Alpha labelings of straight simple polyominal caterpillars

Dalibor Froncek, O'Neill Kingston, Kyle Vezina

Department of Mathematics and Statistics

University of Minnesota Duluth

1117 University Drive

Duluth, MN 55812-3000, U.S.A.

e-mail: {dalibor,kings212,vezin009}@d.umn.edu

Abstract

Barrientos and Minion [1] introduced the notion of snake polyomino graphs and proved that they admit an alpha labeling. We introduce a related family of graphs called straight simple polyominal caterpillars and prove that they also admit an alpha labeling. This implies that every straight simple polyominal caterpillar with n edges decomposes the complete graph K_{2kn+1} for any positive integer k.

Keywords: Alpha labeling, graceful labeling, graph decomposition.

2000 Mathematics Subject Classification: 05C78

1 Introduction

In her talk at the Forty-Fifth Southeastern International Conference on Combinatorics, Graph Theory, and Computing in Boca Raton in March, 2014, Sarah Minion (an undergraduate student) presented her joint results with Christian Barrientos on alpha labelings of snake polyominoes and other related graphs [1]. The first author of this paper then proposed a generalization of their result as a topic for the final project in his graph theory class. The two other co-authors (also undergraduate students) wrote a project report that later developed into the presented paper.

Barrientos and Minion [1] define a *snake polyomino* as a chain of m edge-amalgamated four-cycles $C^1, C^2, \ldots, C^m$ with the property that C^1 shares one edge with C^2, C^m shares one edge with C^{m-1}, and for $i = 2, 3, \ldots, m-1$, each C^i shares one edge with C^{i-1} and another edge with C^{i+1}. Note that no edge appears in more than two of those four-cycles.

We generalize this notion and define a *straight simple polyominal caterpillar* as follows. The *spine* of the caterpillar is a straight snake polyomino

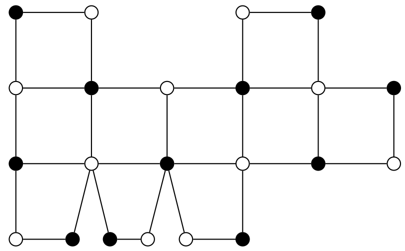

Figure 1: Straight simple polyominal caterpillar

in which the edges of C^i shared with C^{i-1} and C^{i+1} are non-adjacent, which means that every vertex is of degree at most three. The spine can be also viewed as the Cartesian product $P_{m+1}\square P_2$. We denote the vertices of the two paths as $x_0, x_1, \ldots, x_m$ and $y_0, y_1, \ldots, y_m$, respectively. A straight simple polyominal caterpillar than can be constructed by amalgamating at most one four-cycle to each of the edges x_jx_{j+1} and y_ly_{l+1} for $j, l \in \{0, 1, \ldots, m-1\}$. Notice that we can amalgamate the four-cycles to none, one, or both of the two edges x_jx_{j+1} and y_jy_{j+1} for any admissible value of j. The number of four-cycles in the spine will be referred to as the *length* of the caterpillar. An example is shown in Figure 1.

A. Rosa [3] introduced in 1967 certain types of vertex labelings as important tools for decompositions of complete graphs K_{2n+1} into graphs with n edges.

A *labeling* of a graph G with n edges is an injection from $V(G)$, the vertex set of G, into a subset S of the set $\{0, 1, 2, \ldots, 2n\}$ of elements of the additive group Z_{2n+1}. Let ρ be the injection. The *length* of an edge xy is defined as $\ell(x, y) = \min\{\rho(x) - \rho(y), \rho(y) - \rho(x))\}$. The subtraction is performed in Z_{2n+1} and hence $0 < \ell(x, y) \leq n$. If the set of all lengths of the n edges is equal to $\{1, 2, \ldots, n\}$ and $S \subseteq \{0, 1, \ldots, 2n\}$, then ρ is a *rosy labeling* (called originally ρ-valuation by A. Rosa); if $S \subseteq \{0, 1, \ldots, n\}$ instead, then ρ is a *graceful labeling* (called β-valuation by A. Rosa). A graph admitting a graceful labeling is called a *graceful graph.* A graceful labeling ρ is said to be an *α-labeling* if there exists a number λ (called the *boundary value*) with the property that for every edge $xy \in G$ with $\rho(x) < \rho(y)$ it holds that $\rho(x) \leq \lambda < \rho(y)$. Obviously, G must be bipartite to allow an α-labeling. A graph admitting an α-labeling is called an *α-graph.* For an exhaustive survey of graph labelings, see [2] by J. Gallian.

Let G be a graph with at most n vertices. We say that the complete graph K_n has a *G-decomposition* if there are subgraphs $G_0, G_1, G_2, \ldots, G_s$ of K_n, all isomorphic to G, such that each edge of K_n belongs to exactly one G_i. Such a decomposition is called *cyclic* if there exists a graph isomorphism φ such that $\varphi(G_i) = G_{i+1}$ for $i = 0, 1, \ldots, s-1$ and $\varphi(G_s) = G_0$.

Each graceful labeling is of course also a rosy labeling. The following theorem was proved by A. Rosa in [3].

Theorem 1.1. *A cyclic G-decomposition of K_{2n+1} for a graph G with n edges exists if and only if G has a rosy labeling.*

The main idea of the proof is the following. K_{2n+1} has exactly $2n+1$ edges of length i for every $i = 1, 2, \dots, n$ and each copy of G contains exactly one edge of each length. The cyclic decomposition is constructed by taking a labeled copy of G, say G_0, and then adding an element $i \in Z_{2n+1}$ to the label of each vertex of G_0 to obtain a copy G_i for $i = 1, 2, \dots, 2n$.

For graphs with an α-labeling, even stronger result was proved by A. Rosa.

Theorem 1.2. *If a graph G with n edges has an α-labeling, then there exists a G-decomposition of K_{2kn+1} for any positive integer k.*

The proof is based on the observation that if we increase all labels in the partite set with labels exceeding λ by t, then all edge lengths will also stretch by t. Hence, we can take k copies of G and increase the labels in the upper partite set in the j-th copy by jn, where $j = 0, 1, \dots, k-1$. This way we obtain edge lengths $1, 2, \dots, nk$, each exactly once.

2 Amalgamation of alpha labeled graphs

Barrientos and Minion [1] made the following observation.

Theorem 2.1. *If G_1 of order v_1 with n_1 edges and G_2 of order v_2 with n_2 edges are two α-graphs with boundary values λ_1 and λ_2, respectively, then there exist their edge-amalgamation Γ of order $v_1 + v_2 - 2$ with $n_1 + n_2 - 1$ edges that is also an α-graph with boundary value $\lambda = \lambda_1 + \lambda_2$.*

For $i = 1, 2$ let X_i be the partite sets with the lower labels, that is, at most λ_i, and Y_i the sets with the upper labels. Call the respective labelings f_1 and f_2. Further, let $e_1 = x_1y_1$ be the longest edge of G_1 of length n_1 and $e_2 = x_2y_2$ the shortest edge of G_2 of length 1. Then indeed $f_1(x_1) = 0, f_1(y_1) = n_1, f_2(x_2) = \lambda_2$, and $f_2(y_2) = \lambda_2 + 1$.

Barrientos and Minion observed that one can amalgamate x_1 with x_2 and y_1 with y_2 and increase the labels in X_1 and Y_1 by λ_2 and labels in Y_2 by $n_1 - 1$ to obtain the desired graph Γ. The amalgamated edge arising from e_1 and e_2 is called the *link*. Notice that the shortest edge of Γ is in the subgraph arising from G_1 while the longest one is in the subgraph arising from G_2. The edge-amalgamation of G_1 and G_2 as described above will be denoted as $G_1 \| G_2$.

It is easy to observe that this concept can be used for consecutive amalgamation of any number of α-graphs into a larger α-graph. We will use that observation in the next section.

3 Construction

Using the above result, we now prove that every straight simple polyominal caterpillar is an α-graph. The proof will be performed by strong induction. Therefore, we will first need to show suitable α-labelings of certain base cases of our caterpillars.

We define graphs O, B, I, J, U, and Y by their drawings in Figure 2. We also find their α-labelings and present them in Figure 3. Notice that the edge labeled s in each graph in Figure 2 has length 1 induced by the labeling in Figure 3 and the edge labeled l is the longest one induced by the labeling in Figure 3. There is no edge labeled s in graph B since we never use it in the inductive step. The reason is that it does not admit the labeling required by Theorem 2.1.

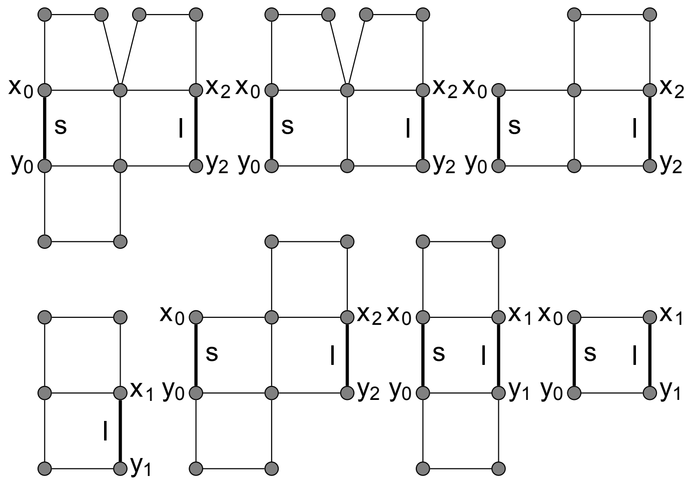

Figure 2: Graphs Y, V, J, O, I, N, B (clockwise from upper left)

The assertion of the Lemma below follows directly from the labelings in Figure 3.

Lemma 3.1. *Graphs O, B, I, J, U, N and Y have α-labelings that induce length 1 on edge $s = x_0y_0$ (except B) and longest length on edge $l = x_1y_1$ (graphs O, B, I) or $l = x_2y_2$ (graphs J, U, N, Y). Moreover, vertices x_0 and y_0 are always labeled so that $f_H(x_0) = \lambda_H$ and $f_H(y_0) = \lambda_H + 1$ for every graph H on the list except graph B.*

We will also need the following Lemma, proved by Rosa [3].

Lemma 3.2. *Let G with n edges admit an α-labeling f. Then the mapping f' defined as $f'(x) = n - f(x)$ is also an α-labeling. Moreover, the length of every edge is the same in both f and f'.*

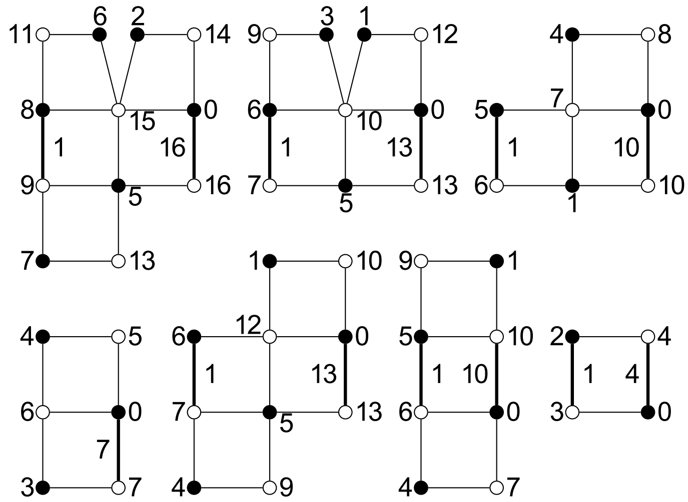

Figure 3: Graphs Y, V, J, O, I, N, B with labelings

We will say that the labeling f' is a dual labeling of f. One can also observe that if the boundary values of f and f' are λ and λ', respectively, then $\lambda' = n - 1 - \lambda$.

For any graph H of the list above, we define its *reflection* H' as the graph horizontally symmetric with H. Although they are mutually isomorphic, we need to treat the graphs and their reflections separately since an amalgamation of two caterpillars H_1 and H_2 may give two non-isomorphic graphs.

Lemma 3.3. *All straight simple polyominal caterpillars of length m, where $m = 1$ or $m = 2$, have an α-labeling f with the property that the longest edge is $x_m y_m$ with $f(x_m) = 0$.*

Proof. There are four caterpillars of length one: O, I, B, and B'. The former three have the labeling by Lemma 3.1 shown in Figure 3, the latter by Lemma 3.2.

There are 16 caterpillars of length two. Four of them, J, U, N and Y have α-labelings as in Figure 3. Their reflections J', U', N' and Y' have α-labelings by Lemma 3.2.

The remaining ones are shown in Figure 4. Caterpillars D and X can be amalgamated of two copies of O and I, respectively, using Theorem 2.1. Caterpillar T and its vertical reflection can be obtained by amalgamating $O\|I$ and $I\|O$, respectively. Caterpillar L arises from amalgamation $B\|O$, and L' follows by Lemma 3.2. Finally, V is amalgamated as $B\|I$ and V' follows again by Lemma 3.2. $\square$

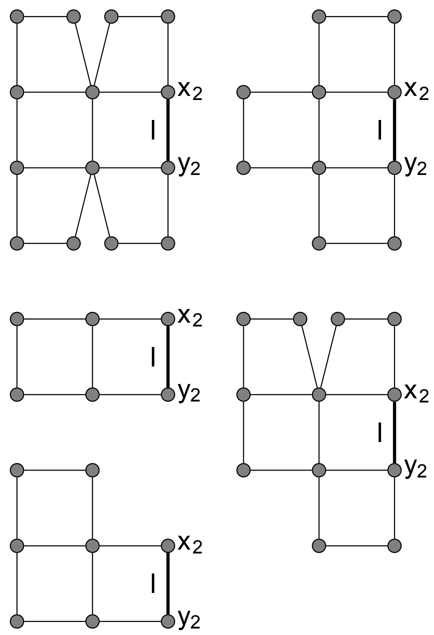

Figure 4: Graphs X, T, V, L, D (clockwise from upper left)

Now we are ready to prove our main result.

Theorem 3.1. *Every straight simple polyominal caterpillar admits an α-labeling.*

Proof. We prove the claim by strong induction on the number of four-cycles in the spine. Let G_m be a straight simple polyominal caterpillar of length m, that is, with m spine four-cycles. As before, we will denote the spine path vertices by $x_0, x_1, \ldots, x_m$ and $y_0, y_1, \ldots, y_m$, respectively. The labeling will be denoted by f.

In fact, we will be proving a stronger statement. Namely, that all such caterpillars admit an α-labeling such that $x_m y_m$ is the longest edge and $f(x_m) = 0$. It is easy to observe that if we have an α-labeling with $f(x_m) > f(y_m) = 0$, we can use the dual f' instead to get $f'(x_m) = 0$. Hence, we can assume that the condition $f(x_m) = 0$ is always satisfied.

By Lemma 3.3, all caterpillars of lengths one and two are α-graphs. Now for $m \geq 3$ we look at the subgraph of G_m induced by the spine vertices $x_{m-1}, x_m, y_{m-1}, y_m$ and possibly the vertices of the four-cycles amalgamated to edges $x_{m-1}x_m$ and $y_{m-1}y_m$, if there are any. If the subgraph is isomorphic to either I or O, then we create G_{m-1} by removing all vertices of that subgraph except x_{m-1} and y_{m-1}. By the inductive hypothesis, G_{m-1} has an α-labeling such that $x_{m-1}y_{m-1}$ is the longest edge and $f(x_{m-1}) = 0$.

We use Theorem 2.1 to amalgamate G_{m-1} with either I or O to obtain the desired α-labeling of G_m with the longest edge $x_m y_m$. We notice that $f(x_m) > f(y_m) = 0$, so we use the dual labeling f' of G_m and we are done.

If the subgraph of G_m induced by the spine vertices $x_{m-1}, x_m, y_{m-1}, y_m$ and the vertices of the four-cycles amalgamated to edges $x_{m-1}x_m$ and $y_{m-1}y_m$ is not isomorphic to either I or O, then it must be isomorphic to B. In this case we will be removing vertices $x_{m-1}, x_m, y_{m-1}, y_m$, the neighbors of either x_{m-1} and x_m or y_{m-1} and y_m belonging to B and possibly the vertices of the four-cycles amalgamated to edges $x_{m-2}x_{m-1}$ and $y_{m-2}y_{m-1}$, if there are any. Then we receive G_{m-2} and a graph H induced by vertices $x_{m-2}, x_{m-1}, x_m, y_{m-2}, y_{m-1}, y_m$ and the vertices of the cycles amalgamated to them.

Because we have excluded some cases above, H must be isomorphic to one of graphs J, U, N, Y or their horizontal reflections J', U', N', Y'. Now G_{m-2} has the required labeling by inductive hypothesis and H by Lemma 3.3. Hence, we amalgamate edge $x_{m-2}y_{m-2}$ with the shortest edge $x'_0 y'_0$ of H of length one to obtain $G_m = G_{m-2} \| H$ which has the desired labeling by Theorem 2.1. It is easy to check that the longest edge is now $x_m y_m$ and $f(x_m) = 0$. This completes the proof. $\square$

The result on decompositions of complete graphs into straight simple polyominal caterpillars now follows directly from Theorems 3.1 and 1.2.

Corollary 3.4. *Every straight simple polyominal caterpillar with n edges decomposes the complete graph K_{2kn+1} for any positive integer k.*

4 Further research

One can similarly define more general polyominal caterpillars as follows. Recall that a *polyominal snake* PS_m of length m is a graph consisting of m edge-amalgamated four-cycles $C^1, C^2, \ldots, C^m$ with the property that C^1 shares one edge with C^2, C^m shares one edge with C^{m-1}, and for $i = 2, 3, \ldots, m-1$, each C^i shares one edge with C^{i-1} and another edge with C^{i+1}. No edge appears in more than two of those four-cycles. This definition is equivalent to that given by Barrientos and Minion in [1].

Obviously, PS_m is an outerplanar graph. We denote by x and y the only two vertices of degree two in C^0 and by x' and y' the only two vertices of degree two in C^m. A *polyominal caterpillar* PC_m of length m is a graph arising from PS_m (called *spine*) by an amalgamation of any number of four-cycles to any of the outer edges of PS_m except xy and $x'y'$. Such caterpillar is *straight* if the spine is the graph $P_{m+1} \square P_2$, and is *simple* if at any outer edge there is at most one amalgamated four-cycle.

The obvious next steps would be to investigate α-labelings of simple polyominal caterpillars, straight polyominal caterpillars, and general polyominal caterpillars. Our guess is that they are listed here in the increasing order of difficulty. Even more general case would be if we allow amalgamation of multiple four-cycles at edges xy and $x'y'$.

References

[1] C. Barrientos and S. Minion, Alpha labelings of snake polyominoes and hexagonal chains, *manuscript*.

[2] J. Gallian, A Dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics*, **16** (2009), # DS6.

[3] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (Intl. Symp. Rome 1966)*, Gordon and Breach, Dunod, Paris (1967), 349–355.