek proved:

Theorem 7 [10]**.** *If $a \equiv b \equiv 0 \pmod 2$, $a \geq 2$ and $b \geq 4$, then a magic rectangle set MRS$(a, b; c)$ exists for every c.*

Observation 8 [10]**.** *If a magic rectangle set MRS$(a, b; c)$ exists, then both MR(a, bc) and MR(ac, b) exist.*

In the following lemmas, we use $C_4 = xuywx$ and we denote the vertices of $K(n; r)$, the complete r-partite graph with n vertices in each part, by $\{v_i^j : i = 1, \ldots, n \text{ and } j = 1, \ldots, r\}$, where we drop the subscript i if $n = 1$.

Lemma 3.2. The Cartesian product $K_n \square C_4$ is not distance magic.

Proof. Notice that $K_n = K(1; n)$. Let $H = K(1; n) \square C_4$. Suppose H is distance magic and ℓ is a distance magic labeling of H with magic constant μ. Let $\ell(v^j, u) + \ell(v^j, w) = a^j_{u,w}$ and $\ell(v^j, x) + \ell(v^j, y) = a^j_{x,y}$ for any $j = 1, \ldots, n$.

Since

$$\begin{aligned} 0 = w(v^j, x) - w(v^h, x) &= \ell(v^h, x) - \ell(v^j, x) + a^h_{u,w} - a^j_{u,w} \\ = w(v^j, y) - w(v^h, y) &= \ell(v^h, y) - \ell(v^j, y) + a^h_{u,w} - a^j_{u,w}, \end{aligned}$$

we obtain $\ell(v^h, x) - \ell(v^h, y) = \ell(v^j, x) - \ell(v^j, y)$ for any $j, h = 1, \ldots, n$. Therefore, $\ell(v^j, x) = k + \ell(v^j, y)$ for some constant k and for any $j = 1, \ldots, n$. On the other hand,

$$\begin{aligned} \mu = w(v^j, y) &= \sum_{p=1, p \neq j}^{r} \ell(v^p, y) + a^j_{u,w} \\ = w(v^j, x) &= \sum_{p=1, p \neq j}^{r} \ell(v^p, x) + a^j_{u,w} \\ &= \sum_{p=1, p \neq j}^{r} (k + \ell(v^p, y)) + a^j_{u,w}, \end{aligned}$$

which implies $k = 0$ and $\ell(v^j, x) = \ell(v^j, y)$, a contradiction. ■

Lemma 3.3. The Cartesian product $K(2; r) \square C_4$ is not distance magic.

Proof. Notice that $K_2 = K(2; 1)$ is not distance magic by Lemma 3.2. Moreover, $K_{2,2} \cong C_4$ and $C_4 \square C_4$ is not distance magic by Theorem 1, so we assume that $r > 2$. Let $H = K(2; r) \square C_4$. Suppose that H is a distance magic graph with distance magic labeling ℓ and magic constant μ. We have

$$\begin{aligned} \mu = w(v_1^j, x) &= \sum_{p=1, p \neq j}^{r} (\ell(v_1^p, y) + \ell(v_2^p, y)) + \ell(v_1^j, u) + \ell(v_1^j, w) \\ = w(v_2^j, x) &= \sum_{p=1, p \neq j}^{r} (\ell(v_1^p, y) + \ell(v_2^p, y)) + \ell(v_2^j, u) + \ell(v_2^j, w), \end{aligned}$$

which implies that $\ell(v_1^j, u) + \ell(v_1^j, w) = \ell(v_2^j, u) + \ell(v_2^j, w)$ for any $j = 1, \ldots, r$. Analogously, we obtain that $\ell(v_1^j, x) + \ell(v_1^j, y) = \ell(v_2^j, x) + \ell(v_2^j, y)$ for any $j = 1, \ldots, r$.

Since $w(v_1^j,x) = w(v_1^j,y)$, we obtain that

$$\sum_{p=1,p\neq j}^{r} (\ell(v_1^p,x)+\ell(v_2^p,x)) = \sum_{p=1,p\neq j}^{r} (\ell(v_1^p,y)+\ell(v_2^p,y))$$

for any $j = 1,2,\dots,r$. Hence

$$(r-1)\sum_{j=1}^{r}(\ell(v_1^j,x)+\ell(v_2^j,x)) = (r-1)\sum_{j=1}^{r}(\ell(v_1^j,y)+\ell(v_2^j,y)),$$

implying that $\ell(v_1^j,x)+\ell(v_2^j,x) = \ell(v_1^j,y)+\ell(v_2^j,y)$, a contradiction. ∎

Lemma 3.4. Let $r > 1$, $n > 2$. The Cartesian product $K(n;r)\Box C_4$ is distance magic if and only if n is even.

Proof. Let $H = K(n;r)\Box C_4$. Notice that $|V(H)| = 4nr$ and H is $[n(r-1)+2]$-regular. Suppose that H is distance magic and ℓ is a distance magic labeling of H with magic constant μ.

Let $\ell(v_i^j,u)+\ell(v_i^j,w) = a_{u,w}^j$ for any $i = 1,\dots,n$, $j = 1,\dots,r$. Then

$$\mu = w(v_i^j,x) = \sum_{p=1,p\neq j}^{r}\sum_{h=1}^{n}\ell(v_h^p,x)+a_{u,w}^j,$$

for any $i = 1,\dots,n$, $j = 1,\dots,r$. Analogously let $\ell(v_i^j,x)+\ell(v_i^j,y) = a_{x,y}^j$ for any $i = 1,\dots,n$, $j = 1,\dots,r$.

Observe that

$$2\mu = w(v_i^h,x)+w(v_i^h,y) = n\sum_{p=1,p\neq h}^{r} a_{x,y}^p + 2a_{u,w}^h \tag{3.1}$$

and

$$2\mu = w(v_i^j,x)+w(v_i^j,y) = n\sum_{p=1,p\neq j}^{r} a_{x,y}^p + 2a_{u,w}^j \tag{3.2}$$

for $j = 1,\dots,r$,$i = 1,\dots,n$.

Thus subtracting equation 3.1 from 3.2 we obtain:

$$n(a_{x,y}^j - a_{x,y}^h) = 2(a_{u,w}^j - a_{u,w}^h),$$

for any $j,h = 1,\dots,r$. Analogously $2(a_{x,y}^j - a_{x,y}^h) = n(a_{u,w}^j - a_{u,w}^h)$ for any $j,h = 1,\dots,r$.

Obviously for any $j,h = 1,\dots,r$ we have $(n-2)(a_{x,y}^j - a_{x,y}^h) = -(n-2)(a_{u,w}^j - a_{u,w}^h)$. Since $n \neq 2$ thus for any $j = 1,\dots,r$ we have $a_{x,y}^j + a_{u,w}^j = a$ for some constant a.

If $a_{x,y}^j = a_{u,w}^j = a/2$ for any $j = 1,2,\dots,r$, then since $\mu = w(v_i^j,z) = \sum_{p=1,p\neq j}^{r}\sum_{h=1}^{n}\ell(v_h^p,z)+a/2$ for any $z \in \{x,y,u,w\}$ and $i = 1,\dots,n$, $j = 1,\dots,r$, it is easy to check that $\sum_{i=1}^{n} \ell(v_i^j, z) = na/4$ for any $z \in \{x, y, u, w\}$ and $j = 1, \ldots, r$. In this situation there exists distance magic labeling for the graph G is and only if there exists a magic rectangle set MRS$(2, n; 2r)$ with all its row sums equal to the constant $a/2$ and with all its column sums equal to the constant $na/4$.

If n is even, then a magic rectangle set MRS$(2, n; 2r)$ exists by Theorem 7. Denote by $z_{i,h}^j$ the entry in the i-th row and h-th column of the j-th rectangle from the set MRS$(2, n; 2r)$, let:

$$\ell(v_i^j, x) = z_{i,1}^j,\ \ \ell(v_i^j, y) = z_{i,2}^j,$$

$$\ell(v_i^j, u) = z_{i,1}^{j+r},\ \ \ell(v_i^j, v) = z_{i,2}^{j+r}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, r$. Obviously the labeling ℓ is distance magic.

Therefore we can assume now that n is odd. Suppose first that $a_{x,y}^j = a/2 - c$ for any $j = 1, 2, \ldots, r$ and some constant c. Thus $a_{x,y}^j = a/2 + c$ for any $j = 1, 2, \ldots, r$ and moreover $\sum_{i=1}^{n}(\ell(v_i^j, x) + \ell(v_i^j, y)) = n(a/2 + c)$, $\sum_{i=1}^{n}(\ell(v_i^j, u) + \ell(v_i^j, v)) = n(a/2 - c)$ for any $j = 1, 2, \ldots, r$. Observe that:

$$2\mu = w(v_i^j, x) + w(v_i^j, y) = n(r-1)(a/2 + c) + 2(a/2 - c),$$
$$2\mu = w(v_i^j, u) + w(v_i^j, v) = n(r-1)(a/2 - c) + 2(a/2 + c).$$

Subtracting the above equation we obtain that $c = 0$, hence $a_{x,y}^j = a_{u,v}^j = a/2$ and a distance magic labeling is impossible since there does not exist a magic rectangle set MRS$(2, n; 2r)$ for n being odd n must be even by Theorem 6 and Observation 8.

Let now $a_{x,y}^j = a/2 - c^j$ and $a_{u,v}^j = a/2 + c^j$ for any $j = 1, 2, \ldots, r$ and some constants c^j. Therefore $\sum_{i=1}^{n}(\ell(v_i^j, x) + \ell(v_i^j, y)) = n(a/2 + c^j)$, $\sum_{i=1}^{n}(\ell(v_i^j, u) + \ell(v_i^j, v)) = n(a/2 - c^j)$ for any $j = 1, 2, \ldots, r$. Notice that

$$2\mu = w(v_i^j, x) + w(v_i^j, y) = n \sum_{p=1, p\neq j}^{r} (a/2 + c^p) + 2(a/2 - c^j) \tag{3.3}$$

and

$$2\mu = w(v_i^h, x) + w(v_i^h, y) = n \sum_{p=1, p\neq h}^{r} (a/2 + c^p) + 2(a/2 - c^h) \tag{3.4}$$

for $j = 1, \ldots, r$,$i = 1, \ldots, n$.

Thus subtracting equation 3.3 from 3.4 we obtain: $(n+2)c^h = (n+2)c^j$ for any $j, h = 1, \ldots, r$. Hence $c^j = c$ for any $j = 1, 2, \ldots, r$ and a distance magic labeling does not exist. ■

As a result of Lemmas 3.2, 3.3 and 3.4, we have the following theorems:

Theorem 9. *The Cartesian product $K(n; r)\Box C_4$ is distance magic if and only if $r > 1$ and $n > 2$ is even.*

Theorem 10. *The Cartesian product $K(n; r)\Box C_4$ is distance magic if and only if there exists a magic rectangle set MRS$(2, n; 2r)$.*

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