Factorizations of complete graphs into trees with at most four non-leave vertices

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Abstract

We give a complete characterization of trees with at most four non-leave vertices, which factorize the complete graph K_{2n}.

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1 Introduction and definitions

In this paper we investigate isomorphic factorizations of complete graphs into trees with at most four non-leave vertices. We completely characterize lobsters with four non-leave vertices of diameter 4, which factorize the complete graph K_{2n}. The paper has three main parts. First in Section 1 we state some basic definitions. Then a recursive method for factorizations of complete graphs into spanning trees is given in Section 2. In the second part (Sections 3 and 4) we give necessary conditions for a spanning tree with four non-leave vertices to factorize the complete graph K_{2n}. We list all lobsters of order $2n$ with four non-leave vertices that do not factorize K_{2n}. The third part (Sections 5

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– 7) of the paper is technical, we check all starting cases for the recursive method case by case.

We consider only graphs with no loops or multiple edges. We begin by stating a well known definition.

Definition 1.1 *Let H be a graph on n vertices. A* decomposition of the graph *H is a set of pairwise edge disjoint subgraphs $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$ of H such that every edge of H belongs to precisely one of the subgraphs G_r. If each subgraph G_r is isomorphic to a graph G we speak about a G-*decomposition of *H. If G is a connected factor of H, then the G-decomposition is often called a G-*factorization.

A G-decomposition of H with $2n$ vertices into $G_1, G_2, \ldots, G_s$ is bi-cyclic *if there exists an ordering $(x_0, x_1, \ldots, x_{n-1}, y_0, y_1, \ldots, y_{n-1})$ of vertices of H and isomorphisms $\phi_i : G_1 \to G_i, i = 1, 2, \ldots, s$ such that $\phi_i(x_j) = x_{i+j}$ and $\phi_i(y_j) = y_{i+j}$ for every $j = 0, 1, \ldots, n-1$, where the subscripts are taken modulo n.*

Thus a bi-cyclic factorization means that there exist a partition of the vertex set of G, $V(G)$, into sets V_0 and V_1 such that $|V_0| = |V_1| = n$ and all the vertices of V_0 and V_1 rotate separately under a cyclic permutation of order n. This way we obtain all factors $G_1, G_2, \ldots, G_n$ of the G-factorization of H. We say that such a bi-cyclic factorization is obtained by a *bi-cyclic rotation*.

There are two obvious necessary conditions for the existence of a factorization of the complete graph K_m into spanning trees isomorphic to a tree T. First, m must be even, since the number of edges of T, $m-1$, must divide $m(m-1)/2$, the number of edges of K_m. Obviously, there will always be $m/2$ factors isomorphic to T. Therefore, we further consider only factorizations of the complete graph K_{2n}. Second, $\Delta(T)$ is bounded by the *degree condition*. The maximum degree of T cannot exceed n, because the number of factors is $n = m/2$, and each vertex must be in every factor (i.e. a spanning tree) of degree at least one.

A tree is a *caterpillar* if the deletion of all leaves (i.e., vertices of degree 1) gives a path or a single vertex, and a *lobster* if the deletion of all leaves gives a caterpillar. Obviously, there are just two different classes of trees with four non-leave vertices — caterpillars of diameter 5 and certain lobsters of diameter 4. The non-leave vertices in a caterpillar are called spinal vertices. A caterpillar R of order $2n$ and diameter four or five we denote by $(\deg(a), \deg(c), \deg(b))$ or $(\deg(A), \deg(a), \deg(b), \deg(B))$-caterpillar, respectively, where a, c, b or A, a, b, B, respectively, are spinal vertices of these caterpillars in order from left to right. If L is a lobster of diameter 4, then the only vertex with eccentricity 2 is called the *central vertex* or the *center* of L, and the remaining non-leaves are called *secondary vertices*. The central vertex is denoted by c and the secondary vertices by x, y, and z. Further, the number of vertices of degree 1 adjacent to the vertex c or x or y or z, respectively, will be denoted C or X or Y or Z. Therefore we denote every lobster L of order $2n$ and diameter 4 with four non-leaves vertices c, x, y, z by $L = (C|X, Y, Z)$ or $(C|X, Y, Z)$, where $C \geq 0$ and $X \geq Y \geq Z \geq 1$. Obviously, $C + X + Y + Z = 2n - 4$. For convenience we denote a few times in Sec. 5 also some caterpillars by $L(C|X, Y, 0)$ allowing $Z = 0$.

A tree with one non-leaf is a star and with two non-leave vertices is a double-star. It is easy to observe that a star $K_{1,2n-1}$ cannot factorize K_{2n} and the only double-star which can (and actually does) factorize K_{2n} must have both the non-leaves of degree n. A tree with three non-leaves is a caterpillar of diameter 4. A complete characterization of caterpillars of diameter 4, which factorize complete graphs, was given by Fronček [4], Kovářová [10,11], and Kubesa [12,17]. For caterpillars of diameter 5 the characterization was given by the authors and Tereza Kovářová in [7]. Therefore, to characterize completely the class mentioned in the title, we only need to investigate lobsters with four non-leaves.

Our constructions are mostly recursive, and based on special types of labelings, which we define below.

Definition 1.2 *Let G be a graph with $V(G) = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = r$. Let λ be an injection, $\lambda : V_i \to \{0_i, 1_i, \ldots, (r-1)_i\}$, $i = 0, 1$.*

The pure length *of an edge (x_i, y_i) with $x_i, y_i \in V_i$, where $i \in \{0, 1\}$, for $\lambda(x_i) = p_i$ and $\lambda(y_i) = q_i$ is defined as*

$$\ell_{ii}(x_i, y_i) = \min\{p - q \pmod r,\ q - p \pmod r\}.$$

The mixed length *of an edge (x_0, y_1) with $x_0 \in V_0$, $y_1 \in V_1$, for $\lambda(x_0) = p_0$ and $\lambda(y_1) = q_1$, is defined as*

$$\ell_{01}(x_0, y_1) = q - p \pmod r,$$

where $p, q \in \{0, 1, \ldots, r-1\}$ are the vertex labels without subscripts. This concept can be extended to any equal-sized partite sets $V_0, V_1, \ldots, V_k$.

Definition 1.3 *A graph G with $4n + 1$ edges has a* blended labeling *(sometimes called* blended ρ-labeling*) if there exists a partition $V(G) = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$, $|V_0| = |V_1| = 2n + 1$ and an injection $\lambda : V_i \to \{0_i, 1_i, \ldots, (2n)_i\}$, where $x_i, y_i \in V_i$, $i = 0, 1$ with the following property:*

(1) $\{\ell_{ii}(x_i, y_i) \colon (x_i, y_i) \in E(G)\} = \{1, 2, \ldots, n\}$ *for* $i = 0, 1$
(2) $\{\ell_{01}(x_0, y_1) \colon (x_0, y_1) \in E(G)\} = \{0, 1, \ldots, 2n\}$.

To simplify our notation, we often unify vertices with their respective labels.

We will say “a vertex i” rather than “a vertex x with $\lambda(x) = i$”. As follows from the definition, a blended labeling can be used for a factorization of a complete graph only if the number of vertices is $2 \pmod 4$.

Theorem 1.4 (Fronček) *Let G with $4n+1$ edges have a blended labeling. Then there exists a bi-cyclic decomposition of K_{4n+2} into $2n+1$ copies of G.*

The proof can be found in [3]. For an example see Figure 1. The arrows stand for the mapping ϕ_1 from Definition 1.1.

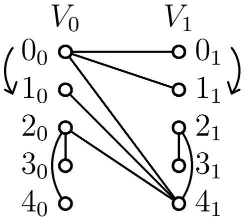

Fig. 1. Blended labeling of a tree on 10 vertices.

A swapping labeling introduced in [11] is often used for factorizations of K_{2n} when the number of vertices is $0 \pmod 4$.

Definition 1.5 *A graph G with $4n-1$ edges has a* swapping blended labeling *(briefly* swapping labeling*) if the following is satisfied. The vertex set $V(G) = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = 2n$. Let λ be an injection, $\lambda : V_i \to \{0_i, 1_i, \dots, (2n-1)_i\}$ for $i = 0, 1$. The pure length ℓ_{ii} for $i \in \{0, 1\}$ and the mixed length ℓ_{01} of an edge are defined as in Definition 1.2 and*

(1) $\{\ell_{ii}(x_i, y_i)\colon (x_i, y_i) \in E(G)\} = \{1, 2, \dots, n\}$, *for $i = 0, 1$,*
(2) there exists an isomorphism φ such that G is isomorphic to G', where $V(G') = V(G)$ and $E(G') = E(G) \setminus \{(k_0, (k+n)_0), (l_1, (l+n)_1)\} \cup \{(k_0, (l+n)_1), ((k+n)_0, l_1)\}$,
(3) $\{\ell_{01}(x_0, y_1)\colon (x_0, y_1) \in E(G)\} = \{0, 1, \dots, 2n-1\} \setminus \{\ell_{01}(k_0, (l+n)_1)\}$.

Notice that G with a swapping labeling can be split into three subgraphs, H_0 and H_1 on the vertices of V_0 and V_1, respectively, and a bipartite subgraph H_{01} with the partite sets V_0 and V_1. The labelings of H_0 and H_1 induced by λ are such that the edges in H_0 or H_1 have all different pure lengths (Condition (1)). The labeling induced by λ on the vertices of H_{01} gives edges of almost all different mixed lengths. Edges of the mixed length $\ell_{01}(k_0, (l+n)_1) = \ell_{01}((k+n)_0, l_1) = l + n - k \pmod{2n}$ is missing (Condition (3)).

Let us decompose a complete graph K_{4n} into copies of G using swapping labeling. It is not difficult to observe that if one lets G rotate bi-cyclically so that the vertices of V_0 or V_1 permute separately under cyclic permutations, then two complete graphs K_{2n} on vertices of V_0 and V_1 are decomposed into $2n$ ‘copies’ of H_0 and H_1, respectively. Half of the ‘copies’ have the edge of length

n missing. But since the number of vertices of V_0 or V_1 is even each edge of the maximum pure length is covered twice. Also the complete bipartite graph $K_{2n,2n}$ with partite sets V_0, V_1 is decomposed into copies of H_{01}, and since an edge of mixed length $\ell_{01}(k_0, (l+n)_1)$ is missing in H_{01}, edges of this length in $K_{2n,2n}$ are not covered at all. Therefore the edges of the maximum pure length remain in H_0 and H_1 only for the first n rotations. In the next n rotations these edges are swapped for the mixed edges of the missing length in H_{01}. The isomorphism φ required by the Condition (2) guarantees that after swapping the edges, isomorphic copies of G are obtained.

Theorem 1.6 (Kovářová) *Let G be a graph on $4n$ vertices with $4n-1$ edges which has a swapping blended labeling. Then there exists a G-decomposition of K_{4n} into $2n$ isomorphic copies of G.*

For a proof of this theorem see [11].

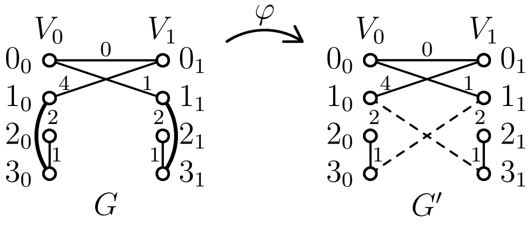

Fig. 2. Swapping labeling of a path on 8 vertices.

However, since the swapping labeling is very difficult to use, in the next section we generalize the notion of bi-cyclic labeling to four sets V_0, V_1, V_2, and V_3 by allowing the sets V_i to be of two different sizes, thus obtaining for $2n = 4k$ four sets of two different odd sizes.

Remark 1.7 *So far we have presented two tools for factorizations (or decompositions in general), namely blended and swapping labelings. Suppose a graph G admits one of these labelings, then there exists a G-factorization (or decomposition) of the corresponding complete graph by Theorems 1.4 and 1.6. We say the factorization (or decomposition) is* given by the labeling.

Some of the claims in Section 1 are stated for decompositions and some for factorizations only. We preferred to give the more general statements, if possible. Starting with Section 2 we deal only with factorizations.

Remark 1.8 *In our proofs we will often present labelings of lobsters in the form of figures. In that case, the vertices in the left column are assumed to have labels (from top to bottom) $0_0, 1_0, \ldots, t_0$ and the vertices in the right column are assumed to have labels $0_1, 1_1, \ldots, t_1$, where t is typically equal to $2n$ or $2n-1$. See an example with labels in Figures 1 and 2.*

2 Recursive constructions

Now we focus on lobsters of order $2n$ with four non-leaves and with diameter 4. For the notation see page 2.

Notice that while so far we were using for an edge between vertices labeled i and j notation (i, j), now we denote the edge between vertices x and y also by xy. We will use both notations alternatively; the former mostly if dealing with labeled vertices, the latter mostly for unlabeled vertices. Whether a vertex is labeled or unlabeled should always be clear from the context.

Definition 2.1 *Let $\mathcal{L} = \{L_1, L_2, \ldots, L_n\}$ be an L-factorization of K_{2n} induced by the isomorphisms $\phi_j, L_j = \phi_j(L)$ for $j = 1, 2, \ldots, n$, where L is a lobster on $2n$ vertices. We say that a vertex v is* semisurjective with respect to the sets V_0 and V_1 *(or briefly just* semisurjective*) if there exists a partition V_0, V_1 of the vertex set of K_{2n} with $|V_0| = |V_1| = n$ such that either $\{\phi_j(v) : j = 1, 2, \ldots, n\}$ is equal to V_0 or V_1. We call an L-factorization of K_{2n}* semisurjective *if every vertex v of L is semisurjective, and* weakly semisurjective *if every non-leave vertex of L is semisurjective.*

Observe that if L has a blended labeling, then the factorization is semisurjective. This follows from the fact that the factorization is bi-cyclic. None of the above is true in general for a swapping labeling, but for certain lobsters it is possible to obtain swapping labelings that give weakly semisurjective factorizations. E.g., this is true if the swapped edge has one vertex of degree 1 which is always the case in our constructions below.

In the following sections we give recursive constructions for labelings of lobsters with four non-leave vertices. Mostly we *reduce* lobsters into smaller lobsters with the same diameter. By *reduction* we mean the following. Suppose that we have a lobster L' with $2n + 2m$ vertices. We select $2m$ vertices of degree one, $v_1, v_2, \ldots, v_m$ and $u_1, u_2, \ldots, u_m$ and remove them to obtain a lobster L with $2n$ vertices. Then we find a semisurjective or weakly semisurjective L-factorization of K_{2n} with vertex partition V_0, V_1, both V_i of size n. If all the vertices $v_1, v_2, \ldots, v_m$ were in L' adjacent to vertices of V_0 while all the vertices $u_1, u_2, \ldots, u_m$ had their neighbors in V_1, then we have reduced L' to L. We then say that L is a *reduced* (also called *base*) lobster. If $\operatorname{diam} L = \operatorname{diam} L'$, then we need the reduced lobster L to have just a weakly semisurjective factorization. As we observe further, the reduced lobster can be shorter. If $\operatorname{diam} L < \operatorname{diam} L'$, then also the diametrical leaves in L arising from non-leave vertices in L' have to be semisurjective.

Definition 2.2 *Let N be the graph obtained from P_4 with vertices y_0, x_0, x_1, y_1 and edges y_0x_0, x_0x_1, x_1y_1 by blowing up in the following way: a copy of K_n is put into each of the inner vertices x_0, x_1 of P_4 and $\overline{K}_m = mK_1$ into the end-*

vertices y_0, y_1. The edge x_0x_1 of P_4 is replaced by $K_{n,n}$ and the edges y_0x_0, x_1y_1 of P_4 are replaced by the edges of $K_{m,n}$ and $K_{n,m}$, respectively. The vertex set of N is then $V(N) = Y_0 \cup X_0 \cup X_1 \cup Y_1$, where $|X_0| = |X_1| = n$, $|Y_0| = |Y_1| = m$, and $|V(N)| = 2n + 2m$.

Let Z be the graph obtained from P'_4 with vertices x_0, y_1, y_0, x_1 and edges x_0y_1, y_1y_0, y_0x_1 by blowing up in the following way: a copy of K_m is put into each of the inner vertices y_0, y_1 of P'_4 and $\overline{K}_n = nK_1$ into the end-vertices x_0, x_1. The edge y_1y_0 of P'_4 is replaced by $K_{m,m}$ and the edges x_0y_1, y_0x_1 of P'_4 are replaced by the edges of $K_{n,m}$ and $K_{m,n}$, respectively.

The vertex set of Z is then the same as the vertex set of N, $V(Z) = V(N)$.

It is easy to observe that the graphs N and Z factorize the complete graph K_{2n+2m} into factors N and Z which may or may not be isomorphic, see Figure 3.

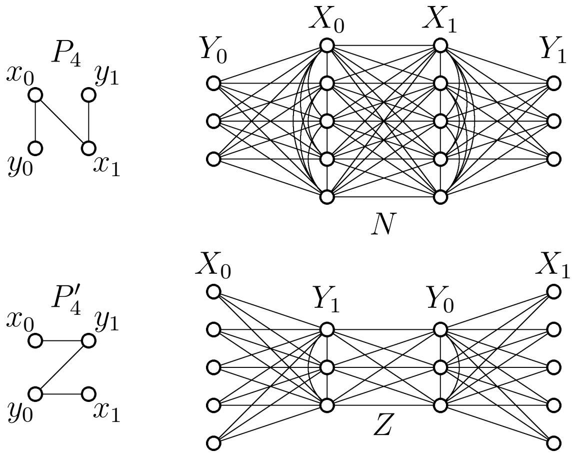

Fig. 3. Graphs N and Z for $n = 5$ and $m = 3$.

The graphs N and Z can be further factorized separately into isomorphic factors, namely into certain lobsters with four non-leave vertices. Further in this text we use the notation N and Z for the graphs defined in Definition 2.2. For factorizations we consider only those lobsters that satisfy the necessary conditions listed in Section 1 on page 2 and given by Theorem 3.2. Hence the following lemma.

Lemma 2.3 *If a graph H decomposes both graphs N and Z then H decomposes the complete graph K_{2n+2m}.*

Lemma 2.4 *Let L' be a lobster of order $2(n+m)$ with four non-leave vertices. If a reduced lobster L of order $2n$ weakly semisurjectively factorizes K_{2n} then the lobster L' factorizes the graph N.*

PROOF. The vertices in sets V_0 and V_1, respectively, we identify with the vertices in the sets X_0 and X_1 of N. The additional vertices $v_1, v_2, \dots, v_m$ and $u_1, u_2, \dots, u_m$, respectively, of degree one we identify with the vertices in Y_0 and Y_1. If we bi-cyclically rotate the vertices in X_0, X_1 and we fix the vertices in Y_0, Y_1 then we obtain an L'-factorization of N (see Fig. 4). □

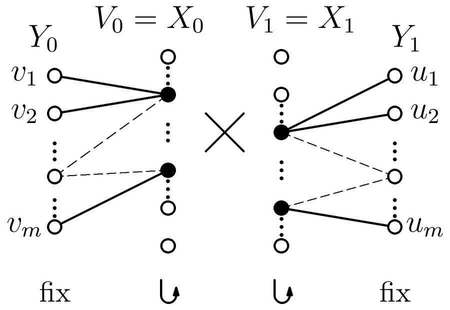

Fig. 4. Factorization of the graph N.

The following lemma it can be proven by the same way (see Fig. 5).

Lemma 2.5 *Let L' be a lobster of order $2(n+m)$ with four non-leave vertices. If a reduced lobster L of order $2m$ weakly semisurjectively factorizes K_{2m} then the lobster L' factorizes the graph Z.*

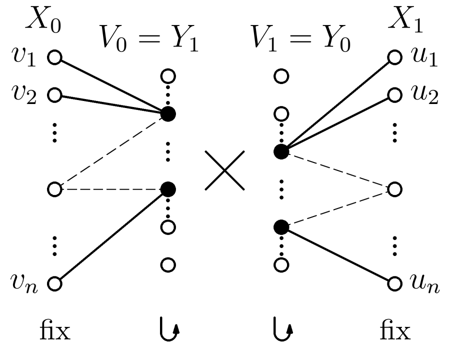

Fig. 5. Factorization of the graph Z.

The method of factorization described in the proof of Lemma 2.4 is called *fixing rotation* or *fixing method*.

3 Necessary conditions

The following proof goes by a simple counting argument. We extend a well known result (see e.g. [4]) to *any* tree with four non-leave vertices, not only to caterpillars with diameter 5. The smallest non-leave degree $\deg(v_4)$ can be 2.

We say two vertices x and y from different factors *meet* at some vertex v of K_{2n}, if $\phi_i(x) = \phi_j(y) = v$ for some $i \neq j \in \{0, 1, \dots, n-1\}$.

Theorem 3.1 *Let T be any tree with exactly four non-leave vertices v_1, v_2, v_3 and v_4 such that* $\deg(v_1) \geq \deg(v_2) \geq \deg(v_3) \geq \deg(v_4) \geq 2$*. If T factorizes K_{2n} then $n + 1 - \deg(v_4) \leq \deg(v_1) \leq n$.*

PROOF. Since T is a factor of K_{2n}, T has $2n-1$ edges and there are exactly

$$\frac{|E(K_{2n})|}{2n-1} = \frac{2n(2n-1)}{2(2n-1)} = n$$

factors in K_{2n} isomorphic to T. The upper bound $\deg v_1 \leq n$ follows from the fact that for every particular vertex v of K_{2n} the degree of v is in every factor at least 1 and the sum of degrees over all n factors is $2n-1$. By $\deg_i(v)$ we denote the degree of v in the factor F_i $(i = 1, 2, \dots, n)$. Thus for every vertex and every i is

$$\begin{aligned}
2n - 1 &= \deg_1(v) + \deg_2(v) + \cdots + \deg_{n-1}(v) + \deg_n(v) \\
2n - 1 &\geq 1 + 1 + \cdots + 1 + \deg_i(v) \\
2n - 1 &\geq n - 1 + \deg_i(v) \\
n &\geq \deg_i(v).
\end{aligned}$$

To prove the lower bound $\deg v_1 \geq n + 1 - \deg(v_4)$ we count the non-leave vertices in two ways. There are four non-leave vertices in every of the n factors. This gives a total of $4n$ non-leave vertices among all factors. Suppose the largest degree is strictly less than n, that is $\deg(v_i) \leq \deg(v_1) < n$. Then at every vertex v of K_{2n} there meet exactly two non-leave vertices from different factors. If there is a vertex v in K_{2n} where three or more non-leave vertices from different factors meet, then by pigeonhole principle there would be a vertex u in K_{2n} where at most one non-leave vertex meets with $n-1$ vertices of degree 1 from other factors and the sum of their degrees would be

$$\deg v_i + 1 + 1 + \cdots + 1 = \deg v_i + (n-1) < n + n - 1 = 2n - 1$$

which is a contradiction, since every v in K_{2n} is of degree $2n-1$.

Now we examine the possibilities for a certain non-leaf vertex v_1 of the largest degree. This vertex meets at some vertex v of K_{2n} with another v_i $(i \in \{1, 2, 3, 4\})$ and $n-2$ vertices of degree 1. Since $\deg v_i \geq \deg(v_4)$ it immediately follows

$$\deg(v_1) + \deg(v_i) + (n-2) \cdot 1 = 2n - 1$$

$$\begin{aligned}\deg(v_1) &= n+1-\deg(v_i)\\ \deg(v_1) &\geq n+1-\deg(v_4).\end{aligned}$$

□

Another extension of the counting argument gives the values of sums of pairs or triples of degrees.

Theorem 3.2 *Let T be any tree with exactly four non-leave vertices v_1, v_2, v_3 and v_4 such that* $\deg(v_1) \geq \deg(v_2) \geq \deg(v_3) \geq \deg(v_4) \geq 2$. *If T factorizes K_{2n} then*

(1) either $\deg(v_1) = n$ *and* $\deg(v_2) + \deg(v_3) + \deg(v_4) = n + 2$,
(2) or $\deg(v_1) + \deg(v_4) = \deg(v_2) + \deg(v_3) = n + 1$.

PROOF. *(1)* If $\deg(v_1) = n$ then there are n vertices in K_{2n} where one vertex of degree n meets with $n-1$ vertices all of degree 1. There are n remaining vertices of K_{2n} where the vertices v_2, v_3, v_4 meet with vertices of degree 1. There are n factors and adding all degrees of n vertices v_2, n vertices v_3 and n vertices v_4 and the remaining $(2n-4)-(n-1)$ vertices of degree 1 ($n-1$ leaves meet with v_1) in each of the n factors, we get n times the degree $2n-1$. Thus

$$\begin{aligned}n \cdot (\deg(v_2) + \deg(v_3) + \deg(v_4) + n - 3) &= n \cdot (2n-1)\\ \deg(v_2) + \deg(v_3) + \deg(v_4) &= n + 2.\end{aligned}$$

(2) If $\deg(v_1) < n$ we can conclude by the same argument as in the proof of Theorem 3.1 that at every vertex v of K_{2n} there meet exactly two non-leave vertices from two distinct factors with $n-2$ leaves from the remaining factors. Thus

$$\begin{aligned}\deg(v_i) + \deg(v_j) + n - 2 = \deg(v_k) + \deg(v_l) + n - 2 &= 2n - 1\\ \deg(v_i) + \deg(v_j) = \deg(v_k) + \deg(v_l) &= n + 1,\end{aligned}$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. For the two equalities to hold, the largest degree is summed up with the lowest and the second largest degree with the second lowest. We have

$$\deg(v_1) + \deg(v_4) = \deg(v_2) + \deg(v_3) = n + 1.$$

□

4 Nonexistence

Consider a lobster $L = (0|n-1, n-k-3, k)$ for $k \geq 1$. Denote the vertices adjacent to the central vertex, c, by x, y, z where $\deg x = n, \deg y = n-k-2, \deg z = k+1$ and by c', x', y', z' vertices of degree one adjacent to c, x, y, z, respectively. Recall that $\phi_i(L)$ is the mapping taking the lobster L to its image $L_i \in \mathcal{L}$.

First we make an easy observation.

Proposition 4.1 *If there is a factorization of K_{2n} by $L = (0|n-1, n-k-3, k)$ and $A = \{\phi_i(x) \colon i = 1, 2, \ldots, n\} = \{x_1, x_2, \ldots, x_s\}$, then $s = n$ and for any $x_i \in A$ always $x_i = \phi_i(x)$ and x_i is of degree one in all other images $L_j, j \neq i$.*

Denote $B = V(K_{2n}) \setminus A$. Then $|B| = n$ and $\phi_i(c), \phi_i(y), \phi_i(z) \in B$ and $\phi_i(x) \notin B$ for all $i = 1, 2, \ldots, n$.

Now we prove a technical lemma. Recall that $\deg y \geq \deg z$.

Lemma 4.2 *If there is a factorization of K_{2n} into $L = (0|n-1, n-k-3, k)$, then there is a vertex $\tilde{w}_j$ which is an image of y and/or z in at most two factors.*

PROOF. We proceed by contradiction and use notation of Proposition 4.1. By $w_j, j = 1, 2, \ldots, n$ denote vertices in B. When we want to stress that a particular w_j is an image of y in at least one factor L_j, we denote it by $\tilde{w}_j$.

Suppose that every vertex in B that is an image of y or z is an image of y and/or z at least three times. Let these vertices be $\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_s \in B$. Obviously, $s \leq 2n/3$, otherwise in n factors we have more than $2n$ occurrences of images of y and z, which is impossible. Also, since we have n images of y then there must be a vertex $\tilde{w}_j$ such that it is an image of y in at least two of the three or more factors. Because we have $\deg(y) + \deg(z) = (n-k-2)+(k+1) = n-1$ and $\deg(y) \geq \deg(z)$, then we obtain $\deg(y) \geq (n-1)/2$. Therefore in these three factors the sum of degrees of $\tilde{w}_j$ is at least $(n-1) + (n-1)/2 = (3n-3)/2$. At the same time we need at least $n-3$ edges incident with $\tilde{w}_j$ in the remaining $n-3$ factors. Hence $(3n-3)/2 \leq (2n-1) - (n-3) = n+1$ and $n \leq 7$. Thus for $n > 7$ every image of y in an L-factorization of K_{2n} is an image of y and/or z at most two times.

If $n = 7$ and $k = 1$, then $\tilde{w}_j$ is at least twice of degree 4 and the sum of its degrees in the three factors is at least $4 + 4 + 2$. This leaves only 3 edges for the remaining 4 factors, which is impossible.

If $n = 7$ and $k = 2$, then the sum of degrees of $\tilde{w}_j$ in some three factors is 9 and in the remaining four factors is 5. Since for $k = 2$ no vertex of L is of degree 2, this is also impossible.

If $n = 6$, we cannot have a vertex w_j which is never an image of y or z. If such a vertex exists, then it is always an image of c which has degree equal to 3, or of a vertex of degree 1. But then the sum of the *odd* degrees of w_1 in six factors is always even, which is impossible, since in K_{12} every vertex is of degree 11.

For $n = 5$ we get a a single lobster of order 10, namely $L = (0|4, 1, 1)$. It was proved by Eldergill [1] that it does not factorize K_{10}. For $n < 5$ there exists no $(0|n-1, n-3-k, k)$-lobster. This completes the proof. □

Theorem 4.3 *The lobster* $L = (0|n-1, n-k-3, k)$ *does not factorize* K_{2n} *for any possible* n.

PROOF. We proceed by contradiction and assume that $L = (0|n-1, n-k-3, k)$ factorizes K_{2n} and $n \geq 5$. We again use notation of Proposition 4.1 and Lemma 4.2. Suppose that $\tilde{w}_j \in B$ is the vertex from Lemma 4.2.

Let $\tilde{w}_1 = \phi_1(y)$ and $x_1 = \phi_1(x)$. Then $x_1\tilde{w}_1 = \phi_1(x)\phi_1(y) \notin L_1$. Since in A are only images of x or leaves x', y', z', every edge $ab \in E(K_{2n})$, where $a \in A$ and $b \in B$, must be covered by an image of an edge xc, xx', yy' or zz' from L in K_{2n}. Because x_1 cannot be in L_i, $i > 1$ an image of x, the edge $x_1\tilde{w}_1$ must be an image of yy' or zz' in some $L_i, i \neq 1$, say L_2. Then $\tilde{w}_1 = \phi_2(y)$ or $\tilde{w}_1 = \phi_2(z)$ and there is a vertex $x_2 \neq x_1$, which is in L_2 an image of x. Obviously, the edge $x_2\tilde{w}_1$ is not in L_2. Suppose that $x_2\tilde{w}_1 \in L_i$ for $i > 2$. From Lemma 4.2 and Proposition 4.1 it follows that both x_2 and $\tilde{w}_1$ are in $L_i, i > 2$ images of leaves from L. Hence, $x_2\tilde{w}_1 \notin L_i$ for $i > 2$ and therefore $x_2\tilde{w}_1 \in L_1$. It is obvious that $x_2\tilde{w}_1$ cannot be an image of either xx' or xc, because x_2 is not an image of x in L_1.

Since both edges $x_1\tilde{w}_1 \in E(L_2)$ and $x_2\tilde{w}_1 \in E(L_1)$ are images of yy' or zz', the edge x_1x_2 cannot belong to either L_1 or L_2. But in $L_i, i > 2$ both vertices x_1 and x_2 are images of leaves from L, therefore the edge cannot belong to any $L_i, i = 1, 2, \ldots, n$. We have a contradiction. □

Theorem 4.4 *The* $(1|1, 1, 1)$*-lobster does not factorize the complete graph* K_8.

PROOF. By contradiction. Let $L = (1|1, 1, 1)$. Suppose that L_1, L_2, L_3, L_4 are factors of an L-factorization of K_8. Let the set A be defined by this way $A = \{a_i \in V(K_8) | \phi_i(c) = a_i, i = 1, 2, 3, 4\}$, where ϕ_i is isomorphism that L maps on L_i. It is evident that $|A| = 4$. If $\deg_j(a_i) > 1$ for some

$j \neq i, j \in \{1,2,3,4\}$, then $\sum_{k=1}^{4} \deg_k(a_i) > 7$, but it is impossible. Therefore $\phi_i(x), \phi_i(y), \phi_i(z) \in B = V(K_8) \setminus A$ for every $i = 1,2,3,4$. Thus we have partitions A, B of $V(K_8)$, $|A| = |B| = 4$, such that all images of c belong to A and all images of x, y, z belong to B. Since $\phi_i(x), \phi_i(y), \phi_i(z)$ are independent, there exists at most one edge $e = bb'$ in L_i, where $b, b' \in B$. Thus each factor covers at most one edge of K_4, which is induced on B. Hence, there are covered at most 4 among the 6 edges of K_4, which is induced on B. This is a contradiction to the existence of an L-factorization of K_8. $\square$

5 Maxend lobsters of diameter 4 with four non-leave vertices

From now on the positive integer $m \geq 3$ relates to the definitions of graphs N and Z in Definition 2.2.

If for a lobster $L = (C|X, Y, Z)$ of order $2n$ is $X = n - 1$ then it is called a *maxend lobster of diameter 4* or a *maxend lobster* for short. Obviously, maxend lobsters of order $2n$ exist only for $n \geq 5$. Further, by Theorem 4.3 a maxend lobster $(0|n-1, n-3-k, k)$ does not factorize K_{2n} for any $n \geq 5$ and $1 \leq k \leq \lfloor \frac{n-3}{2} \rfloor$. Therefore we investigate only maxend lobsters $(C|n-1, Y, Z)$, where $C, Y, Z \geq 1$.

Lemma 5.1 *Let $L = (C|n-1, Y, Z)$ be a maxend lobster of order $2n$, where $C, Y, Z \geq 1$, and let $L' = (C + C'|n+4, Y + Y', Z + Z')$ be a maxend lobster of order $2(n+5)$, where C', Y', Z' are nonnegative integers and $C' + Y' + Z' = 5$. If L weakly semisurjectively factorizes K_{2n} then L' factorizes the graph N for $m = 5$.*

PROOF. Since L weakly semisurjectively factorizes the complete graph K_{2n}, there exist partitions $V_0, V_1, |V_0| = |V_1| = n$ of $V(K_{2n})$ such that the images of vertex x in n factors of this factorization cover every vertex of V_0 exactly once and the images of vertices c, y, z cover every vertex of V_1 each exactly once. Note that the images of vertices c, y, z cannot belong to V_0, because x is of degree n and the degrees of x, y, z are greater than 1. If $m = 5$, additional vertices of degree 1 which are joined to the vertex x and another $m = 5$ to the vertices c, y, z, respectively, will be fixed then we obtain an L'-factorization of the graph N, see Figure 6. $\square$

Lemma 5.2 *If $L' = (C + C'|n + 4, Y + Y', Z + Z')$ is a maxend lobster of order $2(n + 5)$, where $C, Y, Z \geq 1$ and $C' + Y' + Z' = 5$, then there exists an L'-factorization of the graph Z for $m = 5$.*

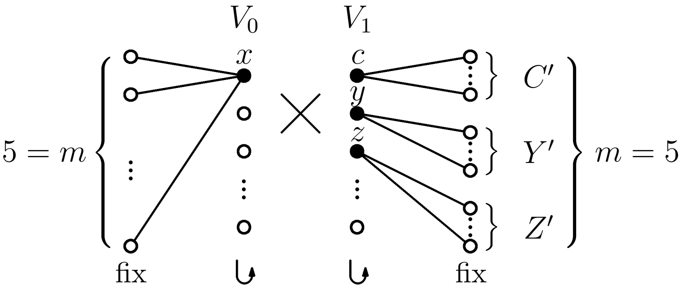

Fig. 6. Factorization of the graph N.

PROOF. We find in L' a subgraph R in the following way. We take a subgraph $M = \langle c, x, y, z \rangle$ (induced on vertices c, x, y, z) of L' and we add 4 neighbors of degree 1 of vertex x, one neighbor of degree 1 of vertex y and one neighbor of degree 1 of vertex c. We see that R is a $(5, 4, 2)$-caterpillar of order 10 and diameter 4. Such caterpillar allows a blended labeling (see Figure 7) and therefore we obtain a bi-cyclic R-factorization of K_{10}, where $x \in V_0 \subset V(R)$ and $c, y, z \in V_1 \subset V(R)$. Further, $n+4-4 = n$ leaves in L' are joined to the vertex $x \in V_0 \subset V(R)$. Another $(C+C'-1)+(Y+Y'-1)+(Z+Z') = C+Y+Z+3 = n$ leaves in L' are joined to the vertices $c, y, z \in V_1 \subset V(R)$. Hence we obtain an L'-factorization of a graph Z for $m = 5$, which is guaranteed by the fixing method. $\square$

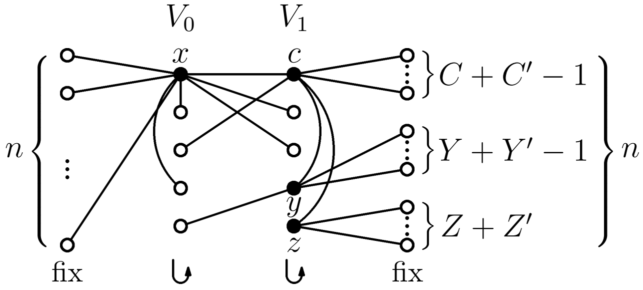

Fig. 7. Factorization of the graph Z.

Theorem 5.3 *Let $L = (C|n-1, Y, Z)$ be a maxend lobster of order $2n$, where $C, Y, Z \geq 1$, and let $L' = (C+C'|n+4, Y+Y', Z+Z')$ be a maxend lobster of order $2(n+5)$, where $C'+Y'+Z' = 5$. If L weakly semisurjectively factorizes K_{2n} then L' weakly semisurjectively factorizes $K_{2(n+5)}$.*

PROOF. From Lemmas 5.1 and 5.2 it follows that the images of all non-leave vertices of L' are identified during an L'-factorization of K_{2n} with all vertices either in the partition $X_0 \cup Y_0$ or in the partition $X_1 \cup Y_1$ of $V(K_{2n})$. Therefore the L'-factorization is weakly semisurjective. $\square$

Lemma 5.4 (Starting cases) *All maxend lobsters $L = (C|n-1, Y, Z)$ of order $2n$, where $C, Y, Z \geq 1$, weakly semisurjectively factorize K_{2n} for every $n = 5, 6, 7, 8, 9$.*

PROOF. $n = 5$: There exists just one maxend lobster, it is $(0|4, 1, 1)$. But we know that $(0|4, 1, 1)$ does not factorize K_{10} (see Theorem 4.3). Therefore we use as the starting case the caterpillar $(1|4, 1, 0)$ of diameter 4. Blended labeling of $(1|4, 1, 0)$ is given in Figure 8.

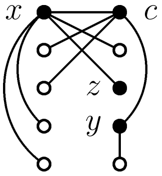

Fig. 8. Blended labeling of the $(1|4, 1, 0)$-caterpillar.

$n = 6$: There exist maxend lobsters $(0|5, 2, 1)$ and $(1|5, 1, 1)$. The $(0|5, 2, 1)$-lobster does not factorize K_{12} by Theorem 4.3 and a swapping labeling of $(1|5, 1, 1)$ is given in Figure 9.

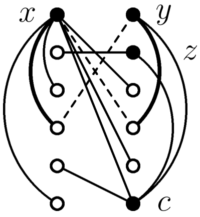

Fig. 9. Swapping labeling of the lobster $L(1|5, 1, 1)$.

$n = 7$: There exist maxend lobsters $(0|6, 3, 1)$, $(0|6, 2, 2)$, $(1|6, 2, 1)$ and $(2|6, 1, 1)$. The lobsters $(0|6, 3, 1)$ and $(0|6, 2, 2)$ do not factorize K_{14} by Theorem 4.3. Blended labelings of $(1|6, 2, 1)$ and $(2|6, 1, 1)$-lobsters are given in Figure 10.

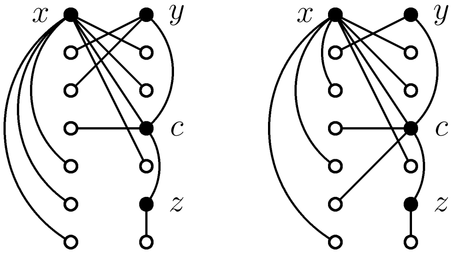

Fig. 10. Blended labeling of the lobsters $L(1|6, 2, 1)$ and $L(2|6, 1, 1)$.

$n = 8$: There exist maxend lobsters $(0|7, 4, 1)$, $(0|7, 3, 2)$, $(1|7, 3, 1)$, $(1|7, 2, 2)$, $(2|7, 2, 1)$, and $(3|7, 1, 1)$. The lobsters $(0|7, 4, 1)$ and $(0|7, 3, 2)$ do not factorize K_{16} by Theorem 4.3. Swapping labelings of $(1|7, 3, 1)$, $(1|7, 2, 2)$, $(2|7, 2, 1)$ and $(3|7, 1, 1)$-lobsters are given in Figure 11.

$n = 9$: There exist maxend lobsters $(0|8, 5, 1)$, $(0|8, 4, 2)$, $(0|8, 3, 3)$, $(1|8, 4, 1)$, $(1|8, 3, 2)$, $(2|8, 3, 1)$, $(2|8, 2, 2)$, $(3|8, 2, 1)$ and $(4|8, 1, 1)$. The lobsters $(0|8, 5, 1)$, $(0|8, 4, 2)$ and $(0|8, 3, 3)$ do not factorize K_{18} by Theorem 4.3. Blended labeling

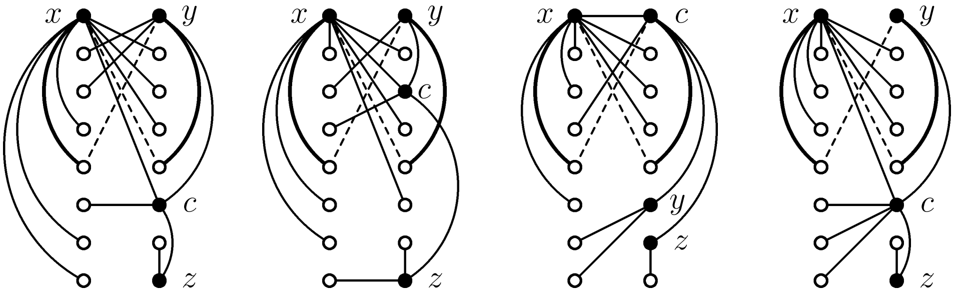

Fig. 11. Swapping labeling of the lobsters $L(1|7,3,1)$, $L(1|7,2,2)$, $L(2|7,2,1)$, and $L(3|7,1,1)$.

of $(1|8,4,1)$, $(1|8,3,2)$, $(2|8,3,1)$, $(2|8,2,2)$, $(3|8,2,1)$ and $(4|8,1,1)$-lobsters are given in Figures 12 and 13. □

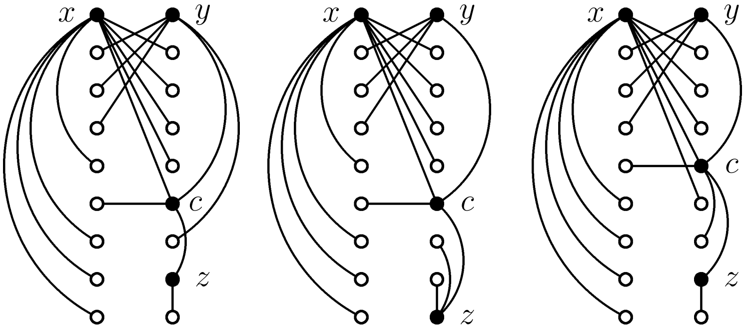

Fig. 12. Blended labeling of the lobsters $L(1|8,4,1)$, $L(1|8,3,2)$, $L(2|8,3,1)$.

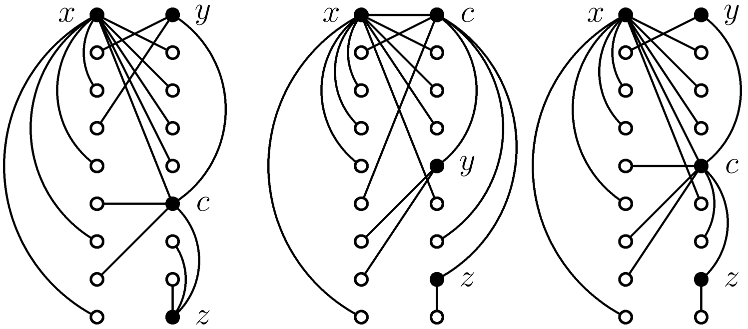

Fig. 13. Blended labeling of the lobsters $L(2|8,2,2)$, $L(3|8,2,1)$, $L(4|8,1,1)$.

Theorem 5.5 *A maxend lobster* $L=(C|n-1,Y,Z)$ *of order* $2n$, $n\geq 5$, *where* $C\geq 0, Y,Z\geq 1$, *factorizes the complete graph* K_{2n} *if and only if* L *is not isomorphic to a maxend lobster* $(0|n-1,n-3-k,k)$ *for every* $1\leq k\leq \lfloor\frac{n-3}{2}\rfloor$.

PROOF. By induction. Let $L'=(C+C'|n+4,Y+Y',Z+Z')$ be a maxend lobster of order $2(n+5)$ and diameter 4, where $n\geq 5$ and $C,X,Y,Z\geq 1$ and $C'+Y'+Z'=5$. Then a lobster (or in one case a caterpillar) $L=(C|n-1,Y,Z)$

of order $2n$ and diameter 4 is also a maxend lobster. Suppose that L weakly semisurjectively factorizes K_{2n}. For $n = 5$ $L = (1|4, 1, 0)$ is a caterpillar and since L bi-cyclically factorizes K_{10} (see Fig. 8) the L-factorization of K_{10} is semisurjective. Now according Theorem 5.3 the lobster L' factorizes $K_{2(n+5)}$. Therefore by Lemma 5.4 each maxend lobster $L = (C|n-1, Y, Z)$ of order $2n$ factorizes K_{2n} for every $n \geq 5$. By Theorem 4.3 no maxend lobster factorizes K_{2n} if $C = 0$. This completes the proof. □

6 Maxcenter lobsters of diameter 4 with four non-leave vertices

If for a lobster $L = (C|X, Y, Z)$ of order $2n$ is $C = n - 3$ then it is called a *maxcenter lobster of diameter 4* or a *maxcenter lobster* for short. Obviously, maxcenter lobsters of order $2n$ exist only for $n \geq 4$.

Lemma 6.1 *Let $L = (n-3|X, Y, Z)$ be a maxcenter lobster of order $2n$, where $X, Y, Z \geq 1$, and let $L' = (n+2|X+X', Y+Y', Z+Z')$ be a maxcenter lobster of order $2(n+5)$, where X', Y', Z' are nonnegative integers and $X'+Y'+Z' = 5$. If L weakly semisurjectively factorizes K_{2n} then L' factorizes the graph N for $m = 5$.*

PROOF. The proof is very similar to the proof of Lemma 5.1 and therefore it is omitted. □

Lemma 6.2 *If $L' = (n+2|X+X', Y+Y', Z+Z')$ is a maxcenter lobster of order $2(n+5)$, where $X, Y, Z \geq 1$ and $X'+Y'+Z' = 5$, then there exists an L'-factorization of the graph Z for $m = 5$.*

PROOF. Let $Y + Y' > 1$. In L' we form a subgraph R such that we take the subgraph $M = \langle c, x, y, z \rangle$ of L' and we add two neighbors of degree 1 to vertex x, two neighbors of degree 1 to vertex y, and two neighbors of degree 1 to vertex c. The subgraph R is a $(3, 5, 3)$-caterpillar of order 10 and diameter 4. We know (see Figure 14) that such a caterpillar allows a blended labeling and therefore we obtain a bi-cyclic R-factorization of K_{10}. Further, $n + 2 - 2 = n$ leaves in L' are joined to the vertex $c \in V_0 \subset V(R)$ and $(X + X' - 2) + (Y + Y' - 2) + (Z + Z') = X + Y + Z + 1 = n$ leaves in L' are joined to the vertices $c, y, z \in V_1 \subset V(R)$. Hence, we obtain an L'-factorization of the graph Z for $m = 5$, which is guaranteed by the fixing method.

Let $Y + Y' = 1$. Then $Z + Z' = 1$ and since $X + X' + Y + Y' + Z + Z' = n + 4$ we get $X + X' = n + 2 > 3$. In L' we form a subgraph R such that we take the subgraph $M = \langle c, x, y, z \rangle$ of L' and we add three neighbors of degree 1 of

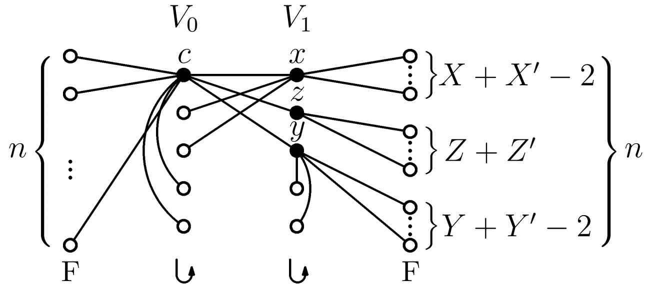

Fig. 14. Factorization of the graph Z, case $Y + Y' > 1$.

vertex x, one neighbor of degree 1 of vertex y, and two neighbors of degree 1 of vertex c. We see that R is a $(4, 5, 2)$-caterpillar of order 10 and diameter 4. We know again (see Figure 15) that such caterpillar allows a blended labeling and therefore we obtain a bi-cyclic R-factorization of K_{10}. We obtain an L'-factorization of the graph Z for $m = 5$ by the same argument as in the previous case. □

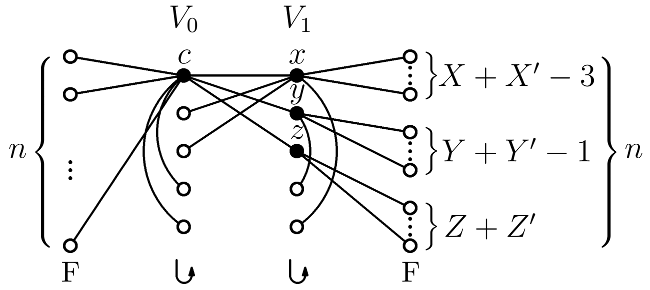

Fig. 15. Factorization of the graph Z, case $Y + Y' = 1$.

Theorem 6.3 *Let* $L = (n - 3|X, Y, Z)$ *be a maxcenter lobster of order* $2n$*, where* $X, Y, Z \geq 1$*, and let* $L' = (n + 2|X + X', Y + Y', Z + Z')$ *be a maxcenter lobster of order* $2(n + 5)$*, where* $X' + Y' + Z' = 5$*. If* L *weakly semisurjectively factorizes* K_{2n} *then* L' *weakly semisurjectively factorizes* $K_{2(n+5)}$*.*

PROOF. Follows from Lemmas 6.1 and 6.2 and Lemma 2.3. □

Lemma 6.4 (Starting cases) *All maxcenter lobsters* $L = (n - 3|X, Y, Z)$ *of order* $2n$*, where* $X, Y, Z \geq 1$*, weakly semisurjectively factorize* K_{2n} *for every* $n = 5, 6, 7, 8, 9$*.*

PROOF. $n = 5$: There exists just one maxcenter lobster $(2|2, 1, 1)$. Blended labeling of $(2|2, 1, 1)$ is given in Figure 16.

$n = 6$: There exist two maxcenter lobsters $(3|3, 1, 1)$ and $(3|2, 2, 1)$. Their swapping labelings are given in Figure 17.

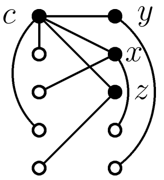

Fig. 16. Blended labeling of the lobster $L(2|2,1,1)$.

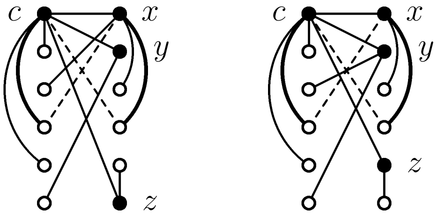

Fig. 17. Swapping labeling of the lobsters $L(3|3,1,1)$, $L(3|2,2,1)$.

$n = 7$: There exist three maxcenter lobsters $(4|4,1,1)$, $(4|3,2,1)$, and $(4|2,2,2)$. Their blended labelings are given in Figure 18.

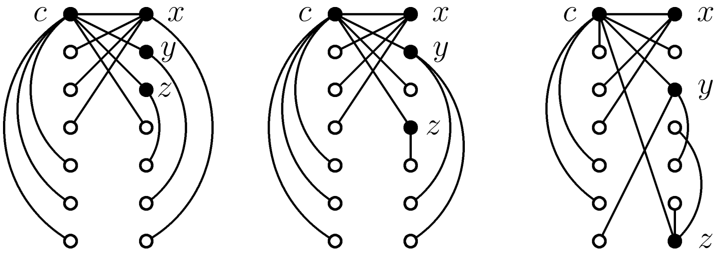

Fig. 18. Blended labeling of the lobsters $L(4|4,1,1)$, $L(4|3,2,1)$, $L(4|2,2,2)$.

$n = 8$: There exist four maxcenter lobsters $(5|5,1,1)$, $(5|4,2,1)$, $(5|3,3,1)$, and $(5|3,2,2)$. Swapping labelings of $(5|5,1,1)$, $(5|4,2,1)$, $(5|3,3,1)$, and $(5|3,2,2)$-lobsters are given in Figure 19.

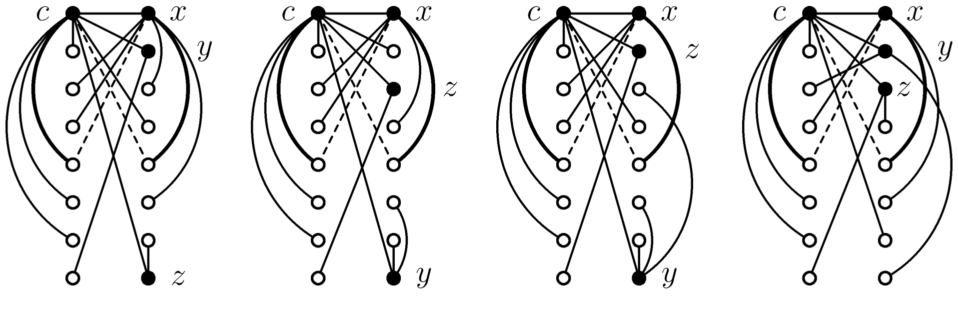

Fig. 19. Swapping labeling of the lobsters $L(5|5,1,1)$, $L(5|4,2,1)$.

$n = 9$: There exist five maxcenter lobsters $(6|6,1,1)$, $(6|5,2,1)$, $(6|4,3,1)$, $(6|4,2,2)$ and $(6|3,3,2)$. Blended labelings of $(6|6,1,1)$, $(6|5,2,1)$, $(6|4,3,1)$, $(6|4,2,2)$, and $(6|3,3,2)$-lobsters are given in Figures 20 and 21. □

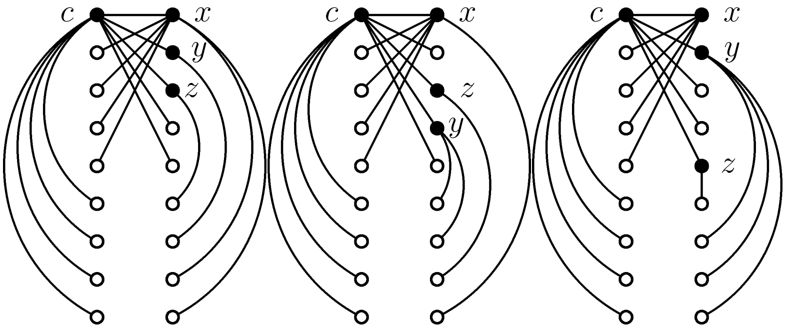

Fig. 20. Blended labeling of the lobsters $L(6|6,1,1)$, $L(6|5,2,1)$, $L(6|4,3,1)$.

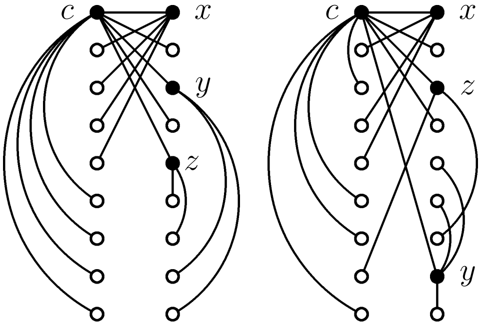

Fig. 21. Blended labeling of the lobsters $L(6|4,2,2)$, $L(6|3,3,2)$.

Theorem 6.5 *A maxcenter lobster* $L = (n-3|X,Y,Z)$ *of order* $2n$*,* $n \geq 4$*, where* $X,Y,Z \geq 1$*, factorizes a complete graph* K_{2n} *if and only if* L *is not isomorphic to a maxcenter lobster* $(1|1,1,1)$ *for* $n = 4$*.*

PROOF. By induction. Let $L' = (n{+}2|X{+}X', Y{+}Y', Z{+}Z')$ be a maxcenter lobster of order $2(n+5)$, where $n \geq 5$ and $X,Y,Z \geq 1$ and $X'+Y'+Z' = 5$. Then a lobster $L = (n-3|X,Y,Z)$ of order $2n$ and diameter 4 is also a maxcenter lobster. Suppose that L weakly semisurjectively factorizes K_{2n}. According to Theorem 6.3 the lobster L' factorizes $K_{2(n+5)}$. Therefore by Lemma 6.4 each maxcenter lobster $L = (n-3|X,Y,Z)$ of order $2n$ factorizes K_{2n} for every $n \geq 5$. For the proof of nonexistence of a $(1|1,1,1)$-factorization of K_8 see Theorem 4.4. □

7 Nonmax lobsters of diameter 4 with four non-leave vertices

If a lobster $L = (C|X,Y,Z)$ of order $2n$ and diameter 4 has $n-3 > C$ and $n-1 > X$ then it is called a *nonmax lobster of diameter 4* or a *nonmax lobster* for short. Obviously, nonmax lobsters of order $2n$ exist only for $n \geq 4$. We know that such lobsters have to satisfy the necessary condition $t_1 + t_4 = t_2 + t_3 = n+1$, where $t_1 \geq t_2 \geq t_3 \geq t_4$ and $t_i \in \{\deg(c) = C+3, \deg(x) =$

$X+1, \deg(y) = Y+1, \deg(z) = Z+1\}$ for every $i = 1,2,3,4$. Therefore we split the set of all nonmax lobsters, which satisfy the necessary condition above, into four classes. First class, called *type A*, contains lobsters for which it holds that $C+3 \geq X+1 \geq Y+1 \geq Z+1$, second class, called *type B*, contains lobsters with $X+1 > C+3 \geq Y+1 \geq Z+1$, third class, called *type C*, contains lobsters with $X+1 \geq Y+1 > C+3 \geq Z+1$ and the last class, called *type D*, contains lobsters with $X+1 \geq Y+1 \geq Z+1 > C+3$. Hence, for type A lobsters it holds that $C+3+Z+1 = n+1$ and $X+1+Y+1 = n+1$, which implies $C+Z = n-3$ and $X+Y = n-1$. For type B lobsters it holds that $X+1+Z+1 = n+1$ and $C+3+Y+1 = n+1$, which implies $X+Z = n-1$ and $C+Y = n-3$. For type C lobsters it holds that $X+1+Z+1 = n+1$ and $C+3+Y+1 = n+1$, which implies $X+Z = n-1$ and $Y+C = n-3$ and for type D lobsters it holds that $X+1+C+3 = n+1$ and $Z+1+Y+1 = n+1$, which implies $X+C = n-3$ and $Y+Z = n-1$. Since the type B and type C lobsters satisfy the same conditions, we will join both types into one type – type BC lobsters.

Lemma 7.1 *Let $L = (C|X,Y,Z)$ be a type A nonmax lobster of order $2n$, where $C+Z = n-3$, $X+Y = n-1$, $C \geq 0, X,Y,Z \geq 1$, and let $L' = (C+C'|X+X', Y+Y', Z+Z')$ be a type A nonmax lobster of order $2(n+3)$, where for some nonnegative integers C', X', Y', Z' it holds that $C'+Z' = 3$ and $X'+Y' = 3$. If L weakly semisurjectively factorizes K_{2n}, then L' factorizes the graph N for $m = 3$.*

PROOF. Since L weakly semisurjectively factorizes K_{2n}, there exist partitions $V_0, V_1, |V_0| = |V_1| = n$ of $V(K_{2n})$ such that the images of vertices c, z in n factors of this factorization cover every vertex of V_0 exactly once and the images of vertices x, y cover every vertex of V_1 exactly once.

We add three vertices of degree 1, join them arbitrarily to the vertices c, z and another three vertices of degree 1 we join to the vertices x, y. The added vertices will be fixed and we obtain an L'-factorization of the graph N for $m = 3$, see Figure 22. The claim follows by the fixing method. □

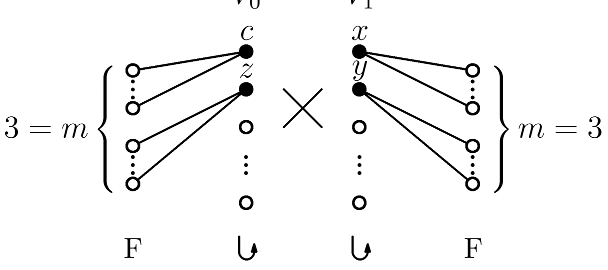

Fig. 22. Factorization of the graph N.

Lemma 7.2 *Let $L = (C|X, Y, Z)$ be a type BC nonmax lobster of order $2n$, where $X + Z = n - 1$, $C + Y = n - 3$, $C \geq 0, X, Y, Z \geq 1$, and let $L' = (C+C'|X+X', Y+Y', Z+Z')$ be a type BC nonmax lobster of order $2(n+3)$, where $X' + Z' = 3$ and $C' + Y' = 3$. If L weakly semisurjectively factorizes K_{2n}, then L' factorizes the graph N for $m = 3$.*

PROOF. The proof is similar to the proof of Lemma 7.1. The only difference is that images of x, z belong to V_0 and images of c, y belong to V_1. Therefore we omit the proof. $\square$

Lemma 7.3 *Let $L = (C|X, Y, Z)$ be a type D nonmax lobster of order $2n$, where $X + C = n - 3$, $Y + Z = n - 1$, $C \geq 0, X, Y, Z \geq 1$, and let $L' = (C+C'|X+X', Y+Y', Z+Z')$ be a type D nonmax lobster of order $2(n+3)$, where $X' + C' = 3$ and $Y' + Z' = 3$. If L weakly semisurjectively factorizes K_{2n}, then L' factorizes the graph N for $m = 3$.*

PROOF. The proof is similar to the proof of Lemma 7.1. The only difference is that images of x, c belong to V_0 and images of y, z belong to V_1. Again we omit the proof. $\square$

Lemma 7.4 *If $L' = (C+C'|X+X', Y+Y', Z+Z')$ is a type A nonmax lobster of order $2(n+3)$, where $C + Z = n - 3$, $X + Y = n - 1$, $C \geq 0, X, Y, Z \geq 1$ and $C' + Z' = 3$, $X' + Y' = 3$, then there exists an L'-factorization of the graph Z for $m = 3$.*

PROOF. In L' we form its subgraph R such that we take the subgraph $M = \langle c, x, y, z \rangle$ of L' and we add one neighbor of degree 1 of vertex x and one neighbor of degree 1 of vertex y. We see that R is a $(2, 3, 2)$-caterpillar of order 6 and diameter 4. We know that such a caterpillar allows a blended labeling and therefore we obtain a bi-cyclic R-factorization of K_6, where $c, z \in V_0 \subset V(R)$ and $x, y \in V_1 \subset V(R)$. Further, $C + C' + Z + Z' = C + Z + 3 = n$ leaves in L' are joined to the vertices $c, z \in V_0 \subset V(R)$ and $X + X' - 1 + Y + Y' - 1 = X + Y + 1 = n$ leaves in L' are joined to the vertices $x, y \in V_1 \subset V(R)$. Hence, we obtain an L'-factorization of a graph Z for $m = 3$, which is guaranteed by the fixing method, see Figure 23. $\square$

Lemma 7.5 *If $L' = (C + C'|X + X', Y + Y', Z + Z')$ is a type BC nonmax lobster of order $2(n+3)$, where $X+Z = n-1$, $C+Y = n-3$, $C \geq 0, X, Y, Z \geq 1$ and $X'+Z' = 3$, $C'+Y' = 3$, then there exists an L'-factorization of the graph Z for $m = 3$.*

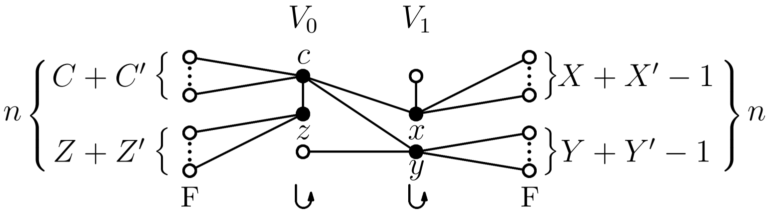

Fig. 23. Factorization of the graph Z, type A.

PROOF. In L' we form its subgraph R such that we take the subgraph $M = \langle c, x, y, z\rangle$ of L' and we add one neighbor of degree 1 of vertex x and one neighbor of degree 1 of vertex z. We see that R is a $(2,3,2)$-caterpillar of order 6 and diameter 4. We know that such a caterpillar allows a blended labeling and therefore we obtain a bi-cyclic R-factorization of K_6, where $c, y \in V_0 \subset V(R)$ and $x, z \in V_1 \subset V(R)$. Further, $C + C' + Y + Y' = C + Y + 3 = n$ leaves in L' are joined to the vertices $c, z \in V_0 \subset V(R)$ and $X + X' - 1 + Z + Z' - 1 = X + Z + 1 = n$ leaves in L' are joined to the vertices $x, z \in V_1 \subset V(R)$. Hence, we obtain an L'-factorization of the graph Z for $m = 3$, which is guaranteed by the fixing method, see Figure 24. □

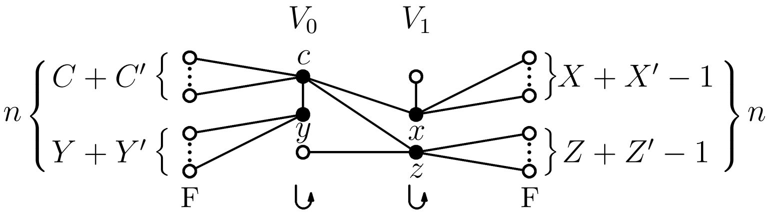

Fig. 24. Factorization of the graph Z, type BC.

Lemma 7.6 *If* $L' = (C+C'|X+X', Y+Y', Z+Z')$ *is a type D nonmax lobster of order* $2(n+3)$*, where* $X + C = n - 3$*,* $Y + Z = n - 1$*,* $C \geq 0, X, Y, Z \geq 1$ *and* $X' + C' = 3$*,* $Y' + Z' = 3$*, then there exists an* L'*-factorization of the graph* Z *for* $m = 3$*.*

PROOF. In L' we form its subgraph R such that we take the subgraph $M = \langle c, x, y, z\rangle$ of L' and we add one neighbor of degree 1 of vertex y and one neighbor of degree 1 of vertex z. We see that R is a $(2,3,2)$-caterpillar of order 6 and diameter 4. We know that such a caterpillar allows a blended labeling and therefore we obtain a bi-cyclic R-factorization of K_6, where $c, x \in V_0 \subset V(R)$ and $y, z \in V_1 \subset V(R)$. Further, $C + C' + X + X' = C + X + 3 = n$ leaves in L' are joined to the vertices $c, x \in V_0 \subset V(R)$ and $Y + Y' - 1 + Z + Z' - 1 = Y + Z + 1 = n$ leaves in L' are joined to the vertices $x, y \in V_1 \subset V(R)$. Hence, we obtain an L'-factorization of the graph Z for $m = 3$, which is guaranteed by the fixing method, see Figure 25. □

Theorem 7.7 *Let* $L = (C|X, Y, Z)$ *be a nonmax lobster of order* $2n$*, where* $C \geq 0, X, Y, Z \geq 1$*, and either* $C+Z = n-3$*,* $X+Y = n-1$ *or* $C+Y = n-3$*,*

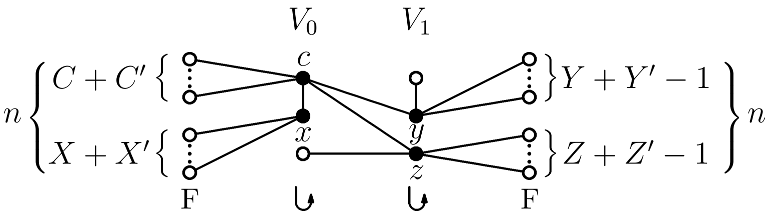

Fig. 25. Factorization of the graph Z, type D.

$X + Z = n - 1$ or $X + C = n - 3$, $Y + Z = n - 1$, respectively. Further, let $L' = (C + C'|X + X', Y + Y', Z + Z')$ be a nonmax lobster of order $2(n + 3)$, where either $C' + Z' = 3$, $X' + Y' = 3$ or $C' + Y' = 3$, $X' + Z' = 3$ or $X' + C' = 3$, $Y' + Z' = 3$. If L weakly semisurjectively factorizes K_{2n}, then L' weakly semisurjectively factorizes $K_{2(n+3)}$.

PROOF. Follows from Lemmas 7.1 through 7.6 and Lemma 2.3. □

Lemma 7.8 (Starting cases) *All nonmax lobsters $L = (C|X, Y, Z)$ of order $2n$, which satisfy the necessary conditions, weakly semisurjectively factorize K_{2n} for every $n = 4, 5, 6$ (type A) or for every $n = 5, 6, 7$ (type BC) or for every $n = 7, 9, 11$ (type D).*

PROOF. *Type A:* Note that for such lobsters the following holds:

(1) $n > C + 3 \geq X + 1 \geq Y + 1 \geq Z + 1$ and
(2) $C + Z = n - 3$ and $X + Y = n - 1$.

$n = 4$: Lobster $(0|2, 1, 1)$ weakly semisurjectively factorizes K_8, see Figure 26.

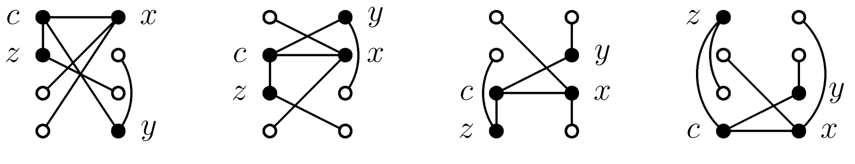

Fig. 26. Weakly semisurjective $L(0|2, 1, 1)$-factorization of K_8.

$n = 5$: There exist three lobsters $(1|3, 1, 1)$, $(1|2, 2, 1)$, and $(0|2, 2, 2)$, which satisfy condition (1). But only two $(0|2, 2, 2)$ and $(1|3, 1, 1)$-lobsters satisfy condition (2) because for $(1|2, 2, 1)$ it holds that $C + Z = 1 + 2 = 3 > 2 = n - 3$. Blended labelings of $(0|2, 2, 2)$ and $(1|3, 1, 1)$-lobsters are given in Figure 27.

$n = 6$: There exist five lobster $(2|4, 1, 1)$, $(2|3, 2, 1)$, $(2|2, 2, 2)$, $(1|3, 3, 1)$, and $(1|3, 2, 2)$, which satisfy condition (1). But only $(2|4, 1, 1)$, $(1|3, 2, 2)$ and $(2|3, 2, 1)$-lobsters satisfy condition (2) because for $(2|2, 2, 2)$ and $(1|3, 3, 1)$ it holds either $C + Z = 2 + 2 = 4 > 3 = n - 3$ or $C + Z = 1 + 1 = 2 < 3 = n - 3$. Obviously, to find an L-factorization of K_{2n} when $m = n = 3$ it is enough to

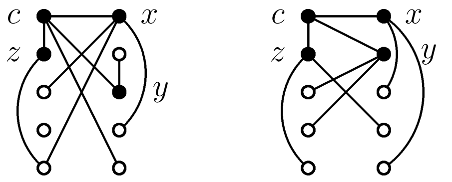

Fig. 27. Blended labeling of the lobsters $L(1|3,1,1)$, $L(0|2,2,2)$.

describe an L-factorization of one of the graphs N or Z. The weakly semisurjective factorization of K_{12} into $(2|4,1,1)$, $(1|3,2,2)$ and $(2|3,2,1)$-lobsters follows from Figures 28 and 29 by the fixing method.

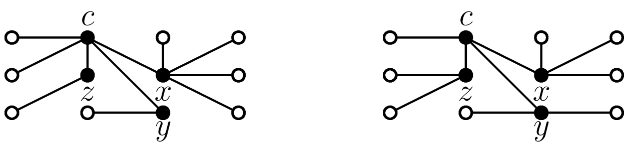

Fig. 28. Weakly semisurjective $L(2|4,1,1)$- and $L(1|3,2,2)$-factorization of K_{12} is guaranteed by the fixing method.

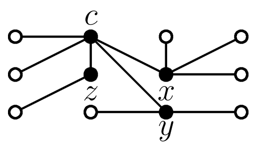

Fig. 29. Weakly semisurjective $L(2|3,2,1)$-factorization of K_{12} is guaranteed by the fixing method.

Type BC: Note that for such lobsters it holds that

(1) $n > X+1 > C+3 \geq Y+1 \geq Z+1$ or
(2) $n > X+1 \geq Y+1 > C+3 \geq Z+1$ and
(3) $X+Z = n-1$ and $C+Y = n-3$.

$n = 5$: There exists only one type B nonmax $(0|3,2,1)$-lobster, which satisfies conditions (1) and (3). A blended labeling of the $(0|3,2,1)$-lobster is given in Figure 30.

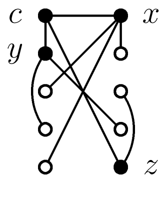

Fig. 30. Blended labeling of the lobster $L(0|3,2,1)$.

$n = 6$: There exist four lobsters $(1|4,2,1)$, $(0|4,3,1)$, $(0|4,2,2)$, and $(0|3,3,2)$, which satisfy condition (1) or (2). The lobster $(0|4,2,2)$ does not satisfy condition (3), because $C+Y = 0+2 = 2 < 3 = n-3$. Again the weakly semisurjective factorization of K_{12} into $(1|4,2,1)$ (type B), $(0|4,3,1)$ (type C), and $(0|3,3,2)$-lobsters (type C) follows from Figures 31 and 32.

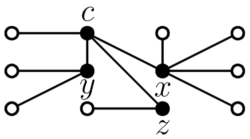

Fig. 31. Weakly semisurjective $L(1|4,2,1)$-factorization of K_{12} is guaranteed by the fixing method.

Fig. 32. Weakly semisurjective $L(0|4,3,1)$- and $L(0|3,3,2)$-factorization of K_{12} is guaranteed by the fixing method.

$n = 7$: There exist nine lobsters $(2|5,2,1)$, $(1|5,3,1)$, $(1|5,2,2)$, $(0|5,4,1)$, $(0|5,3,2)$, $(1|4,4,1)$, $(1|4,3,2)$, $(0|4,4,2)$, and $(0|4,3,3)$, which satisfy condition (1) or (2). The lobsters $(1|5,2,2)$, $(0|5,3,2)$, $(1|4,4,1)$ and $(0|4,3,3)$ do not satisfy condition (3), because in all cases $C + Y$ is not equal to $4 = n - 3$. Blended labelings of $(2|5,2,1)$ (type B), $(1|5,3,1)$ (type B), $(0|5,4,1)$ (type C), $(1|4,3,2)$ (type B), and $(0|4,4,2)$-lobsters (type C) are given in Figures 33 and 34.

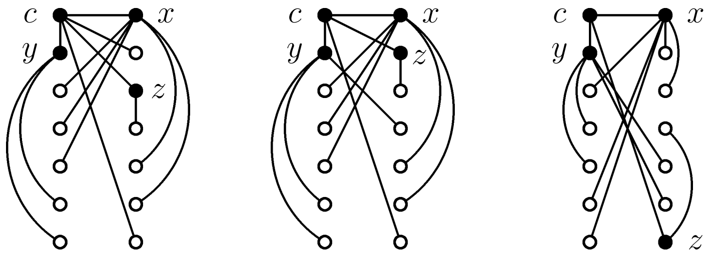

Fig. 33. Blended labeling of the lobsters $L(2|5,2,1)$, $L(1|5,3,1)$, $L(0|5,4,1)$.

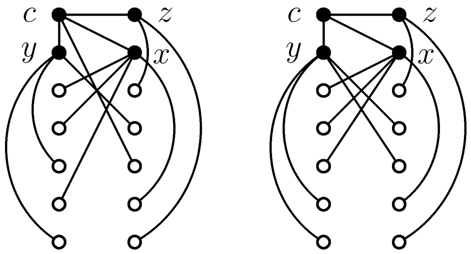

Fig. 34. Blended labeling of the lobsters $L(1|4,3,2)$, $L(0|4,4,2)$.

Type D: Note that for such lobsters it holds that

(1) $n > X + 1 \geq Y + 1 \geq Z + 1 > C + 3$ and
(2) $X + C = n - 3$ and $Y + Z = n - 1$.

We see that $\deg(z) = Z + 1$ must be at least 4 and therefore such caterpillars exist just for orders $n \geq 7$.

$n = 7$: There is only one type D nonnmax lobster. It is the $(0|4, 3, 3)$-lobster and its blended labeling is given in Figure 35.

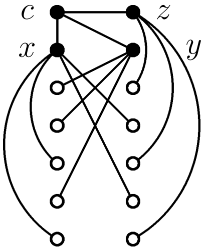

Fig. 35. Blended labeling of the lobster $L(0|4, 3, 3)$.

It is not difficult to observe that for $n = 8$ there does not exist any type D nonmax lobster. Therefore we need starting cases for $n = 9, 11$.

$n = 9$: There exists just one type D nonmax lobster. It is the $(0|6, 4, 4)$-lobster and its blended labeling is given in Figure 36.

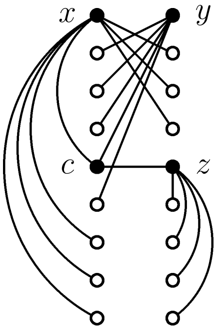

Fig. 36. Blended labeling of the lobster $L(0|6, 4, 4)$.

$n = 11$: There exist three type D nonmax lobsters: $(0|8, 6, 4)$, $(0|8, 5, 5)$, and $(1|7, 5, 5)$. Blended labelings of $(0|8, 6, 4)$, $(0|8, 5, 5)$, and $(1|7, 5, 5)$ are given in Figure 37. $\square$

The following lemma will be useful to prove Theorem 7.11, the main result of this section.

Lemma 7.9 *Let* $n, \alpha, \beta, \gamma, \delta$ *be integers such that* $\alpha \geq \beta \geq \gamma \geq \delta$, $\alpha+\delta = \beta+\gamma = n+2$ *and* $n \geq 5$. *Then there exist integers* a, b, c, d, *where* $a+d = b+c = 3$, *such that* $\alpha - a \geq \beta - b \geq \gamma - c \geq \delta - d$ *and* $\delta - d \geq 1$.

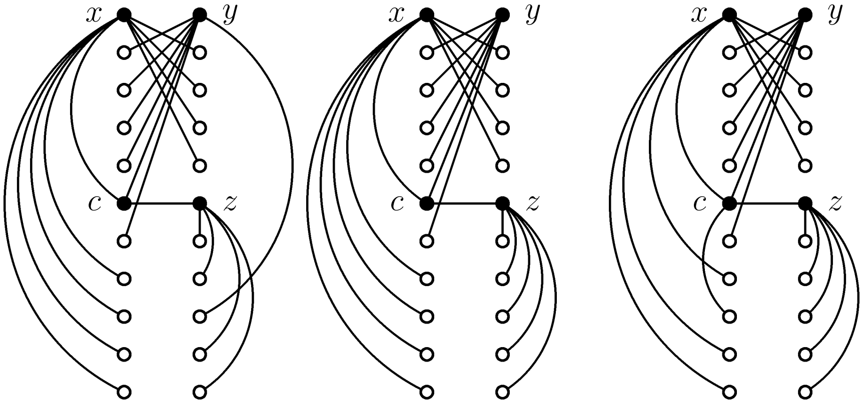

Fig. 37. Blended labeling of the lobsters $L(0|8,6,4)$, $L(0|8,5,5)$, $L(1|7,5,5)$.

PROOF. If $\delta \geq 2$ then $\alpha - 2 \geq \beta - 2 \geq \gamma - 1 \geq \delta - 1$ for $\beta > \gamma$. Thus $a = b = 2$ and $c = d = 1$ is a solution. If $\beta = \gamma$ then either $\alpha > \beta = \gamma > \delta$ or $\alpha = \beta = \gamma = \delta$. If $\alpha > \beta = \gamma > \delta$ then we can set $a = c = 2$ and $b = d = 1$. If $\alpha = \beta = \gamma = \delta$ then $\alpha = \beta = \gamma = \delta = \frac{n+2}{2} \geq 4$ and we can set $a = b = 1$ and $c = d = 2$. If $\delta = 1$ then $\alpha = n + 1 \geq 6$ and $\beta \geq \frac{n+2}{2} \geq 4$ and we can take $a = b = 3$ and $c = d = 0$ if $\beta \geq \gamma + 3$. Let $\gamma + 2 \geq \beta > \gamma$ and $\delta = 1$. Then the integers can be $a = 3, b = 2, c = 1, d = 0$, because $\gamma > \delta$ and $\alpha = n + 1 > \frac{n+4}{2} \geq \beta$ for every $n \geq 5$. Let $\beta = \gamma$ and $\delta = 1$. Then the integers can be $a = 3, b = 1, c = 2, d = 0$, because $\alpha = n + 1 > \frac{n+4}{2} = \beta + 1$ and $\gamma = \frac{n+2}{2} \geq 4 > \delta + 1$ for every $n \geq 5$. □

Corollary 7.10 *Let $L' = (C_1|X_1, Y_1, Z_1)$ be a nonmax lobster of an arbitrary type (A, BC, or D) and order $2(n+3)$ then there exist nonnegative integers C', X', Y', Z' which satisfy either $C'+Z' = X'+Y' = 3$ for type A or $C'+Y' = X'+Z' = 3$ for type BC, or $C'+X' = Y'+Z' = 3$ for type D, such that the lobster $L = (C|X, Y, Z)$ of order $2n$, where $C_1 = C + C'$, $X_1 = X + X'$, $Y_1 = Y + Y'$, $Z_1 = Z + Z'$, is of the same type as L'.*

PROOF. Every type of a nonmax lobster L' of order $2(n+3)$ is characterized by a 4-tuple $(\alpha, \beta, \gamma, \delta)$ satisfying the inequality $\alpha \geq \beta \geq \gamma \geq \delta$, where $\alpha, \beta, \gamma, \delta \in \{C_1 + 3, X_1 + 1, Y_1 + 1, Z_1 + 1\} = \{C + C' + 3, X + X' + 1, Y + Y' + 1, Z + Z' + 1\}$. Lemma 7.9 shows how to choose C', X', Y', and Z' so that the inequality holds also after subtracting C', X', Y', and Z'. □

Theorem 7.11 *A nonmax lobster $L = (C|X, Y, Z)$ of order $2n$, where $n \geq 4$ and $C \geq 0, X, Y, Z \geq 1$, factorizes a complete graph K_{2n} if and only if L satisfies*

(i) $C + Z = n - 3$ *and* $X + Y = n - 1$ *for type A,*
(ii) $C + Y = n - 3$ *and* $X + Z = n - 1$ *for type B or C, and*

(iii) $C + X = n - 3$ *and* $Y + Z = n - 1$ *for type* D.

PROOF. For $L = (0|2, 1, 1)$ of order 8 and type A, the L-factorization of K_8 is given in Figure 26.

By induction. Let $L' = (C + C'|X + X', Y + Y', Z + Z')$ be a nonmax lobster of order $2(n + 3)$ and diameter 4, where $n \geq 5$ and $C \geq 0, X, Y, Z \geq 1$, of arbitrary type where C', X', Y', Z' are the integers specified in Corollary 7.10 for every particular type. Then a nonmax lobster $L = (C|X, Y, Z)$ of order $2n$, where $n \geq 5$ and $C \geq 0, X, Y, Z \geq 1$, is of the same type as L'. Suppose that L weakly semisurjectively factorizes K_{2n}. Then according to Theorem 7.7 the lobster L' also weakly semisurjectively factorizes $K_{2(n+3)}$. Therefore by Lemma 7.8 each nonmax lobster $L = (C|X, Y, Z)$ of order $2n$ factorizes K_{2n} for every $n \geq 5$. $\square$

8 Conclusion

We repeat the necessary conditions for lobsters $L = (C|X, Y, Z)$ of order $2n$ and diameter 4 with four non-leave vertices that factorize K_{2n}:

(1) $\Delta(L) \leq n$.
(2) Either $t_1 = n$ and $t_2 + t_3 + t_4 = n + 2$ or $t_1 + t_4 = t_2 + t_3 = n + 1$ if $t_1 \geq t_2 \geq t_3 \geq t_4$ and $t_i \in \{\deg(c), \deg(x), \deg(y), \deg(z)\}$ for every $i = 1, 2, 3, 4$.

Theorem 8.1 *A lobster* $L = (C|X, Y, Z)$ *of order* $2n$, $n \geq 4$, *and diameter 4 with four non-leave vertices, where* $C \geq 0, X, Y, Z \geq 1$, *factorizes a complete graph* K_{2n} *if only if* L *satisfies the above necessary conditions and there is not isomorphic to* $(1|1, 1, 1)$ *and* $(0|n - 1, n - 3 - k, k)$*-lobsters for every* $1 \leq k \leq \lfloor \frac{n-3}{2} \rfloor$.

PROOF. For maxend lobsters the claim follows from Theorem 5.5. For maxcenter lobsters the claim follows from Theorem 6.5. For nonmax lobsters the claim follows from Theorem 7.11. $\square$

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