

DECOMPOSITIONS OF COMPLETE GRAPHS INTO KAYAK PADDLES WITH TWO ODD CYCLES

DALIBOR FRONCEK AND LEAH TOLLEFSON

University of Minnesota Duluth

ABSTRACT. A canoe paddle is a pair of cycles joined by a path. It was shown by Truszczyński that all canoe paddles are graceful and therefore decompose complete graphs. A kayak paddle is a pair of cycles joined by a path. We prove that the complete graph K_{2n+1} is decomposable into kayak paddles with n edges whenever at least both cycles are odd.

1. INTRODUCTION

In 1967 A. Rosa [6] introduced some important types of vertex labelings serving as tools for finding decompositions of complete graphs. Graceful labeling (called β -valuation by Rosa) and rosy labeling (called ρ -valuation by Rosa) are useful tools for decompositions of complete graphs K_{2n+1} into graphs with n edges. Let G be a graph with at most n vertices. We say that the complete graph K_n has a G -decomposition (or that it is G -decomposable) if there are subgraphs $G_0, G_1, G_2, \dots, G_s$ of K_n , all isomorphic to G , such that each edge of K_n belongs to exactly one G_i .

A labeling of a graph G with n edges is an injection ρ from $V(G)$, the vertex set of G , into a subset S of the set $\{0, 1, 2, \dots, 2n\}$ of elements of the additive group Z_{2n+1} . The length of an edge $e = xy$ with endvertices x and y is defined as $\ell(xy) = \min\{\rho(x) - \rho(y), \rho(y) - \rho(x)\}$. Notice that the subtraction is performed in Z_{2n+1} and hence $1 \leq \ell(e) \leq n$. If the set of all lengths of the n edges is equal to $\{1, 2, \dots, n\}$, then ρ is a rosy labeling; if moreover $S \subseteq \{0, 1, \dots, n\}$, then ρ is a graceful labeling. A graceful labeling α is called α -labeling if there exists a number α_0 such that for every edge e in G with endvertices x and y and with $\alpha(x) < \alpha(y)$ it

1991 *Mathematics Subject Classification.* 05C78.

Key words and phrases. Graph decomposition, graceful labeling, rosy labeling.

holds that $\alpha(x) \leq \alpha_0 < \alpha(y)$. Obviously, G must be bipartite to allow an α -labeling. For an exhaustive survey of graph labelings, see [5] by J. Gallian. For a survey of Rosa-type labelings, see [2] by S. El-Zanati and C. Vanden Eynden.

A. Rosa observed that if a graph G with n edges has a graceful or rosy labeling, then K_{2n+1} can be cyclically decomposed into $2n + 1$ copies of G . It is so because K_{2n+1} has exactly $2n + 1$ edges of length i for every $i = 1, 2, \dots, n$ and each copy of G contains exactly one edge of each length. The cyclic decomposition is constructed by taking a labeled copy of G , say G_0 , and then adding a non-zero element $i \in \mathbb{Z}_{2n+1}$ to the label of each vertex of G_0 to obtain a copy G_i for $i = 1, 2, \dots, 2n$.

A *canoe paddle* (or a *kite* or *dragon*) is a cycle attached an end-vertex of a path. It was shown by Truszczyński [8] that all canoe paddles are graceful. A *kayak paddle* (or a *double kite* or *double dragon*) is a pair of cycles joined by a path. In particular, $KP(r, s, l)$ stands for cycles of lengths r and s joined by a path of length l .

It was proved by the authors [3] that every kayak paddle $KP(r, s, l)$ with $n = r + s + l$ edges and r even has a rosy labeling. In this paper we complete the characterization and prove that also every $KP(r, s, l)$ with $n = r + s + l$ edges and r, s odd has a rosy labeling and hence K_{2n+1} is $KP(r, s, l)$ -decomposable for every $r \geq 3, s \geq 3, l \geq 1$.

2. BASIC NOTIONS AND AUXILIARY RESULTS

Some special cases of the rosy labeling that are “close” to graceful labeling were introduced by several authors. Rosa [6] introduced a $\hat{\rho}$ -labeling (Frucht [4] used the term *nearly graceful labeling*). In a nearly graceful labeling the vertices are labeled by elements of the set $\{0, 1, 2, \dots, n, n + 1\}$ while the edge lengths are either $\{1, 2, \dots, n\}$ or $\{1, 2, \dots, n - 1, n + 1\}$. To avoid confusion, we note that the edge length was in these papers defined as $\hat{\ell}(xy) = \hat{\rho}(y) - \hat{\rho}(x)$ for $0 \leq \hat{\rho}(x) < \hat{\rho}(y) \leq n + 1$. We observe that for an edge with $\hat{\ell}(xy) = n + 1$ the vertices would be labeled 0 and $n + 1$, respectively. Using our definition of the edge length, we would conclude that $\ell(xy) = \min\{\rho(y) - \rho(x), \rho(x) - \rho(y)\} = n$, since the subtraction is carried out in \mathbb{Z}_{2n+1} . We will use this observation later in our constructions.

Seoud and Elsakhawi [7] have shown that all cycles are nearly graceful, and Barrientos [1] proved that C_n is nearly graceful with edge lengths $\{1, 2, \dots, n - 1, n + 1\}$ if and only if $n \equiv 1$ or $2 \pmod{4}$.

The authors defined in [3] a *gap graceful labeling*, denoted $\tilde{\beta}$. The vertices are labeled by elements of the set $\{0, 1, 2, \dots, n + 1\}$ while the set of the edge lengths is an n -element subset of $\{1, 2, \dots, n + 1\}$. Here we generalize

this definition one step further. For $1 \leq p < n + 1$, a *p-gap graceful labeling* of G is an injection from the vertex set of G to the set $\{0, 1, 2, \dots, n + 1\}$ while the set of the edge lengths is an n -element set $\{1, 2, \dots, n, n + 1\} \setminus \{p\}$.

For $1 \leq p < q \leq n + 1$, a *(p, q)-gap graceful labeling* of G is an injection from the vertex set of G to the set $\{0, 1, 2, \dots, n + 1, n + 2\}$ while the set of the edge lengths is an n -element set $\{1, 2, \dots, n + 1, n + 2\} \setminus \{p, q\}$. A *p-gap α -labeling* or a *(p, q)-gap α -labeling* can be defined analogously. One can notice that in our terminology, the nearly graceful labeling is an n -gap graceful labeling.

The following observations are analogues of a well-know construction described independently by many authors.

Observation 1. *Let G be a p -gap graceful graph with t edges, $\beta_p(x_0) = 0$, $\beta_p(x_1) = t + 1$ and H a graph with an α -labeling with $\alpha(y_0) = \alpha_0$, $\alpha(y_1) = \alpha_0 + 1$. For $i \in \{0, 1\}$ denote by $F_i = (G \odot H)_i$ the graph with $V(F_i) = V(G) \cup V(H)$ where the vertex x_i is identified with y_i . Then each of F_0 and F_1 has a p -gap graceful labeling. Moreover, when $H \cong P_s$, the labeling can be chosen so that the label of the single vertex of degree one is $n + s$.*

Observation 2. *Let G be a graph with t edges and a (p, q) -gap graceful labeling β such that $\beta(x_0) = 0$, $\beta(x_1) = t + 2$ and H a graph with an α -labeling with $\alpha(y_0) = \alpha_0$, $\alpha(y_1) = \alpha_0 + 1$. Let $F_i = (G \odot H)_i$. Then each of F_0 and F_1 has a (p, q) -gap graceful labeling. Moreover, when $H \cong P_s$, the labeling can be chosen so that the label of the single vertex of degree one is $n + s + 1$.*

The existence of rosy labelings of kayak paddles with at least one even cycle was proved by the authors in [3].

Theorem 3. *Let r be even, $r \geq 4$, $s \geq 3$, $l \geq 1$, and $n = r + s + l$. Then the kayak paddle $KP(r, s, l)$ has a rosy labeling and the complete graph K_{2n+1} is $KP(r, s, l)$ -decomposable.*

3. SOME USEFUL LEMMAS

We first prove some lemmas that we will need in our constructions of rosy kayak paddles later.

Lemma 4. *The cycle C_{4k+1} has a $(2i - 1)$ -gap graceful labeling for every $k > 0$ and $1 \leq i \leq k$ or $k + 2 \leq i \leq 2k + 1$.*

Proof. For $1 \leq i \leq k$, the vertex labels of the cycle are $\{0, 1, 2, \dots, 4k + 1, 4k + 2\} \setminus \{2k + i, 3k + 1\}$. First part of the cycle is labeled

$$(0)4k + 2[4k + 1](1)[4k](4k + 1) \dots \\ \dots(3k + 3)[2k + 3](k)[2k + 2](3k + 2)[2k + 1](k + 1).$$

For $i = k$, we have a $(2k - 1)$ -gap graceful labeling and the second part is labeled

$$(k + 1)[2k - 2](3k - 1)[2k - 3](k + 2)[2k - 4](3k - 2) \dots \\ \dots(2k - 2)[4](2k + 2)[3](2k - 1)[2](2k + 1)[1](2k)[2k](0).$$

For $1 < i < k$, we have a $(2k - i)$ -gap graceful labeling and the second part is labeled

$$(k + 1)[2k - 1](3k)[2k - 2](k + 2)[2k - 3](3k - 1) \dots \\ \dots(2k - i)[2i + 1](2k + i + 1)[2i](2k - i + 1)[2i - 2](2k + i - 1) \dots \\ \dots(2k - 2)[4](2k + 2)[3](2k - 1)[2](2k + 1)[1](2k)[2k](0).$$

For $i = 1$, we have a 1-gap graceful labeling and the second part is labeled

$$(k + 1)[2k - 1](3k)[2k - 2](k + 2)[2k - 3](3k - 1) \dots \\ \dots(2k - 2)[5](2k + 3)[4](2k - 1)[3](2k + 2)[2](2k)[2k](0).$$

For $k + 1 \leq i \leq 2k + 1$, the vertex labels of the cycle are $\{0, 1, 2, \dots, 4k + 1, 4k + 2\} \setminus \{2k - i + 2, 3k + 2\}$. The “lower” part of the cycle is labeled

$$(3k + 3)[2k + 2](k + 1)[2k](3k + 1)[2k - 1](k + 2)[2k - 2](3k) \dots \\ (2k + 3)[3](2k)[2](2k + 2)[1](2k + 1)[2k + 1](0)$$

For $i = 2k + 1$, the “upper” part of the cycle is labeled

$$(0)4k + 2[4k](2)[4k - 1](4k + 1)[4k - 2](3)[4k - 3](4k) \dots \\ (k - 1)[2k + 5](3k + 4)[2k + 4](k)[2k + 3](3k + 3).$$

For $k + 2 \leq i < 2k + 1$, it is labeled

$$(0)4k + 2[4k + 1](1)[4k](4k + 1)[4k - 1](2)[4k - 2](4k) \dots \\ (2k + i + 2)[2i](2k - i + 1)[2i - 2](2k + i + 1)[2i - 3](2k - i + 3)[2i - 4] \\ \dots \\ (k - 1)[2k + 5](3k + 4)[2k + 4](k)[2k + 3](3k + 3).$$

For $i = k + 1$, it is labeled

$$(0)4k + 2[4k + 1](1)[4k](4k + 1)[4k - 1](2)[4k - 2](4k) \dots \\ (k - 2)[2k + 6](3k + 4)[2k + 5](k - 1)[2k + 4](3k + 3). \quad \square$$

Lemma 5. *The cycle C_{4k+1} has a $(1, 2k+1)$ -gap graceful labeling for every $k > 0$.*

Proof. For $k = 1$, the cycle C_5 is labeled

$$(0)7[6](1)[5](6)[4](2)[2](0).$$

For $k \geq 2$, the vertex labels of the cycle are $\{0, 1, 2, \dots, 4k+2, 4k+3\} \setminus \{2k+1, 3k+1, 3k+2\}$. The cycle is labeled

$$\begin{aligned} &(0)4k+3[4k+2](1)[4k+1](4k+2) \dots \\ &(k)[2k+3](3k+3)[2k+2](k+1)[2k-1](3k)[2k-2](k+2) \\ &\dots \\ &(2k-2)[5](2k+3)[4](2k-1)[3](2k+2)[2](2k)[2k](0). \quad \square \end{aligned}$$

Lemma 6. *The cycle C_{4k+3} has a $(2i-1, 2i)$ -gap graceful labeling for every $1 \leq i \leq k$.*

Proof. The vertex labels of the cycle are $\{0, 1, 2, \dots, 4k+4, 4k+5\} \setminus \{k+2, 2k+i+2, 2k+i+3\}$. First part of the cycle is labeled

$$\begin{aligned} &(0)4k+5[4k+4](1)[4k+3](4k+4)[4k+2](2)[4k+1](4k+3) \dots \\ &(k)[2k+5](3k+5)[2k+4](k+1)[2k+3](3k+4)[2k+1](k+3). \end{aligned}$$

For $i = k$, we have a $(2k, 2k-1)$ -gap graceful labeling and the second part is labeled

$$\begin{aligned} &(k+3)[2k-2](3k+1)[2k-3](k+4)[2k-4](3k)[2k-5](k+5) \dots \\ &(2k)[4](2k+4)[3](2k+1)[2](2k+3)[1](2k+2)[2k+2](0). \end{aligned}$$

For $1 < i < k$, we have a $(2i-1, 2i)$ -gap graceful labeling and the second part is labeled

$$\begin{aligned} &(k+3)[2k](3k+3)[2k-1](k+4)[2k-2](3k+2)[2k-3](k+5) \dots \\ &(2k+i+4)[2i+1](2k-i+3)[2i-2](2k+i+1)[2i-3](2k-i+4) \dots \\ &(2k)[4](2k+4)[3](2k+1)[2](2k+3)[1](2k+2)[2k+2](0). \end{aligned}$$

For $i = 1$, we have a $(1, 2)$ -gap graceful labeling and the second part is labeled

$$\begin{aligned} &(k+3)[2k](3k+3)[2k-1](k+4)[2k-2](3k+2)[2k-3](k+5) \dots \\ &(2k)[6](2k+6)[5](2k+1)[4](2k+5)[3](2k+2)[2k+2](0). \quad \square \end{aligned}$$

Lemma 7. *The cycle C_{4k+3} has a $(2, 2k+1)$ -gap graceful labeling for every $k \geq 0$.*

Proof. The vertex labels of the cycle are $\{0, 1, 2, \dots, 4k+4, 4k+5\} \setminus \{k+2, 2k+2, 3k+4\}$. The cycle is labeled

$$\begin{aligned} &(0)4k+5[4k+4](1)[4k+3](4k+4)[4k+2](2)[4k+1](4k+3) \dots \\ &(k)[2k+5](3k+5)[2k+4](k+1)[2k+2](3k+3)[2k](k+3)[2k-1] \dots \\ &(2k+5)[4](2k+1)[3](2k+4)[1](2k+3)[2k+3](0). \quad \square \end{aligned}$$

4. MAIN RESULTS

Theorem 8. *The kayak paddle $KP(4k+3, 4m+3, l)$ has a rosy labeling for any $k, m \geq 0, l \geq 1$.*

Proof. First we prove the case $k = m = 0$. For $l = 1$, we label C_3 by $0, 2, 5$, C'_3 by $6, 7, 13$ and join the vertices 2 and 6. One can check that this $KP(3, 3, 1)$ has a rosy labeling.

When $l > 1$, we again use the $(1, 4)$ -gap labeling of C_3 as above and construct $(1, 4)$ -gap labeling of $C_3 \odot P_l$ with the vertex of degree one labeled $l+4$ and longest edge $l+4$. Then we label C'_3 by $l+8, l+9, 2l+9$ obtaining edge lengths $1, l, l+1$ and join vertices $l+4$ and $l+8$ by the edge with the unused length 4. Because $2l+9 < 2(3+3+l) = 2n$, this is a rosy labeling.

Now we assume $k > m$. Hence, we have $2m+2 \leq 2k$ and by Lemma 6, K_{4k+3} has a $(2m+1, 2m+2)$ -gap graceful labeling. By Observation 2, $K_{3k+3} \odot P_l$ has also a $(2m+1, 2m+2)$ -gap graceful labeling. Moreover, we can choose the labeling so that the vertex of degree one (or the vertex of K_{4k+3} with the highest label, if $l = 1$) is labeled $4k+l+4$. Now we take a path P_{4m+3} with α -labeling

$$\begin{aligned} &(0)4m+2[4m+1](1)[4m](4m+1)[4m-1](2) \dots \\ &(2m)[4](2m+4)[3](2m+1)[2](2m+3)[1](2m+2) \end{aligned}$$

and increase each of the labels $0, 1, 2, \dots, 2m+1$ by $4k+l+5$ and all remaining labels by $(4k+l+5) + (4k+l+4) = 8k+2l+9$. The edge lengths in P_{4m+3} are now $4k+l+5, 4k+l+6, \dots, 4k+4m+l+6$ and we use the so far unused edge length $2m+1$ to join vertices $4k+l+5$ and $4k+2m+l+6$ to close the cycle C_{4m+3} . Finally, we use the last unused edge length $2m+2$ and join the vertex of degree one of $K_{3k+3} \odot P_l$ labeled $4k+l+4$ with vertex $4k+2m+l+6$ in C_{4m+3} . The resulting graph is $KP(4k+1, 4m+1, l)$ as desired.

Next we check the case $k = m > 0$. We know by Lemma 7 that C_{4k+3} has a $(2, 2k+1)$ -gap graceful labeling, hence for $l > 1$ the graph $C_{4k+3} \odot P_l$ also has a $(2, 2k+1)$ -gap graceful labeling with the vertex of degree one labeled $4k+l+4$. Notice that for $l = 1$ the highest label of C_{4k+3} is also $4k+l+4$. We use the same labeling of $C_{4m+3} = C'_{4k+3}$ as above. The only unused edge length is now 2, hence we join the vertex $4k+l+4$ of degree one with $4k+l+6$, which belongs to $C_{4m+3} = C'_{4k+3}$. Therefore, we have constructed $KP(4k+1, 4k+1, l)$.

In both cases the highest vertex label is $8k+4m+2l+11 < 2(4k+4m+l+6) = 2n$, hence the labeling is rosy. \square

Theorem 9. *The kayak paddle $KP(4k+1, 4m+1, l)$ has a rosy labeling for any $k > m \geq 1, l \geq 1$.*

Proof. First we solve the case $k > m$. By Lemma 4, C_{4k+1} has a $(2i-1)$ -gap graceful labeling for any $i \leq k$, hence it has a $(2m+1)$ -gap labeling when $m < k$. Similarly as in the previous proof, we find a $(2m+1)$ -gap labeling of $C_{4k+1} \odot P_{l+1}$ with the vertex of degree one labeled $4k+l+2$. Then we label a path P_{4m+1} as

$$\begin{aligned} & (4k+l+2)[4k+4m+l+3](8k+4m+2l+5)[4k+4m+l+1] \\ & (4k+l+4)[4k+4m+l](8k+4m+2l+4)[4k+4m+l-1](4k+l+5) \\ & \dots \\ & (4k+l+2m+1)[4k+l+6](8k+2m+2l+7)[4k+l+5] \\ & (4k+l+2m+2)[4k+l+4](8k+2m+2l+6)[4k+l+3] \\ & (4k+l+2m+3) \end{aligned}$$

and join vertices $(4k+l+2)$ and $(4k+l+2m+3)$ by an edge with the unused length $2m+1$. Notice that the actual length of the edge between vertices $4k+l+2$ and $8k+4m+2l+5$ is $4k+4m+l+2$, as the subtraction is performed in Z_{2n+1} with $n = 4k+4m+l+2$.

We check that the maximum label satisfies $8k+4m+2l+5 < 8k+8m+2l+2 = 2n$, since $m > 0$, and conclude that the labeling is rosy.

For $k = m$, we use Lemma 5. Because C_{4k+1} has a $(1, 2k+1)$ -gap graceful labeling, similarly as above, we find a $(1, 2k+1)$ -gap labeling of $C_{4k+1} \odot P_l$ with the vertex of degree one labeled $4k+l+2$. Then we label a path P_{4m+1} as

$$\begin{aligned} & (4k+l+3)[8k+l+3](12k+2l+6)[8k+l+1] \\ & (4k+l+5)[8k+l](12k+2l+5)[8k+l-1](4k+l+6) \\ & \dots \end{aligned}$$

$$(6k + l + 2)[4k + l + 6](10k + 2l + 8)[4k + l + 5]$$

$$(6k + l + 3)[4k + l + 4](10k + 2l + 7)[4k + l + 3](6k + l + 4)$$

and join vertices $(4k + l + 3)$ and $(6k + l + 4)$ by an edge with the unused length $2k + 1$. The actual length of the edge between vertices $(4k + l + 3)$ and $(6k + l + 4)$ is $8k + l + 2 = 4k + 4m + l + 2$, as the subtraction is performed in Z_{2n+1} with $n = 8k + l + 2 = 4k + 4m + l + 2$.

The maximum label satisfies $12k + 2l + 6 \leq 16k + 2l + 2 = 2n$, since $k \geq 1$, and the labeling is rosy. This completes the proof \square

Theorem 10. *The kayak paddle $KP(4k + 3, 4m + 1, l)$ has a rosy labeling for any $k \geq 0, m \geq 1, l \geq 1$.*

Proof. First we look at the case $k \geq m$. By Lemma 6, C_{4k+3} has a $(2i - 1, 2i)$ -gap graceful labeling for $1 \leq i \leq k$. Hence, $C_{4k+3} \odot P_l$ has a $(2m - 1, 2m)$ -gap graceful labeling with the vertex of degree one labeled $4k + l + 4$. We label a path P_{4m+1} as

$$(4k + l + 5)[4k + 4m + l + 4](8k + 4m + 2l + 9)[4k + 4m + l + 3]$$

$$(4k + l + 6)[4k + 4m + l + 2](8k + 4m + 2l + 8)[4k + 4m + l + 1]$$

$$(4k + l + 7)$$

$$\dots$$

$$(4k + 2m + l + 3)[4k + l + 8](8k + 2m + 2l + 11)[4k + l + 7]$$

$$(4k + 2m + l + 4)[4k + l + 6](8k + 2m + 2l + 10)[4k + l + 5]$$

$$(4k + 2m + l + 5)$$

and join vertices $(4k + l + 5)$ and $(4k + 2m + l + 5)$ by an edge with the unused length $2m$. Then we join the vertex of $C_{4k+3} \odot P_l$ of degree one, $(4k + l + 4)$, with the vertex of C_{4m+1} labeled $(4k + 2m + l + 3)$ by an edge with the last unused length $2m - 1$.

As the maximum used label satisfies $8k + 4m + 2l + 9 < 8k + 8m + 2l + 8 = 2n$ for every $m \geq 1$, the labeling is rosy.

Now we solve the case when $k < m$. By Lemma 4, the cycle C_{4m+1} has a $(2i - 1)$ -gap graceful labeling for every $1 \leq i \leq m$. Because $k < m$, we have a $(2k + 1)$ -gap graceful labeling of C_{4m+1} and also of $C_{4m+1} \odot P_{l+1}$ with the vertex of degree one labeled $4m + l + 2$.

Then we label a path P_{4k+1} as

$$\begin{aligned}
& (4m+l+2)[4m+4k+l+4](8m+4k+2l+6)[4m+4k+l+3] \\
& (4m+l+3)[4m+4k+l+2](8m+4k+2l+5)[4m+4k+l+1] \\
& (4m+l+4) \\
& \dots \\
& (4k+l+2m+1)[4m+l+6](8m+2k+2l+7)[4m+l+5] \\
& (4m+2k+l+2)[4m+l+4](8m+2k+2l+6)[4m+l+3] \\
& (4m+2k+l+3)
\end{aligned}$$

and join the vertices $(4m+l+2)$ and $(4m+2k+l+3)$ by an edge with the unused length $2k+1$.

Because the largest label satisfies $8m+4k+2l+6 < 8m+8k+2l+8 = 2n$ for every $m \geq 1, k \geq 0$, the labeling is rosy. \square

Our main result now follows immediately.

Theorem 11. *Let r, s be odd, $r \geq s \geq 3, l \geq 1$, and $n = r + s + l$. Then the complete graph K_{2n+1} is decomposable into kayak paddles $\text{KP}(r, s, l)$.*

Combining Theorem 11 with the result obtained by the authors in [3], we can see that every kayak paddle has a rosy labeling and decomposes the corresponding complete graph.

Theorem 12. *Let $r \geq s \geq 3, l \geq 1$, and $n = r + s + l$. Then the kayak paddle $\text{KP}(r, s, l)$ has a rosy labeling and therefore the complete graph K_{2n+1} is $\text{KP}(r, s, l)$ -decomposable.*

REFERENCES

1. C. Barrientos, *Graceful labelings of cyclic snakes*, *Ars Combin.* **60** (2001), 85–96.
2. S. El-Zanati, C. Vanden Eynden, *On Rosa-type labelings and cyclic graph decompositions*, *Math. Slovaca* **59** (2009), 1–18.
3. D. Froncek, L. Tollefson, *Decompositions of complete graphs into kayak paddles*, accepted by J. Indonesian Math. Soc..
4. R. Frucht, *Nearly graceful labelings of graphs*, *Scientia* **5** (1992–1993), 47–59.
5. J.A. Gallian, *A dynamic survey of graph labeling*, *Electronic Journal of Combinatorics* **DS6** (2010).
6. A. Rosa, *On certain valuations of the vertices of a graph*, *Theory of Graphs (Intl. Symp. Rome 1966)*, Gordon and Breach, Dunod, Paris, 1967, pp. 349–355.
7. M. A. Seoud and E. A. Elsayhawi, *On almost graceful, felicitous and elegant graphs*, *J. Egyptian Math. Soc.* **7** (1999), 137–149.
8. M. Truszczyński, *Graceful unicyclic graphs*, *Demonstratio Mathematica* **17** (1984), 377–387.