

# Edge reconstruction and the swapping number of a graph

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**Abstract**

We define the swapping number of an arbitrary simple graph, which is related to edge reconstruction, and involves a weakening of the concept of a graph automorphism. We classify all 1-swappable trees and unicyclic graphs, and prove that the expected value of the swapping number grows linearly with the size of the graph.

*Key words:* swapping number,  $k$ -swappable, near automorphism, edge reconstruction, unicyclic graph, graph automorphism

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## 1 Introduction

In what follows, all graphs are finite, undirected, and simple unless otherwise specified. If  $G = (V, E)$  is a graph and  $S \subseteq V$ , then we will understand  $G - S$  to mean the graph induced by  $G$  on the vertex set  $V - S$ . By  $\binom{V}{2}$ , we mean the set of all 2-combinations of  $V$ .

We recall that the *edge-deck* of  $G$  is the multiset  $ED(G)$  of all isomorphism classes of graphs of the form  $(V(G), E(G) - e)$ , where  $e$  is an edge of  $G$ . The *Edge-Reconstruction Conjecture* (Harary, 1964) states that every graph on 4 or more edges is uniquely reconstructible, up to isomorphism, from its edge-deck (see e.g. [3]).

If the Edge-Reconstruction Conjecture fails, then there are two non-isomorphic graphs  $G$  and  $H$  such that every member of  $ED(G)$  can be extended to  $G$  by adding some edge  $e_1$ , and to  $H$  by adding some edge  $e_2$ , where  $e_1 \neq e_2$ . By requiring this latter situation to occur when  $H = G$ , we arrive at the definition of a *1-swappable* graph: namely, a graph in which every edge can be replaced with a non-edge (which may depend on the edge chosen) to produce a graph isomorphic to the original graph. More generally, we have the following:

**Definition 1** Let  $G = (V, E)$  be a graph and let  $k$  be a positive integer. We say that  $G$  is *k-swappable* if for every edge  $e$  of  $G$ , there are two sets  $A \subseteq E$  and  $B \subseteq \binom{V}{2} - E$  such that  $e \in A$ ,  $|A| \leq k$ , and  $G \cong (V, (E \cup B) - A)$ . The *swapping number* of  $G$  is the minimum  $k$  such that  $G$  is *k-swappable*.

In the situation above, for a given edge  $e \in E$ , we will usually write  $G' = (V, (E \cup B) - A)$  and denote the required isomorphism by  $\sigma_e : G \rightarrow G'$ . When  $k = 1$ , we refer to  $\sigma_e$  as a *swapping map*; in this case, we can write  $B = \{e'\}$ , and we call  $e'$  the *replacement* for  $e$ . We also say that  $e$  is *swappable* with  $e'$ .

We note that  $G$  is 1-swappable iff every member of  $ED(G)$  can be extended to  $G$  in at least two different ways.

**Example 1** Any path on three or more vertices is 1-swappable. The swapping number of an  $n$ -cycle is 2 when  $n \geq 4$ . However, any complete graph on two or more vertices (which includes the cases of a path on two vertices and a 3-cycle) is not  $k$ -swappable for any  $k$ , since there are no non-edges to serve as replacements.

The previous example generalizes to the following class of 1-swappable graphs:

**Example 2** Let  $T = (V, E)$  be an edge-transitive graph, and let  $e'$  be any edge of  $T$ . Then  $G = (V, E - \{e'\})$  is 1-swappable. For if  $e \in E(G)$ , then we may replace  $e$  by  $e'$  to produce a graph  $G'$  with  $G' \cong G$ .

By taking  $T$  to be a cycle in Example 2, we get a path as in Example 1. Next we see that almost every 1-swappable tree is a path. Although the concept of swapping number was not explicit in earlier work, its close relationship to the edge-reconstruction problem allows us to classify all 1-swappable trees using a result of Harary and Laury.

We introduce some terminology to facilitate our discussion:

**Definition 2** A *comet* is a tree  $T$  in which exactly one vertex  $x$  has degree greater than 2. We call  $x$  the *central vertex* of  $T$ . An *arm* of a comet  $T$  is a connected component of  $T - \{x\}$ .

We denote a comet with central vertex of degree  $d$  by  $\mathfrak{C}(a_1, a_2, \dots, a_d)$ , where  $a_1, a_2, \dots, a_d$  are the numbers of vertices in the arms; thus  $|V(\mathfrak{C}(a_1, a_2, \dots, a_d))| = 1 + a_1 + a_2 + \dots + a_d$ .

**Theorem 1** *Let  $T$  be a tree on two or more vertices. Then  $T$  is 1-swappable if and only if one of the following holds:*

- (1)  $T$  is a path on at least three vertices;
- (2)  $T \cong \mathfrak{C}(2, 1, 1)$ ;
- (3)  $T \cong \mathfrak{C}(3, 2, 1)$ .

**PROOF.** Theorem 4.4 of [2] gives that the graphs enumerated above (called *quasipaths* in that paper) are the only trees with the property that every edge meeting an endnode of the graph is swappable with some non-edge. Conversely, one sees easily that in fact every edge of the comets  $\mathfrak{C}(2, 1, 1)$  and  $\mathfrak{C}(3, 2, 1)$  is swappable with some non-edge.

An alternate characterization of swappability comes from the following definition:

**Definition 3** Let  $G = (V, E)$  be a graph. A *near-automorphism* of  $G$  of *discrepancy*  $k$  is a bijection  $\sigma : V \rightarrow V$  such that  $|\sigma(E) - E| = k$ .

Note that a near-automorphism of discrepancy 0 is an automorphism. (We use here our assumption that our graphs are finite.)

**Lemma 1** A graph  $G$  is  $k$ -swappable if and only if for every  $e \in E$ , there is a near-automorphism  $\sigma$  of  $G$ , of discrepancy at most  $k$ , such that  $e \notin \sigma(E)$ .

**PROOF.** Let  $G = (V, E)$  be a graph. ( $\Rightarrow$ ) Suppose that  $G$  is  $k$ -swappable, and let  $e \in E$ . Then there are sets  $A \subseteq E$  and  $B \subseteq \binom{V}{2} - E$  such that  $e \in A$ ,  $|A| \leq k$ , and  $G \cong G' = (V, (E \cup B) - A)$ . Let  $\sigma : V \rightarrow V$  be an isomorphism of  $G$  onto  $G'$ . Now  $|E(G')| = |E(G)|$ , so we must have  $|B| = |A|$ . We have  $\sigma(E) - E = E(G') - E(G) = B$ , so  $|\sigma(E) - E| = |B| = |A| \leq k$ . Thus,  $\sigma$  is a near-automorphism of  $G$  of discrepancy at most  $k$ . Since  $e \in A$ , it follows that  $e \notin \sigma(E)$ .

( $\Leftarrow$ ) Let  $e \in E$ . By hypothesis, there is a near-automorphism  $\sigma$  of  $G$ , of discrepancy

at most  $k$ , such that  $e \notin \sigma(E)$ . Set  $B = \sigma(E) - E$  and  $A = E - \sigma(E)$ . Then  $e \in A$ ,  $|A| = |B| \leq k$ , and  $G \cong (V, (E \cup B) - A)$ , as required.

## 2 1-swappable Unicyclic Graphs

Let  $G$  be a connected 1-swappable unicyclic graph having an  $n$ -cycle  $C$  as an induced subgraph, with  $n \geq 3$ . Let  $v_1, \dots, v_n$  be the vertices of  $C$  with subscripts in  $\mathbf{Z}/n\mathbf{Z}$ , and with  $v_i$  adjacent to  $v_{i+1}$  in  $C$ . Let  $G_i$  be the unique component of  $G - \{v_i\}$  disjoint from  $C$ , and let  $T_i$  be the subgraph of  $G$  induced by  $V(G_i) \cup \{v_i\}$ . Thus  $T_i$  is the tree pendant from  $v_i$  in  $G$ .

We first focus on the possibilities for the ordered sequence  $(t_1, \dots, t_n)$ , where  $t_i = |V(T_i)|$ . For each edge  $e$  of  $G$ , fix a corresponding swapping map  $\sigma_e : G \rightarrow G'$ . If  $e \notin E(C)$ , then  $E(C) \subseteq E(G')$ , so, as  $G$  is unicyclic, we must have  $\sigma_e(V(C)) = V(C)$ ; that is,  $\sigma_e$  restricts to an automorphism of  $C$ , namely, an element of the dihedral group  $D_{2n}$ . For such edges  $e$ , we will consider  $\sigma_e$  to act on the elements of  $\mathbf{Z}/n\mathbf{Z}$  via  $\sigma_e(i) = j$  where  $\sigma_e(v_i) = v_j$ . Recall that  $D_{2n}$  consists of  $n$  rotations (shifts) and  $n$  reflections.

The following lemma says that a complete collection of swapping maps for the leaf-edges of a tree cannot have a fixed point.

**Lemma 2** *Let  $T$  be a tree with more than one vertex, and let  $v$  be any vertex of  $T$ . Suppose that for each leaf  $\ell \neq v$  of  $T$ , we are given a corresponding isomorphism  $\sigma_\lambda : T \rightarrow (V(T), E(T) - \lambda + \lambda')$ , where  $\lambda$  is the leaf-edge of  $T$  containing  $\ell$  and  $\lambda' \in \binom{V}{2} - E(T)$ . Then there is some  $\lambda$  such that  $\sigma_\lambda(v) \neq v$ .*

**PROOF.** Suppose for a contradiction that  $T$  is a minimal non-trivial ( $|V(T)| > 1$ ) tree with a distinguished vertex  $v$  for which the lemma fails to hold. Let  $C$  be

a minimal component of the graph  $G = T - \{v\}$ . We must have  $|V(C)| > 1$ , since a leaf-edge attached to  $v$  cannot be swapped when  $v$  is required to be fixed (consider the degree of  $v$  before and after the swap). Let  $v_1$  be the unique vertex in  $V(C)$  which is adjacent to  $v$  in  $T$ . Pick any vertex  $\ell$  in  $V(C) - \{v_1\}$  which is a leaf in  $C$ . Then  $\ell$  is also a leaf in  $T$ ; let  $e \in E(C)$  be the edge containing  $\ell$ . By hypothesis, there is an  $e' \in \binom{V(T)}{2} - E(T)$  and an isomorphism  $\sigma : T \rightarrow T'$  such that  $\sigma(v) = v$ , where  $T' = (V(T), (E(T) \cup \{e'\}) - \{e\})$ . Since  $\sigma(v) = v$  and we chose  $C$  to be minimal, we must have  $e' \subseteq V(C)$ . Now  $\sigma$  permutes the components of  $G$  as vertex sets; let  $(V(C_1), V(C_2), \dots, V(C_k))$  be the cycle of  $\sigma$  containing  $V(C)$ , where  $C_1 = C$ . Let  $v_i$  be the unique vertex in  $V(C_i)$  which is adjacent to  $v$  in  $T$ , and let  $C'_i$  be the component of  $T' - \{v\}$  containing  $v_i$ . Note that  $C_i = C'_i$  if  $i \neq 1$ . So we have  $C_1 \cong C_2 \cong \dots \cong C_k \cong C'_1$ , where each isomorphism is induced by  $\sigma$ . We have  $\sigma(v_i) = v_{i+1}$  if  $1 \leq i < k$ , and  $\sigma(v_k) = v_1$ . Thus we get an isomorphism  $C_1 \cong C'_1$  sending  $v_1$  to itself. Now we see that  $C_1$  violates the lemma with  $v_1$  as the fixed vertex, contradicting the minimality of  $T$ .

**Corollary 2** *For every  $i \in \mathbf{Z}/n\mathbf{Z}$  with  $t_i > 1$ , there is a leaf  $\ell$  of  $T_i$  meeting an edge  $e$  of  $T_i$  such that the swapping map  $\sigma_e$  replaces  $e$  with an edge meeting  $V(T_j)$  for some  $i \neq j$ . We have  $t_j = t_i - 1$ . In particular,  $\sigma_e$  acts non-trivially on  $C$ .*

**PROOF.** Fix  $i \in \mathbf{Z}/n\mathbf{Z}$  with  $t_i > 1$ . For an edge  $e$  of  $T_i$ , if  $\sigma_e(v_i) = v_i$  then  $\sigma_i$  acts on  $T_i$  as a swapping map. By Lemma 2, this cannot happen for every leaf-edge  $e$  of  $T_i$ , so there is an  $e$  whose replacement does not lie in  $\binom{V_i}{2}$ . The result follows.

From now on, for each  $i \in \mathbf{Z}/n\mathbf{Z}$  with  $t_i > 1$ , we fix a swapping map  $\sigma_i$  with the property of Corollary 2. We also let  $\rho_i$  denote the restriction of  $\sigma_i$  to  $C$ , and  $\phi(i)$  denote the value  $j$  of Corollary 2.

**Corollary 3** *If  $i, j \in \mathbf{Z}/n\mathbf{Z}$  with  $t_i, t_j > 1$  and  $i \neq j$ , then  $\rho_i \neq \rho_j$ . Further, if  $\rho_i$  is a reflection, then  $\phi(i) = \rho_i(i)$ .*

**PROOF.** Choose  $i \in \mathbf{Z}/n\mathbf{Z}$  with  $t_i > 1$ . For  $k \in \mathbf{Z}/n\mathbf{Z}$ , let  $t'_k$  denote the size of the tree pendant from  $v_k$  in the range  $G'$  of  $\sigma_i$ . Let  $\rho = \rho_i$ . Then we have

$$t'_k = \begin{cases} t_k - 1, & \text{if } k = i; \\ t_k + 1, & \text{if } k = \phi(i); \\ t_k, & \text{otherwise.} \end{cases}$$

Since we have  $t_k = t'_{\rho(k)}$  for all  $k$ , we see that

$$t_{\rho(k)} - t_k = \begin{cases} 1, & \text{if } \rho(k) = i; \\ -1, & \text{if } \rho(k) = \phi(i); \\ 0, & \text{otherwise.} \end{cases}$$

Thus  $i$  is uniquely determined by  $\rho_i$ . If  $\rho$  is a reflection, then  $\rho$  has order 2, so, taking  $k = \rho^{-1}(i) = \rho(i)$ , we have  $t_i - t_{\rho(i)} = 1$ , or  $t_{\rho(i)} - t_i = -1$ , which implies  $\rho(i) = \phi(i)$ , as desired.

**Definition 4** Let  $d \in \mathbf{Z}/n\mathbf{Z}$ . A  $d$ -segment in  $d \in \mathbf{Z}/n\mathbf{Z}$  is an arithmetic progression in  $\mathbf{Z}/n\mathbf{Z}$  with common difference  $d$ .

**Definition 5** Let  $j, f, k \in \mathbf{Z}$  with  $f|n$ , let

$$S_k = \{i \in \mathbf{Z}/n\mathbf{Z} : t_i = k\},$$

and let

$$B_{j,f} = \{i \in \mathbf{Z}/n\mathbf{Z} : i \equiv j \pmod{f}\}.$$

Let

$$O_{j,f,k} = B_{j,f} \cap S_k.$$

We say that  $O_{j,f,k}$  is full, empty, or partial if the cardinality of  $O_{j,f,k}$  is  $|B_{j,f}|$ , 0, or neither, respectively. We call  $B_{j,f}$  an  $f$ -band. For  $f$  dividing  $n$ , let  $P_{f,k}$  be the union over  $j$  of the partial  $O_{j,f,k}$  sets.

**Lemma 3** *Suppose that  $k > 1$  and  $S_k$  is non-empty. Then  $P_{f,k}$  is non-empty if  $f < n$ . Further, there exists  $d \in \mathbf{Z}/n\mathbf{Z} - \{0\}$  such that  $P_{f,k}$  is the disjoint union of at most two  $d$ -segments when  $f = \gcd(d, n)$ . If the rotation by  $d$  is equal to  $\rho_i$  for some  $i \in S_k$ , then  $P_{f,k}$  is a single  $d$ -segment, where again  $f = \gcd(d, n)$ .*

**PROOF.** Choose  $k \in \mathbf{Z}$  such that  $S_k$  is non-empty. For a rotation  $\rho \in D_{2n}$ , set

$$\Delta(\rho) = \{i \in \mathbf{Z}/n\mathbf{Z} : t_i = k \text{ and } t_{\rho(i)} \neq k\}.$$

Let  $d$  be the amount of rotation; i.e.,  $\rho(i) = i + d$  for all  $i$ . Let  $f = \gcd(d, n)$ . Then the orbits of  $\rho$  on  $\mathbf{Z}/n\mathbf{Z}$  are the bands  $B_{j,f}$  for  $j : 0 \leq j < f$ . We have that  $\Delta(\rho)$  intersects  $B_{j,f}$  non-trivially iff  $O_{j,f,k}$  is partial, and we see that  $|\Delta(\rho)|$  is the smallest number  $\delta$  such that the union of the partial  $O_{j,f,k}$  sets can be written as the disjoint union of  $\delta$  distinct  $d$ -segments.

If  $|S_k| = 1$  then the result is trivial, so assume  $|S_k| > 1$ , and let  $a, b \in S_k$ . First suppose that  $\rho_a$  is a rotation, and let  $\rho = \rho_a$ . If  $i \in \mathbf{Z}/n\mathbf{Z}$  and  $t_i \neq t_{\rho(i)}$ , then  $\rho(i) \in \{a, \phi(a)\}$ ; if  $\rho(i) = a$ , then  $t_i = t_{\rho(i)} - 1 = t_a - 1 = k - 1$ , while if  $\rho(i) = \phi(a)$ , then  $t_i = t_{\rho(i)} + 1 = t_{\phi(a)} + 1 = t_a = k$ . Thus we have  $\Delta(\rho) = \{\rho^{-1}(\phi(a))\}$ , so  $|\Delta(\rho)| = 1$ .

So we may assume that  $\rho_a$  and  $\rho_b$  are reflections. Let  $\sigma = \rho_a \circ \rho_b$ . Then  $\sigma$  is a rotation, and by Corollary 3 and the fact that reflections have order 2,  $\sigma$  is not the identity. Let  $i \in \Delta(\sigma)$ . Then  $t_{\sigma(i)} \neq t_i$ , so either  $t_{\rho_b(i)} \neq t_i$  or  $t_{\rho_a\rho_b(i)} \neq t_{\rho_b(i)}$ . Therefore,  $i \in \{b, \rho_b(b), \rho_b(a), \rho_b\rho_a(a)\}$ . Now  $t_a = t_b = t_i = k$ , so  $i \neq \rho_b(b)$ . If  $i \neq b$ , then we must have  $t_{\rho_b(i)} = t_i = k$  and  $t_{\rho_a\rho_b(i)} = k - 1$ ; so  $\rho_b(i) = a$  and  $i = \rho_b(a)$ . Therefore,  $\Delta(\sigma) \subseteq \{b, \rho_b(a)\}$ , so  $|\Delta(\sigma)| \leq 2$ .

Finally, suppose that  $f|n$  and  $f < n$ . The bands  $B_{j,f}$ , for  $j : 0 \leq j < f$ , partition  $\mathbf{Z}/n\mathbf{Z}$ , so the sets  $O_{j,f,k}$  partition  $S_k$ . Suppose that  $O_{j,f,k}$  is full. Then  $|O_{j,f,k}| = |B_{j,f}| = n/f \geq 2$ , so we can choose distinct elements  $a, b \in O_{j,f,k}$ . Then we have the isomorphism  $\sigma_a : G \rightarrow G'$ , with  $t'_a = t_a - 1 = k - 1$  and  $t'_b = t_b = k$ , where  $t'_i$  is as in the proof of Corollary 3. Therefore, the band  $O'_{j,f,k}$  in  $G'$  is partial. Since  $\sigma_a$  is an isomorphism and  $\rho_a$  permutes the  $f$ -bands in  $C$ , then  $G$  must contain a partial  $f$ -band, namely  $\rho_a^{-1}(O'_{j,f,k})$ .

**Lemma 4** *Let  $k > 1$  with  $S_k$  non-empty, let  $d \in \mathbf{Z}/n\mathbf{Z}$  be as in Lemma 3, and let  $f = \gcd(d, n)$ .*

(i) *If  $P_{f,k}$  is a single  $d$ -segment, then we have  $|P_{f,k}| \in \{1, 2, n/f - 1\}$ .*

(ii) *Otherwise,  $P_{f,k}$  is the disjoint union of two  $d$ -segments  $I_1$  and  $I_2$  with  $|I_1| \geq |I_2|$  and  $(I_1, I_2) \in \{(1, 1), (2, 1)\}$ .*

**PROOF.** (i) Suppose that  $P_{f,k}$  is a single  $d$ -segment. Then we can write  $P_{f,k} = \{i, i + d, i + 2d, \dots, i + \ell d\}$  with  $t_j = k$  for  $j = i + md$  with  $0 \leq m \leq \ell$  and  $t_{i-d} \neq k, t_{i+(\ell+1)d} \neq k$ . We may suppose that  $|P_{f,k}| > 2$ . Choose an interior point  $j = i + md$  with  $1 \leq m \leq \ell - 1$ , and consider the swapping map  $\sigma_j : G \rightarrow G'$ . We must have that  $P'_{f,k}$  is a single  $d$ -segment in  $G'$ . Now the only way to fill the gap in  $P_{f,k}$  left by lowering  $t_j$  to  $t_j - 1$  is by having  $\phi(j) = i - d = i + (\ell + 1)d$ , so we must have  $|P_{f,k}| = |B_{j,f}| - 1 = n/f - 1$ .

(ii) By Lemma 3, we can say that  $P_{f,k}$  is the disjoint union of two  $d$ -segments  $I_1 = \{i_1, i_1 + d, \dots, i_1 + \ell_1 d\}$  and  $I_2 = \{i_2, i_2 + d, \dots, i_2 + \ell_2 d\}$ , where without loss of generality,  $|I_1| \geq |I_2|$ . Suppose for a contradiction that  $|I_1| \geq 3$ . Then choosing a point  $j$  of  $I_1$  and considering  $\sigma_j$ , we see by counting the number of partial  $d$ -segments in  $S_k$  and  $S'_k$  that  $I_1$  and  $I_2$  must lie in the same  $f$ -band, with  $\phi(j) \in X := \{i_1 - d, i_2 - d, i_1 + (\ell_1 + 1)d, i_2 + (\ell_2 + 1)d\}$ . Choosing  $j$  to be an

interior point of  $I_1$ , we see further that  $|X| \leq 3$  and  $|I_2| \leq |I_1| - 2$ . On the other hand, considering  $j = i_1$  when  $i_2 + (\ell_2 + 1)d = i_1 - d$  and  $j = i_1 + (\ell_1 + 1)d$  when  $i_2 - d = i_1 + (\ell_1 + 1)d$ , we find that  $|I_2| + 1 = |I_1|$ , a contradiction.

Finally, we eliminate the possibility that  $|I_1| = |I_2| = 2$ . So suppose that  $I_1 = \{i_1, i_1 + d\}$  and  $I_2 = \{i_2, i_2 + d\}$ . The only way to preserve two partial  $d$ -segments of length 2 under the swapping map  $\sigma_{i_1+d}$  is by moving an edge from  $T_{i_1+d}$  to  $T_{i_1-d}$ , so we must have  $\phi(i_1 + d) = i_1 - d$ . By Lemma 3, we can say that  $\sigma_{i_1+d}$  is a reflection, so by Corollary 3, we have  $\phi(i_1 + d) = \rho_{i_1+d}(i_1 + d)$ . Since a reflection in  $D_{2n}$  has the form  $x \mapsto -x + r$  for some fixed  $r \in \mathbf{Z}/n\mathbf{Z}$ , we find that  $\rho_{i_1+d}(x) = -x + 2i_1$  for all  $x \in \mathbf{Z}/n\mathbf{Z}$ . Now as  $T_{i_2}$  and  $T_{i_2+d}$  are not affected by swapping an edge from  $T_{i_1+d}$  to  $T_{i_1-d}$ , we can say that  $\rho_{i_1+d}$  acts as a 2-cycle on  $\{i_2, i_2 + d\}$  (a non-identity reflection in  $D_{2n}$  cannot have 3 fixed points), so  $\rho_{i_1+d}(i_2) = -2i_2 + 2i_1 = i_2 + d$ , and

$$2i_1 - 3i_2 = d \tag{1}$$

in  $\mathbf{Z}/n\mathbf{Z}$ . On the other hand, considering  $\sigma_{i_1}$  shows that  $\rho_{i_1}(i_1) = i_1 + 2d$ , so  $\rho_{i_1}(x) = -x + 2i_1 + 2d$  for any  $x \in \mathbf{Z}/n\mathbf{Z}$ . Since we must have  $\rho_{i_1}(i_2) = i_2 + d$ , this gives

$$-2i_1 + 2i_2 = d. \tag{2}$$

By symmetry, we also have

$$2i_2 - 3i_1 = d \tag{3}$$

and

$$-2i_2 + 2i_1 = d \tag{4}$$

Subtracting (1) from (4) gives  $i_2 = 0$ , while subtracting (3) from (2) gives  $i_1 = 0$ , so  $i_1 = i_2$ , a contradiction.

**Lemma 5** *Let  $k \in \mathbf{Z}$ ,  $k > 1$ , and suppose that  $S_k$  is non-empty. Let  $d$  be as in the statement of Lemma 3 and let  $f = \gcd(d, n)$ . Set*

$$F_k = \{\sigma \in D_{2n} : \sigma \neq e \text{ and } \sigma(S_k) = S_k\},$$

*where  $e$  is the identity element of  $D_{2n}$ ; that is,  $F_k$  is the set of all non-identity elements of  $D_{2n}$  which fix  $S_k$  as a set. Then  $|F_k| \leq 1$ .*

**PROOF.** Let  $\sigma \in F_k$ . We proceed using the five cases listed in Lemma 4, writing  $I_1$  and  $I_2$  for the (up to) two  $d$ -segments comprising  $P_{f,k}$ , with  $|I_1| \geq |I_2|$ , and possibly with  $|I_2| = 0$ .

Case 1:  $|I_1| = 1$  and  $|I_2| = 0$ .

Then there is a unique  $j$  such that  $O_{j,f,k}$  is non-empty, and we have  $O_{j,f,k} = \{a\}$  for some  $a \in \mathbf{Z}/n\mathbf{Z}$ . Because  $\sigma$  permutes the  $f$ -bands of  $\mathbf{Z}/n\mathbf{Z}$ , this forces  $\sigma(B_{j,f}) = B_{j,f}$ , and since  $\sigma(S_k) = S_k$ , we must have  $\sigma(a) = a$ . Thus  $\sigma$  is the unique reflection about  $a$  in  $D_{2n}$ .

Case 2:  $|I_1| = 2$  and  $|I_2| = 0$ .

Then there is a unique  $j$  such that  $O_{j,f,k}$  is non-empty, and we can write  $O_{j,f,k} = \{a, b\}$  for some  $a, b \in \mathbf{Z}/n\mathbf{Z}$  with  $a \neq b$ . Again we must have  $\sigma(B_{j,f}) = B_{j,f}$ , and so  $\sigma$  acts on  $\{a, b\}$ . There is a unique reflection  $\tau$  in  $D_{2n}$  which sends  $a$  to  $b$ . Suppose that  $\sigma \neq \tau$ . Then either  $\sigma$  is a reflection which fixes  $a$  and  $b$ , or else  $\sigma$  is a rotation which sends  $a$  to  $b$  and  $b$  to  $a$ . In both cases,  $n$  must be even, say  $n = 2m$ , and  $b = a + m$ . We must have  $d \neq m$ , since otherwise, we would have  $f = d$ , and the  $f$ -band containing  $a$  would be full, whereas we know that  $a \in P_{f,k}$ . It follows that  $a + d \neq a - d$  in  $\mathbf{Z}/n\mathbf{Z}$ , and  $a + d, a - d \notin S_k$ . Likewise, we have  $b + d \neq b - d$  in  $\mathbf{Z}/n\mathbf{Z}$ , and  $b + d, b - d \notin S_k$ . Consider the graph  $G'$  which is the result of applying the swapping map  $\sigma_a$ . In  $G'$ , at least one  $d$ -neighbor of  $b$ , that is,

one of  $b + d$  or  $b - d$ , is still not in  $S_k$ , and  $b \in S'_k$ , so  $b \in P'_{f,k}$ . Since  $P_{f,k}$  is closed under addition by  $m$ , so is  $P'_{f,k}$ , and it follows that  $b + m \in P'_{f,k}$ , i.e.,  $a \in P'_{f,k}$ . So  $t'_a = k$ , contradicting  $t'_a = k - 1$ .

Case 3:  $|I_1| = n/f - 1$  and  $|I_2| = 0$ .

Then we can write  $P_{f,k} = O_{j,f,k}$  for a unique  $j$ , and we have  $B_{j,f} - O_{j,f,k} = \{a\}$  for some  $a \in \mathbf{Z}/n\mathbf{Z}$ . It follows that  $\sigma(B_{j,f}) = B_{j,f}$  and  $\sigma(a) = a$ . Thus  $\sigma$  is the unique reflection about  $a$ .

Case 4:  $|I_1| = |I_2| = 1$ .

Then write  $I_1 = \{a\}$  and  $I_2 = \{b\}$ . As in Case 2, we have that  $\sigma$  acts on  $\{a, b\}$ , so we can proceed as in that case.

Case 5:  $|I_1| = 2$  and  $|I_2| = 1$ .

Then write  $I_2 = \{a\}$ . We have  $t_a = k$ ,  $t_{a-d} \neq k$ ,  $t_{a+d} \neq k$ . Further,  $a$  is the unique element of  $\mathbf{Z}/n\mathbf{Z}$  with this property. Since  $\sigma(S_k) = S_k$  and  $\sigma$  sends  $d$ -segments to  $d$ -segments, then  $\sigma$  fixes  $a$ , so  $\sigma$  is the reflection about  $a$ .

**Corollary 4** *We have  $|S_k| = 0$  if  $k \geq 4$ . Further, if  $|S_3| > 0$ , then we have  $|S_3| = |S_2| = 1$ .*

**PROOF.** Let  $k \in \mathbf{Z}$ ,  $k \geq 3$ , and suppose that  $S_k$  is non-empty. Let  $H = \{\rho_i : 2 \leq t_i < k\}$  and choose  $\rho \in H$ . Set  $c = \sum_{2 \leq j < k} |S_j|$ . By Corollary 3, we have  $|H| = c$ . Also, for  $i \in \mathbf{Z}/n\mathbf{Z}$ , we have that  $t_i = t_{\rho(i)}$  unless  $i < k$ . Therefore,  $\rho(S_k) = S_k$ . It follows from Lemma 5 that  $c \leq 1$ . Since  $|S_j| > 0 \implies |S_{j-1}| > 0$  (by Corollary 2), we also have  $c \geq k - 2$ , so the desired result follows.

In the following theorem,  $\Delta(G)$  is, as usual, the degree of the graph  $G$ .

**Theorem 5** *Let  $G$  be a connected unicyclic graph with an  $n$ -cycle  $C$ ,  $V(C) = v_1, v_2, \dots, v_n$ ,  $\{v_i, v_{i+1}\} \in E(C)$ . Let  $t_j$  be the number of vertices in the tree  $T_i$  pendant from  $v_i$  in  $G$ . Then  $G$  is 1-swappable if and only if it is isomorphic to one of the following types:*

(i)  $n = 2m$  and  $t_j = 2$  for  $j \in \{4, 6, \dots, 2m\}$ ,  $t_j = 1$  otherwise;

(ii)  $n = 3$ ,  $t_1 = 2$ ,  $t_2 = 1$ ,  $t_3 = 1$ ;

(iii)  $n = 3$ ,  $\Delta(G) = 3$ ,  $t_1 = 3$ ,  $t_2 = 2$ ,  $t_3 = 1$ .

(iv)  $n = 5$ ,  $\Delta(G) = 3$ ,  $t_1 = 2$ ,  $t_2 = 3$ ,  $t_j = 1$  otherwise;

(v)  $n = 5$ ,  $\Delta(G) = 3$ ,  $t_1 = 2$ ,  $t_3 = 3$ ,  $t_j = 1$  otherwise.

We prove Theorem 5 using three claims. Some notation:  $H(x, y)$  is the graph  $G$  with the edge  $\{x, y\}$  removed. Once  $\{x, y\}$  is specified, we may use only  $H$ .  $G'(x, y)$  is the graph isomorphic to  $G$  arising from  $H(x, y)$ . The vertices on the cycle  $C_n$  are denoted  $1, 2, \dots, n$ , and a vertex of degree one in  $T_i$  is denoted by  $i'$  or  $i''$ . A vertex of degree two in  $T_i$  is denoted by  $i^*$ . By  $\text{dist}_{G'}(T_i, T_k)$  we mean  $\text{dist}_{G'}(\sigma(i), \sigma(k))$ . Without loss of generality, in Claims 1 and 2 we assume that  $n \geq 2j$ .

**Claim 1** *Suppose  $|S_k| = 0$  for  $k > 3$ ,  $|S_3| = |S_2| = 1$ ,  $|S_1| = n - 2$ ,  $t_n = 2$ ,  $t_j = 3$ . Then  $\Delta(G) \neq 4$ .*

**PROOF.** We observe that if  $n = 2j$ , then we cannot find  $G'(n, n')$  since  $\sigma(j) = j$  and  $\sigma(n)$  cannot be equal to  $n$ . Hence,  $n > 2j$  and  $\text{dist}_G(n, j) = j$ .

To show that  $j \leq 2$ , we proceed by contradiction. Let  $j \geq 3$ . Then  $n \geq 7$ . Observe that in  $H(j - 1, j)$  there are two vertices of degree 3, namely  $n$  and  $j$ . If we want to complete  $H(j - 1, j)$  into  $G'$ , we must join one of  $n, j$  to a vertex of degree

one. Because  $\text{ecc}_H(n) \leq n - 2$ , joining  $n$  to any vertex will produce a cycle  $C_k$  for  $k < n$ , a contradiction. Hence we must join  $j$  to a vertex of degree one. The edges  $\{j - 1, j\}, \{j, j'\}, \{j, j''\}$  all belong to  $G$  and  $\{j, n'\}$  produces a cycle  $C_k$  for  $k < n$ , a contradiction.

Now let  $j = 2$ , and first suppose  $n > 5$ . Then in  $H(2, 3)$  we need to join one of the vertices of degree three, namely  $2$  or  $n$ , to a vertex of degree one. All edges  $\{2, 3\}, \{2, 2'\}, \{2, 2''\}$  belong to  $G$  while adding  $\{2, n'\}$  produces  $C_4$ , a contradiction. Joining  $n$  to  $2'$  or  $2''$  produces  $C_4$ , joining  $n$  to  $3$  produces  $C_{n-2}$ , and  $\{n, n'\}$  belongs to  $G$ . Therefore,  $n \leq 5$ . Because  $n$  cannot be equal to  $4$  for the reasons stated above, we have  $n = 5$ . One can check that  $G$  with these parameters is not swappable.

Finally, we need to show that  $j$  cannot be equal to one. Suppose that  $j = 1$ , so  $n \geq 3$ , and remove the edge  $\{n, 1\}$ . Then  $H(n, 1)$  has one vertex of degree three, namely  $1$ , and all other vertices of degree less than three. To complete  $H(n, 1)$  into  $G'$ , we need to join  $1$  to a vertex of degree two other than  $n$ . This produces  $C_k$  for  $k < n$  or a multiple edge, a contradiction.

**Claim 2** *Suppose  $|S_k| = 0$  for  $k > 3$ ,  $|S_3| = |S_2| = 1$ ,  $|S_1| = n - 2$ ,  $t_n = 2$ ,  $t_j = 3$  and  $\Delta(G) = 3$ . Then  $n = 3$  and  $j = 1$ , or  $n = 5$  and  $j \in \{1, 2\}$ .*

**PROOF.** As in Claim 1, we have  $n > 2j$  and  $\text{dist}_G(n, j) = j$ .

Assume  $j \geq 4$  and observe that to complete  $H(j - 1, j)$  into  $G'$ , we need to join one of the vertices of degree one, namely  $n', j - 1, j'$  to a vertex of degree two to obtain  $C_n$  in  $G'$ . The only vertex at distance  $n - 1$  from  $j - 1$  is  $j$ , but  $\{j - 1, j\} \in E(G)$ . At the same time,  $\text{ecc}_H(n') = \max\{n - j + 3, j\} = n - j + 3$ . For  $j \geq 5$ , this is less than  $n - 1$ , hence adding an edge adjacent to  $n'$  produces  $C_k$  for  $k < n$ . For  $j = 4$  the unique vertex at distance  $n - 1$  from  $n'$  is  $j'$ , which is also of degree

one. Finally, consider  $j'$ . The only possibility here is to join  $j'$  to  $j - 3$ , but then  $\text{dist}(T'_n, T'_3) = 3$ , which is not allowed.

If  $j = 3$ , we proceed similarly and investigate  $H(2, 3)$ . We need to join one of  $n', 2, 3'$  to a vertex of degree two to close  $C_n$ . The only vertex at distance  $n - 1$  from 2 is 3, but  $\{j - 1, j\} \in E(G)$ . The only vertex at distance  $n - 1$  from  $3'$  is  $n$ , which is of degree three. The only vertex at distance  $n - 1$  from  $n'$  is  $3^*$ , and in the graph  $H'$  arising from  $H(2, 3)$  by adding the edge  $\{n', 3^*\}$  we have  $\text{dist}_{H'}(T_n, T_j) = 2 < \text{dist}_G(T_n, T_j) = j = 3$ . This is impossible.

Therefore,  $j < 3$ . Suppose  $j = 2$ . One can check that if  $n = 5$ , then  $G$  is swappable. If  $n > 5$ , look at  $H(n, 1)$ . We again need to add an edge joining a vertex of degree one,  $n', 1$ , or  $2'$ , to a vertex of degree two. The only vertex at distance  $n - 1$  from 1 is  $n$ , but  $\{n, 1\} \in E(G)$ . The only vertex at distance  $n - 1$  from  $n'$  is 2, but it is of degree three. The only vertex at distance  $n - 1$  from  $2'$  is  $n - 1$ , but by adding the edge  $(2', n - 1)$  to  $H(n, 1)$  we obtain  $H'$  in which  $\text{dist}_{H'}(T_{n-1}, T_2) = \min\{3, n - 3\}$ . Because  $\text{dist}_G(T_n, T_j) = j = 2$ , we must have  $n - 3 = 2$  and  $n = 5$ , which contradicts our assumption that  $n > 5$ .

Finally, suppose  $j = 1$ . For  $n = 3$ ,  $G$  is swappable. If  $n = 4$ , then  $H(1, 2)$  cannot be completed, so  $G$  with these parameters is not swappable.

For  $n = 5$ ,  $G$  is again swappable. For  $n \geq 6$ , we check  $H(1, 2)$  and try to join one of the vertices of degree one,  $n', 1', 2$ , to a vertex of degree two at distance  $n - 1$ . There is no such vertex for  $n'$ , since the only vertex at distance  $n - 1$  is 2. The only vertex at distance  $n - 1$  from  $1'$  is 4. Adding the edge  $\{1', 4\}$  we obtain  $H'$ . But then  $\text{dist}_{H'}(T_n, T_4) = \min\{4, n - 4\}$ , which must equal  $\text{dist}_G(T_n, T_j) = j = 1$ . So we must have  $n - 4 = 1$  and  $n = 5$ , which contradicts our assumption that  $n \geq 6$ . The only vertex of degree 2 at distance  $n - 1$  from 2 is 1 and because  $\{1, 2\} \in E(G)$ , this edge cannot be added. This completes the proof.

**Claim 3** Suppose  $|S_k| = 0$  for  $k > 2$  and  $|S_2| \geq 2$ . Then  $n = 2m$ , and  $G$  is isomorphic to the unicyclic graph with  $t_j = 2$  for  $j \in \{4, 6, \dots, 2m\}$  and  $t_j = 1$  otherwise.

**PROOF.** We prove this Claim in four steps.

*Subclaim 1* Suppose  $t_n = t_j = 2$  and  $t_1 = t_2 = \dots = t_{j-1} = 1$ . Then  $j \leq 4$ .

Assume that  $j > 4$  and look at  $H(2, 3)$ . We need to join two vertices of degree one to obtain  $C_n$  in  $G'$ . However, for any  $i' \in V(H(2, 3))$  we have  $\text{ecc}_H(i') < n - 1$  and hence none of them can be used to produce  $C_n$ . Thus we are left only with 2 and 3, but  $\{2, 3\} \in E(G)$ . Therefore,  $j \leq 4$ .

*Subclaim 2* Suppose  $t_n = t_1 = 2$ . Then  $G$  is not swappable.

In  $H(n, 1)$  the number of vertices of degree one is the same as in  $G$ . Therefore, we need to join two vertices of degree greater than one to produce  $C_n$ . However, the only such pair of vertices with eccentricities at least  $n - 1$  is  $n, 1$ , and  $\{n, 1\} \in E(G)$ . Hence,  $G$  is not swappable.

*Subclaim 3* Suppose  $t_n = t_3 = 2$  and  $t_1 = t_2 = 1$ . Then  $G$  is not swappable.

In  $H(2, 3)$  the number of vertices of degree one is one more than in  $G$ . Therefore, we need to join a vertex of degree one to a vertex of degree two to produce  $C_n$ . There are only two vertices of degree two with eccentricity at least  $n - 1$  in  $H(2, 3)$ , namely 1 and 3. Vertex 3 would have to be joined back to 2, which is not allowed. The only vertex at distance  $n - 1$  from 1 in  $H(2, 3)$  is  $3'$ , but adding the edge  $\{1, 3'\}$  would produce two neighboring vertices of degree three in  $G'$ . By Claim 2,  $G' \cong G$  cannot contain such vertices.

*Subclaim 4* Suppose  $t_n = t_4 = 2$ ,  $t_1 = t_2 = t_3 = 1$ , and there exists  $j \geq 4$  such

that  $t_{j+1} = t_{j+2} = t_{j+3} = 1$ . Then  $G$  is not swappable.

By Subclaim 1 we have  $t_{j+4} = 2 = t_j$ . We can assume without loss of generality that the segments of three consecutive vertices of degree two are “closest” to each other in  $G$  in the sense that if there are  $k \neq s$  such that  $t_k = t_{k+4} = 2, t_{k+1} = t_{k+2} = t_{k+3} = 1$  and  $t_s = t_{s+4} = 2, t_{s+1} = t_{s+2} = t_{s+3} = 1$ , then  $\text{dist}_G(k, s) \geq j$ .

As before, in  $H(3, 4)$  the number of vertices of degree one is one more than in  $G$ . Therefore, we need to join a vertex of degree one to a vertex of degree two to produce  $C_n$ . There are only two vertices of degree two with eccentricity at least  $n - 1$  in  $H(3, 4)$ , namely 2 and 4. Vertex 4 would have to be joined back to 3, which is not allowed. The only vertex at distance  $n - 1$  from 2 in  $H(3, 4)$  is  $4'$ . If  $j = 4$ , then adding the edge  $\{2, 4'\}$  would produce five consecutive vertices of degree two in  $G'$ , namely  $4', 4, 5, 6, 7$ , which is impossible by Subclaim 1. If  $j > 4$ , then by Subclaim 2 we have  $j \geq 6$ . But then we have two segments of three consecutive vertices of degree two at distance  $j - 2$ , namely  $4', 4, 5$  and  $j + 1, j + 2, j + 3$ . This contradicts our choice of  $j$  as the minimum distance between them.

Therefore, we can conclude that there is only one segment of three consecutive vertices of degree two and the assertion follows immediately.

### 3 Asymptotic Growth of the Swapping Number

In this section, we give a linear upper bound on the swapping number of a graph in terms of the size of its vertex set, and then we show that the average swapping number comes close to this worst-case bound, in the sense that it grows linearly.

**Lemma 6** *Let  $G = (V, E)$  be a graph on  $n$  vertices ( $n \geq 5$ ), not isomorphic to  $K_n$ . Then the swapping number  $k$  of  $G$  satisfies  $k \leq 2n - 7$ .*

**PROOF.** Let  $e = \{x, y\}$  be an edge of  $G$ .

Case 1:  $\deg(x) < n - 1$  or  $\deg(y) < n - 1$ . Say without loss of generality that  $\deg(x) < n - 1$ , and let  $a$  be a vertex not adjacent to  $x$ . Let  $\sigma : V \rightarrow V$  be the permutation  $(y, a)$  switching  $y$  with  $a$ . The discrepancy of  $\sigma$  is  $|\sigma(E) - E| = |E - \sigma(E)| = |\{q \in V : q \text{ is adjacent to exactly one of } a \text{ or } y\}| \leq n - 2$ .

Case 2:  $\deg(x) = \deg(y) = n - 1$ . Let  $\{a, b\} \in \binom{V}{2} - E$ . Let  $\sigma : V \rightarrow V$  be the permutation  $(x, a)(y, b)$  switching  $x$  with  $a$  and  $y$  with  $b$ . Let  $Q = V - \{a, b, x, y\}$ . The only edge in  $E - \sigma(E)$  contained in  $\{a, b, x, y\}$  is  $\{x, y\}$ . All other edges in  $E - \sigma(E)$  meet  $\{a, b, x, y\}$  in one vertex, and the discrepancy of  $\sigma$  is  $1 + \{q \in Q : q \text{ is not adjacent to } a\} + \{q \in Q : q \text{ is not adjacent to } b\} \leq 1 + 2|Q| = 2n - 7$ .

We note that this bound is achieved when  $G$  is the complete bipartite graph  $K_{2,r}$  ( $r > 1$ ) with the edge in the partite set of size 2 added.

**Theorem 6** *There is a positive constant  $\varepsilon$  such that  $\Pr(\text{sw}(G_n) > \varepsilon n) \rightarrow 1$  as  $n \rightarrow \infty$ , where  $G_n$  is a graph chosen uniformly at random from the set of all graphs on  $n$  vertices  $\{1, 2, \dots, n\}$ ,  $\text{sw}(G)$  is the swapping number of  $G$ , and  $\Pr$  is the probability function.*

**PROOF.** Fix a positive integer  $n$  and set  $V = [n] = \{1, 2, \dots, n\}$ . Let  $k$  be a positive integer,  $k \leq \binom{n}{2}/2$ . Let  $\sigma$  be a permutation of  $V$ . Let us count the number of graphs  $G$  with  $V(G) = V$  for which  $\sigma$  has discrepancy  $k$ .

Form a directed graph  $\Gamma$  (with loops allowed) as follows. Set  $V(\Gamma) = \binom{[n]}{2}$ , the set of all potential edges of  $G$ . Note that  $\sigma$  acts on  $V(\Gamma)$  in a natural way, namely  $\sigma(\{a, b\}) = \{\sigma(a), \sigma(b)\}$ . Set  $E(\Gamma) = \{(e, \sigma(e)) : e \in V(\Gamma)\}$ . Thus,  $\Gamma$  is a collection of disjoint cycles, namely, the orbits of  $\sigma$  on  $\binom{[n]}{2}$ .

Let  $\Gamma_1, \dots, \Gamma_c$  be the components of  $\Gamma$ , and choose a representative  $m_i \in V(\Gamma_i)$  for each  $i$ . Let  $f$  be the number of fixed points of  $\sigma$  acting on  $V$ . Let  $c_1$  be the number of components of  $\Gamma$  of size 1. Note that  $c_1 = \binom{f}{2} + t$ , where  $t$  is the number of 2-cycles of  $\sigma$  acting on  $V$ . We have  $t \leq (n - f)/2$  and  $c \leq c_1 + \left(\binom{n}{2} - c_1\right)/2 \leq \left(\binom{f}{2} + \binom{n}{2} + \frac{n-f}{2}\right)/2$ . Writing  $r = n - f$  and simplifying, we find  $c \leq \binom{n}{2} - \frac{r(n-2)}{4}$ .

We next designate the elements of the symmetric difference  $S = (E - \sigma(E)) \cup (\sigma(E) - E)$ , by selecting a set of  $2k$  vertices of  $\Gamma$ . The only restriction is that there must be an *even* number of elements of  $S$  in each component of  $\Gamma$ . In particular, we cannot choose elements of  $S$  to be edges both of whose endpoints are fixed by  $\sigma$ , and so the number of choices for  $S$  is at most  $\binom{\binom{n}{2} - \binom{f}{2}}{2k}$ . Since  $\binom{n}{2} - \binom{f}{2} = \frac{r(2n-r-1)}{2} < nr$ , the number of choices for  $S$  is at most  $\binom{nr}{2k}$ .

Finally, we determine  $G$  completely by specifying, for each  $i$ , whether or not  $m_i$  is an edge of  $G$ . The number of choices here is  $2^c$ .

Therefore, the number of graphs  $G$  for which  $\sigma$  has discrepancy  $k$  is at most  $2^{\binom{n}{2} - \frac{r(n-2)}{4}} \binom{nr}{2k}$ . The number of permutations of  $V$  with  $f$  fixed points is at most  $\binom{n}{f} (n - f)!$  (first choose the  $f$  fixed points, then choose a permutation of the remaining  $n - f$  points)  $= \binom{n}{r} r! = \frac{n!}{(n-r)!} \leq n^r$ . Let  $g(n, k)$  denote the total number of graphs  $G$  such that  $V(G) = V$  and  $G$  has swapping number  $k$ . Summing over all  $\sigma$ , we find

$$g(n, k) \leq \sum_{r=2}^n n^r \cdot 2^{\binom{n}{2} - \frac{r(n-2)}{4}} \binom{nr}{2k} = 2^{\binom{n}{2}} \sum_{r=2}^n \left(2^{\frac{1}{2} + \log_2(n) - \frac{n}{4}}\right)^r \binom{nr}{2k}.$$

For  $n$  large enough, we have  $\frac{1}{2} + \log_2(n) - \frac{n}{4} < -\frac{n}{5}$ , so we can write

$$g(n, k) \leq 2^{\binom{n}{2}} \sum_{r=2}^n 2^{-rn/5} \binom{nr}{2k}.$$

From Stirling's formula,  $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ , we deduce

$$\binom{nr}{\varepsilon nr} \leq (\varepsilon^{-\varepsilon}(1-\varepsilon)^{\varepsilon-1})^{nr}$$

for any fixed  $\varepsilon$  with  $0 \leq \varepsilon < 0.5$ ,  $r \geq 2$ , and  $n$  large enough. Since

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-\varepsilon}(1-\varepsilon)^{\varepsilon-1} = 1,$$

we can choose an  $\varepsilon$  small enough that  $\varepsilon^{-\varepsilon}(1-\varepsilon)^{\varepsilon-1} \leq 2^{1/10}$  and  $\varepsilon < 0.1$ .

Let  $\mathfrak{G}(n, k)$  denote the number of all graphs  $G$  such that  $V(G) = V$  and  $G$  has swapping number *at most*  $k$ . Fix a  $k$  with  $k \leq \varepsilon n$ . Then we have  $\mathfrak{G}(n, k) = \sum_{\ell=1}^k g(n, \ell)$ . If  $1 \leq \ell \leq k$ , then  $\binom{nr}{2\ell} / \binom{nr}{\ell-1} = (nr - 2\ell + 1)/(2\ell) > (nr - 2\ell)/(2\ell) \geq (1 - \varepsilon)/\varepsilon > (1 - 0.1)/0.1 = 9$ . So we have

$$\sum_{\ell=1}^k \binom{nr}{2\ell} \leq \sum_{q=0}^{k-1} \binom{nr}{2k} 9^{-q} < \binom{nr}{2k} / (1 - 1/9) = \frac{9}{8} \binom{nr}{2k}.$$

Therefore,

$$\begin{aligned} \mathfrak{G}(n, k) &\leq 2^{\binom{n}{2}} \sum_{r=2}^n 2^{-rn/5} \frac{9}{8} \binom{nr}{2k} \leq 2^{\binom{n}{2}} \sum_{r=2}^n 2^{-rn/5} \frac{9}{8} 2^{nr/10} \\ &= \frac{9}{8} 2^{\binom{n}{2}} \sum_{r=2}^n 2^{-rn/10} < \frac{9}{8} 2^{\binom{n}{2}} \frac{2^{-2n/10}}{1 - 2^{-n/10}}. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} \mathfrak{G}(n, k) / 2^{\binom{n}{2}} = 0,$$

which completes the proof.

## 4 Conclusion

The classification of  $k$ -swappable trees for  $k \geq 2$  requires further study. In addition, we are unaware of any infinite families of 1-swappable graphs other than those described by Example 2.

Until now, the prevailing definitions of network reliability have focussed on preserving connectivity of the network, and not on preserving its logical structure [1]. By changing this focus, a possible application of 1-swappable graphs to network design is as follows. Assuming that the physical location of nodes in a network is immaterial, but that maintaining the logical topology of the network is critical, and that connections between nodes are expensive to add or remove, then a 1-swappable graph gives an optimal design for the problem of correcting for an arbitrary faulty connection. This is because a 1-swappable graph provides a network topology in which a faulty connection may be replaced by a single new connection (as the faulty one is being repaired, say) to produce an equivalent logical topology.

It is possible to define several natural variations of the notion of swappability. To define *vertex swappability*, we require that for every  $v \in V(G)$ , there is an element  $w \notin V(G)$  and an isomorphism  $G \rightarrow G' = (V', E')$ , where  $V' = (V(G) - \{v\}) \cup \{w\}$ ,  $E(G - v) \subseteq E'$ , and the neighborhoods  $N_G(v)$  and  $N_{G'}(w)$  are distinct. In view of this definition, we use the term *edge-swappable* to refer to the notion which we have employed in the preceding sections of this paper.

Stronger than vertex swappability but weaker than 1-edge-swappability is the condition that for every  $v \in V(G)$ , there is an edge  $e \in E(G)$  meeting  $v$  such that  $e$  can be replaced by a non-edge of  $G$  to produce a graph isomorphic to  $G$ . We may term this condition *weak edge-swappability*.

To strengthen the notion of edge-swappability, we define  $G$  to be *r-fold 1-edge-swappable* if for every  $e \in E(G)$ , there are at least  $r$  distinct elements  $e' \in \binom{V}{2} - E(G)$  such that  $e'$  is swappable with  $e$ . A somewhat trivial example of an  $r$ -fold 1-edge-swappable graph is the graph on  $n$  vertices with only one edge, where we may take  $r = \binom{n}{2} - 1$ .

## References

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