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Singular Perturbations and Time Scales in Guidance and Control of Aerospace Systems: A Survey

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I. Introduction

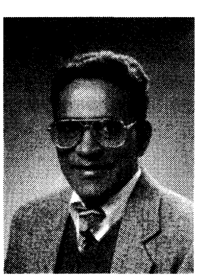
A FUNDAMENTAL problem in the theory of systems and control is the mathematical modeling of a physical system. The realistic representation of many systems calls for high-order dynamic equations. The presence of some parasitic parameters, such as small time constants, resistances, inductances, capacitances, moments of inertia, and Reynolds number, is often the source for the increased order and stiffness of these systems. The stiffness, attributed to the simultaneous occurrence of slow and fast phenomena, gives rise to time scales. The systems in which the suppression of a small parameter is responsible for the degeneration (or reduction) of dimension (or order) of the system are labeled as singularly perturbed systems, which are a special representation of the general class of time scale systems. The curse of dimensionality coupled with stiffness poses formidable computational complexities for the analysis and design of multiple time scale systems.

A. Singular Perturbations in Mathematics and Fluid Dynamics

Singular perturbations has its birth in the boundary layer theory in fluid dynamics due to Prandtl.[300] In a paper, given at the Third International Congress of Mathematicians in Heidelberg in 1904, he pointed out that, for high Reynolds numbers, the velocity in incompressible viscous flow past an object changes very rapidly from zero at the boundary to the value as given by the solution of the Navier–Stokes equation. This change takes place in a region near the wall, which is called the *boundary layer*, the thickness of which is proportional to the inverse of the square root of the Reynolds number. Boundary-layer theory was further developed into an important topic in fluid dynamics.[102,173] The term *singular perturbations* was first introduced by Friedrichs and Wasow.[122] In Russia, mainly at Moscow State University, research activity on singular perturbations for ordinary differential equations, originated and developed by Tikhonov[374] and his students, especially Vasiléva,[378] continues to be vigorously pursued even today.[382] An excellent survey of the historical development of singular perturbations is found in a recent book by O'Malley.[290] Other historical surveys concerning the research activity in singular perturbation theory at Moscow State University and elsewhere are found in Refs. 379, 380.

In studying singular perturbation problems in fluid dynamics, Kaplun[173] introduced several notions such as degenerate solution, limit process, nonuniform convergence, inner and outer expansions, and matching. Fluid dynamics is still an abundant source of many challenging problems. Attention is drawn to the important works on singular perturbations in fluid dynamics in Refs. 90, 102, 106, 111, 152–154, 157, 173, 219, 231, 233, 282, 284, 297, and 303. Reference 219 is a survey on the essential ideas not on the literature.

In Ref. 303, the boundary value technique (BVT), advanced by Roberts,[313] is extended to the solution of the Navier–Stokes equation at high Reynolds numbers. Three standard flows, uniform flow past a plate, flow with a linearly adverse external velocity, and shear flow past a flat plate, were considered. The BVT is different from the method of matched asymptotic expansion (MAE) [also called the



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coefficient matching technique (CMT)] in evaluating the boundary conditions of the outer solution. The principal difference is that "the adjoining of the outer and inner solutions in the BVT is carried out at a *point* in the domain of the problem, this point being found interactively, while in the CMT the inner and the outer expansions are matched *asymptotically*."

The fundamental concepts of matching and boundary layers are revisited by Eckhaus[105] and Van Dyke[103] with a clear exposition of the two main approaches in matching, that is, those of Kaplun–Lagerstrom and Van Dyke. Van Dyke[103] in particular presents an excellent history of boundary-layer ideas and summarizes the applications of matching to problems in hydrostatics, hydrodynamics, elasticity, electrostatics, and acoustics.

B. Singular Perturbations and Time Scales (SPaTS) in Control

The methodology of singular perturbations and time scales (SPaTS), gifted with the remedial features of both dimensional reduction and stiffness relief, is considered as a boon to systems and control engineers. The technique has now attained a high level of maturity in the theory of continuous-time and discrete-time control systems described by ordinary differential and difference equations, respectively. From the perspective of systems and control, Kokotovic and Sannuti[325] were the first to explore the application of the theory of singular perturbations for ordinary differential equations to optimal control, both open-loop formulation leading to two-point boundary value problem[209] and closed-loop formulation leading to the matrix Riccati equation (see Ref. 328). The growth of research activity in the field of SPaTS resulted in the publication of excellent survey papers (see Refs. 66, 129, 165, 202, 203, 207, 217, 219, 259, 261, 263, 267, 268, 274, 323, 378 and references therein) reports and proceedings of special conferences,[10,106,208] and research monographs and books (see Refs. 1, 2, 35, 36, 83, 102, 104, 125–127, 135, 161, 173, 184–186, 205, 206, 218, 240, 257, 278, 283, 289, 290, 296, 331, 381, 382, 398 and references therein).

In this paper we present a survey of the applications of the theory and techniques of SPaTS in guidance and control of aerospace systems. In particular, emphasis will be placed on problem formulation and solution approaches that are useful in applying the theory for various types of problems arising in aerospace systems. A unique feature of this survey is that it assumes no prior knowledge in SPaTS and, hence, provides a brief introduction to the subject. Further, the survey covers related fields such as fluid dynamics, space structures, and space robotics.

II. Modeling

A. Singularly Perturbed Systems

In this section, we present some basic definitions and mathematical preliminaries of SPaTS. For simplicity, consider a system described by a linear, second-order, initial value problem

$$\epsilon\ddot{x}(t,\epsilon)+\dot{x}(t,\epsilon)+x(t,\epsilon)=0$$
$$x(t=0)=x(0), \qquad \dot{x}(t=0)=\dot{x}(0) \tag{1}$$

where the small parameter ϵ multiplies the highest derivative. Here and in the rest of this paper, dot and double dot indicate first and second derivatives, respectively, with respect to t. The degenerate (outer or reduced-order) problem is obtained by suppressing the small parameter ϵ in Eq. (1) as

$$\dot{x}^{(0)}(t)+x^{(0)}(t)=0, \qquad x^{(0)}(t=0)=x(0) \tag{2}$$

with the solution as

$$x^{(0)}(t)=x^{(0)}(0)e^{-t}=x(0)e^{-t} \tag{3}$$

Because the degenerate problem in Eq. (2) is only of first order and cannot be expected to satisfy both the given initial conditions given in Eq. (1), one of the initial conditions $\dot{x}(0)$ has been sacrificed in the process of degeneration. The problem given by Eq. (1), where the small parameter ϵ is multiplying the highest derivative is called a singularly perturbed (singular perturbation) problem,[398] where the order of the problem becomes lower for $\epsilon=0$ than for $\epsilon\neq 0$.

B. Continuous-Time Control Systems

We now introduce the idea of singular perturbations from the systems and control point of view. When the state variable representation is used for a general case, a linear time-invariant system is identified as

$$\dot{\boldsymbol{x}}(t,\epsilon)=\boldsymbol{A}_{11}\boldsymbol{x}(t,\epsilon)+\boldsymbol{A}_{12}\boldsymbol{z}(t,\epsilon)+\boldsymbol{B}_1\boldsymbol{u}(t,\epsilon)$$
$$\boldsymbol{x}(t=0)=\boldsymbol{x}_0\in\Re^n$$
$$\epsilon\dot{\boldsymbol{z}}(t,\epsilon)=\boldsymbol{A}_{21}\boldsymbol{x}(t,\epsilon)+\boldsymbol{A}_{22}\boldsymbol{z}(t,\epsilon)+\boldsymbol{B}_2\boldsymbol{u}(t,\epsilon)$$
$$\boldsymbol{z}(t=0)=\boldsymbol{z}_0\in\Re^m \tag{4}$$

where, $\boldsymbol{x}(t,\epsilon)$ and $\boldsymbol{z}(t,\epsilon)$ are n- and m-dimensional state vectors, respectively, $\boldsymbol{u}(t,\epsilon)$ is an r-dimensional control vector, and ϵ is a small, scalar parameter. The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are of appropriate dimensions. The system given by Eq. (4) is said to be in the singularly perturbed form in the sense that by making $\epsilon=0$ in Eq. (4) the degenerate system

$$\dot{\boldsymbol{x}}^{(0)}(t)=\boldsymbol{A}_{11}\boldsymbol{x}^{(0)}(t)+\boldsymbol{A}_{12}\boldsymbol{z}^{(0)}(t)+\boldsymbol{B}_1\boldsymbol{u}(t), \qquad \boldsymbol{x}^{(0)}(t=0)=\boldsymbol{x}_0$$
$$0=\boldsymbol{A}_{21}\boldsymbol{x}^{(0)}(t)+\boldsymbol{A}_{22}\boldsymbol{z}^{(0)}(t)+\boldsymbol{B}_2\boldsymbol{u}(t), \qquad \boldsymbol{z}^{(0)}(t=0)\neq\boldsymbol{z}_0 \tag{5}$$

is a combination of a differential system in $\boldsymbol{x}^{(0)}(t)$ of order n and an algebraic system in $\boldsymbol{z}^{(0)}(t)$ of order m. The effect of degeneration is not only to cripple the order of the system from $(n+m)$ to n by dethroning $\boldsymbol{z}(t)$ from its original state variable status, but also to desert its initial conditions $\boldsymbol{z}_0$. This is a harsh punishment on $\boldsymbol{z}(t)$ for having a close association (multiplication) with the singular perturbation parameter ϵ. We assume that the matrix $\boldsymbol{A}_{22}$ is nonsingular. However, an important contribution[189] deals about the situation where $\boldsymbol{A}_{22}$ may be singular. We can also view the degeneration as equivalent to letting the forward gain of the system go to infinity.

In the nonlinear case, the singularly perturbed system is represented as

$$\dot{\boldsymbol{x}}(t,\epsilon)=\boldsymbol{f}[\boldsymbol{x}(t,\epsilon),\boldsymbol{z}(t,\epsilon),\boldsymbol{u}(t,\epsilon),\epsilon,t], \qquad \boldsymbol{x}(t=0)=\boldsymbol{x}_0$$
$$\epsilon\dot{\boldsymbol{z}}(t,\epsilon)=\boldsymbol{g}[\boldsymbol{x}(t,\epsilon),\boldsymbol{z}(t,\epsilon),\boldsymbol{u}(t,\epsilon),\epsilon,t], \qquad \boldsymbol{z}(t=0)=\boldsymbol{z}_0 \tag{6}$$

In the preceding discussion, we assumed an initial value problem. As a boundary value problem, we have the conditions as $\boldsymbol{x}(t=0)=\boldsymbol{x}_0$ and $\boldsymbol{z}(t=t_f)=\boldsymbol{z}_f$ or other sets of boundary conditions.

The important features of singular perturbations are summarized as follows.

1) The degenerate problem, also called the unperturbed problem, is of reduced order and cannot satisfy all of the given boundary conditions of the original (full or perturbed) problem.

2) There exists a boundary layer where the solution changes rapidly. It is believed that the boundary conditions that are lost during the process of degeneration are buried inside the boundary layer.

3) To recover the lost initial conditions, it is required to stretch the boundary layer using a stretching transformation such as $\tau=t/\epsilon$.

4) The degenerate problem, also called the unperturbed problem, is of reduced order and cannot satisfy all of the given boundary conditions of the original problem.

5) The singularly perturbed problem described by Eq. (1) has two widely separated characteristic roots giving rise to slow and fast components (modes) in its solution. Thus, the singularly perturbed problem possesses a two-time scale property. The simultaneous presence of slow and fast phenomena makes the problem stiff from the numerical solution point of view.

To illustrate these features, reconsider the simple problem given by Eq. (1) in singularly perturbed, state variable form as

$$\frac{\mathrm{d}x(t,\epsilon)}{\mathrm{d}t}=z(t,\epsilon), \qquad x(t=0)=x(0)$$
$$\epsilon\frac{\mathrm{d}z(t,\epsilon)}{\mathrm{d}t}=-x(t,\epsilon)-z(t,\epsilon), \qquad z(t=0)=z(0) \tag{7}$$

For this problem, with some specific values of $\epsilon=0.1$, $x(0)=2$, and $z(0)=3$, Fig. 1 shows various solutions. Note the following points.

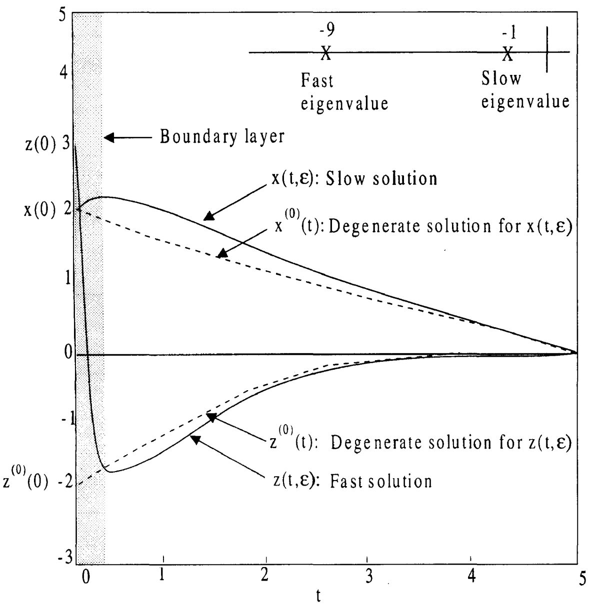

Fig. 1 Basic concepts of singular perturbations and time scales.

1) For $\epsilon = 0.1$, the eigenvalues for Eq. (7) are (approximately) -1 and -9 corresponding to slow and fast solutions, respectively.

2) The predominantly slow solution is $x(t, \epsilon)$ and the predominantly fast solution is $z(t, \epsilon)$, which has been associated (multiplied) with ϵ, obtained by solving the full-order or the exact problem given by Eq. (7).

3) The boundary layer (or region of rapid transition) exists near the initial point $t = 0$.

4) Here, $x^{(0)}(t)$ and $z^{(0)}(t)$ are the degenerate solutions of $x(t, \epsilon)$ and $z(t, \epsilon)$, respectively, obtained by solving the degenerate problem, with $\epsilon = 0$ in Eq. (7) as

$$\frac{\mathrm{d}x^{(0)}(t)}{\mathrm{d}t} = z^{(0)}(t), \qquad x^{(0)}(t = 0) = x(0)$$

$$0 = -x^{(0)}(t) - z^{(0)}(t) \tag{8}$$

5) Here $z^{(0)}(t) = -x^{(0)}(t)$ and $z^{(0)}(t = 0) \neq z(0)$, in general.

6) The degenerate solution $z^{(0)}(t)$ is close to its exact solution $z(t, \epsilon)$ only outside the boundary layer.

7) One $[z(0)]$ of the given two conditions $x(0)$ and $z(0)$ is destroyed in the process of degeneration or $z(t, \epsilon)$ has lost its initial condition $z(0)$ while letting $\epsilon \to 0$.

C. Discrete-Time Control Systems

Similar to continuous-time systems, there are singularly perturbed, discrete-time control systems. Basically, there are two sources of modeling the discrete-time systems.[278,257]

Source 1: Pure Difference Equations

Consider a general linear, time-invariant discrete-time system,

$$\boldsymbol{x}(k+1) = \boldsymbol{A}_{11}\boldsymbol{x}(k) + \epsilon^{1-j}\boldsymbol{A}_{12}z(k) + \boldsymbol{B}_1\boldsymbol{u}(k)$$

$$\epsilon^{2i}z(k+1) = \epsilon^{j}\boldsymbol{A}_{21}\boldsymbol{x}(k) + \epsilon\boldsymbol{A}_{22}z(k) + \epsilon^{j}\boldsymbol{B}_2\boldsymbol{u}(k) \tag{9}$$

where $i \in \{0, 1\}$, $j \in \{0, 1\}$, $\boldsymbol{x}(k)$ and $z(k)$ are n- and m-dimensional state vectors, respectively, and $\boldsymbol{u}(k)$ is an r-dimensional control vector. Depending on the values for i and j, the three limiting cases of Eq. (9) are 1) C model ($i = 0$, $j = 0$), where the small parameter ϵ appears in the column of the system matrix, 2) R model ($i = 0$, $j = 1$), where we see the small parameter ϵ in the row of the system matrix, and 3) D model ($i = 1$, $j = 1$), where ϵ is positioned in an identical fashion to that of the continuous-time system given by Eq. (4) described by differential equations. For further details, see Refs. 24, 42, 91, 234, 257, 276, 278, 299, 304, and 370.

Source 2: Discrete-Time Modeling of Continuous-Time Systems

Here either numerical solution or sampling of singularly perturbed continuous-time systems results in discrete-time models. Consider the continuous-time system given by Eq. (4). When a block diagonalization transformation is applied,[235] the original state variables $\boldsymbol{x}(t)$ and $z(t)$ can be expressed in terms of the decoupled system consisting of slow and fast variables $\boldsymbol{x}_s(t)$ and $z_f(t)$, respectively. Using a sampling device with the decoupled continuous-time system, we get a discrete-time model which critically depends on the sampling interval T (Ref. 172).

Depending on the sampling interval, we get a fast (subscripted by f) or slow (subscripted by s) sampling model. In a particular case, when $T_f = \epsilon$, we get the fast sampling model as

$$\boldsymbol{x}(n+1) = (\boldsymbol{I}_s + \epsilon\boldsymbol{D}_{11})\boldsymbol{x}(n) + \epsilon\boldsymbol{D}_{12}z(n) + \epsilon\boldsymbol{E}_1\boldsymbol{u}(n)$$

$$z(n+1) = \boldsymbol{D}_{21}\boldsymbol{x}(n) + \boldsymbol{D}_{22}z(n) + \boldsymbol{E}_2\boldsymbol{u}(n) \tag{10}$$

where n denotes the fast sampling instant (not to be confused with the system order described earlier). Similarly, if $T_s = 1$, we obtain the slow sampling model as

$$\boldsymbol{x}(k+1) = \boldsymbol{E}_{11}\boldsymbol{x}(k) + \epsilon\boldsymbol{E}_{12}z(k) + \boldsymbol{E}_1\boldsymbol{u}(k)$$

$$z(k+1) = \boldsymbol{E}_{21}\boldsymbol{x}(k) + \epsilon\boldsymbol{E}_{22}z(k) + \boldsymbol{E}_2\boldsymbol{u}(k) \tag{11}$$

where k represents the slow sampling instant and $n = k[1/\epsilon]$. Also, the $\boldsymbol{D}$ and $\boldsymbol{E}$ matrices are related to the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, and transformation matrices.[172] Note that the fast sampling model given by Eq. (10) can be viewed as the discrete analog (either by exact calculation using the exponential matrix or by using the Euler approximation) of the continuous-time system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \epsilon\boldsymbol{A}_{11}\boldsymbol{x}(\tau) + \epsilon\boldsymbol{A}_{12}z(\tau) + \epsilon\boldsymbol{B}_1\boldsymbol{u}(\tau)$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \boldsymbol{A}_{21}\boldsymbol{x}(\tau) + \boldsymbol{A}_{22}z(\tau) + \boldsymbol{B}_2\boldsymbol{u}(\tau) \tag{12}$$

which itself is obtained from the continuous-time system given by Eq. (4) using the stretching transformation $\tau = t/\epsilon$. It is usually said that the singularly perturbed continuous-time systems shown in Eqs. (4) and (12) are the slow time scale, t, and fast time scale, τ, versions, respectively. Also, note that the slow sampling model given by Eq. (11) is the same as the state space C model.

See a recent result in Refs. 131, 155, 156 for stability bounds on the singular perturbation parameter.

III. Singular Perturbation Techniques

A. Basic Theorems

Consider the nonlinear initial value problem given by Eq. (6). Also, to make the analysis simple, let us consider Eq. (6) without input function u as

$$\dot{\boldsymbol{x}}(t, \epsilon) = \boldsymbol{f}[\boldsymbol{x}(t, \epsilon), z(t, \epsilon), \epsilon, t], \qquad \boldsymbol{x}(t = 0) = \boldsymbol{x}_0$$

$$\epsilon\dot{z}(t, \epsilon) = \boldsymbol{g}[\boldsymbol{x}(t, \epsilon), z(t, \epsilon), \epsilon, t], \qquad z(t = 0) = z_0 \tag{13}$$

Here, we follow the seminal works of Tikhonov[374] and Vasiléva.[378] When the small parameter $\epsilon = 0$ is set in Eq. (13), the degenerate problem is given by

$$\dot{\boldsymbol{x}}^{(0)}(t) = \boldsymbol{f}\big[\boldsymbol{x}^{(0)}(t), z^{(0)}(t), 0, t\big] \tag{14}$$

$$0 = \boldsymbol{g}\big[\boldsymbol{x}^{(0)}(t), z^{(0)}(t), 0, t\big] \tag{15}$$

Assuming that we are able to solve the algebraic Eq. (15), we have

$$z^{(0)}(t) = \boldsymbol{h}\big[\boldsymbol{x}^{(0)}(t), t\big] \tag{16}$$

When Eq. (16) is used in Eq. (14), the reduced-order problem becomes

$$\dot{\boldsymbol{x}}^{(0)}(t) = \boldsymbol{f}^0\big(\boldsymbol{x}^{(0)}(t), t\big), \qquad \boldsymbol{x}^{(0)}(t = 0) = \boldsymbol{x}_0 \tag{17}$$

From Eq. (16), it is evident that $z^{(0)}(0)$ is not in general equal to z_0. The two main features of singular perturbation theory are degeneration and asymptotic expansion.

Some important assumptions are given before we state the main results.[8,398]

Assumption 1: The functions $\boldsymbol{f}$ and $\boldsymbol{g}$ in Eq. (13) must depend on ϵ in a regular way.

Assumption 2: The root $z^{(0)}(t)$ of Eq. (16) is called an isolated root, if there exists an $\epsilon > 0$ such that Eq. (15) has no solution other than $\boldsymbol{h}[\boldsymbol{x}^{(0)}(t), t]$ for $|z^{(0)}(t) - \boldsymbol{h}(\boldsymbol{x}^{(0)}(t), t)| < \epsilon$.

Assumption 3: The solution $z^{(0)}(t)$ from Eq. (15) is an asymptotically stable equilibrium point of the boundary-layer equation

$$\frac{\mathrm{d}z(\tau)}{\mathrm{d}\tau} = \boldsymbol{g}\left[\boldsymbol{x}^{(0)}(t), z(\tau), 0, t\right] \tag{18}$$

as $\tau \to \infty$. This means that the Jacobian matrix $\boldsymbol{g}_z$ of Eq. (18) has all eigenvalues with negative real parts and that the boundary conditions are in the domain of influence of the equilibrium point.

In degeneration, our interest is to find the conditions under which the solution of the full problem given by Eq. (13) tends to the solution of the degenerate problem of Eq. (17). A theorem due to Tikhonov[5] concerning degeneration is given next (for details see Refs. 8, 257, and 398).

Theorem 1: The exact solutions $\boldsymbol{x}(t, \epsilon)$ and $z(t, \epsilon)$ of the full problem given by Eq. (13) are related to the solutions $\boldsymbol{x}^{(0)}(t)$ and $z^{(0)}(t)$ of the degenerate problem given by Eqs. (14) and (15) as

$$\lim_{\epsilon \to 0} [\boldsymbol{x}(t, \epsilon)] = \boldsymbol{x}^{(0)}(t), \qquad 0 \le t \le \mathcal{T}$$

$$\lim_{\epsilon \to 0} [z(t, \epsilon)] = z^{(0)}(t), \qquad 0 < t \le \mathcal{T} \tag{19}$$

under certain assumptions.[257,398] Here, $\mathcal{T}$ is any number such that $z^{(0)}(t) = \boldsymbol{h}[\boldsymbol{x}^{(0)}(t), t]$ is an isolated stable root of Eq. (15) for $0 \le t \le \mathcal{T}$. The convergence is uniform in $0 \le t \le \mathcal{T}$ for $\boldsymbol{x}(t, \epsilon)$, and in any interval $0 < t_1 \le t \le \mathcal{T}$ for $z(t, \epsilon)$, and the convergence of $z(t, \epsilon)$ will usually be nonuniform at $t = 0$.

The second feature in singular perturbation theory is the asymptotic expansion for the solutions. [Note that, in general, a function $f(\epsilon)$ has the asymptotic power series expansion,[290] if it can be expressed as

$$f(\epsilon) = \sum_{j=0}^{N} f_j \epsilon^j + \mathcal{O}\left(\epsilon^{N+1}\right); \qquad \epsilon \longrightarrow 0$$

where $\mathcal{O}$ is Landau order symbol.] The main result was given by Vasiléva,[378] Wasow,[398] and Naidu[257] in the form of the following theorem.

Theorem 2: There exist an $\epsilon_0 > 0$, with $0 \le \epsilon \le \epsilon_0$, and $R(t, \epsilon)$ and $S(t, \epsilon)$ having (regular) asymptotic expansions and uniformly bounded in the interval considered, such that

$$\boldsymbol{x}(t, \epsilon) = \sum_{i=0}^{j} \left[\boldsymbol{x}^{(i)}(t) + \bar{\boldsymbol{x}}^{(i)}(\tau) - \underline{\boldsymbol{x}}^{(i)}(\tau)\right]\epsilon^i + R(t, \epsilon)\epsilon^{j+1}$$

$$z(t, \epsilon) = \sum_{i=0}^{j} \left[z^{(i)}(t) + \bar{z}^{(i)}(\tau) - \underline{z}^{(i)}(\tau)\right]\epsilon^i + S(t, \epsilon)\epsilon^{j+1} \tag{20}$$

where $\tau = t/\epsilon$, $\boldsymbol{x}^{(i)}(t)$, and $z^{(i)}(t)$ are the outer or degenerate series solutions (so called because these solutions are valid outside the boundary layer), $\bar{\boldsymbol{x}}^{(i)}(\tau)$ and $\bar{z}^{(i)}(\tau)$ are the inner solutions (so called because these solutions are valid inside the boundary layer), and $\underline{\boldsymbol{x}}^{(i)}(\tau)$ and $\underline{z}^{(i)}(\tau)$ are the intermediate solutions (so called because of the common part of the outer and inner solutions).

The details of obtaining these various series solutions are given in Refs. 257 and 398. The inner and intermediate series solutions are obtained from the stretched system

$$\frac{\mathrm{d}\boldsymbol{x}(\tau)}{\mathrm{d}\tau} = \epsilon \boldsymbol{f}[\boldsymbol{x}(\tau), z(\tau), \epsilon, \epsilon\tau], \qquad \frac{\mathrm{d}z(\tau)}{\mathrm{d}\tau} = \boldsymbol{g}[\boldsymbol{x}(\tau), z(\tau), \epsilon, \epsilon\tau] \tag{21}$$

obtained by using the stretching transformation $\tau = t/\epsilon$ in Eq. (13). Alternatively, the solution is obtained as

$$\boldsymbol{x}(t, \epsilon) = \boldsymbol{x}_0(t, \epsilon) + \boldsymbol{x}_c(\tau, \epsilon), \qquad z(t, \epsilon) = z_0(t, \epsilon) + z_c(\tau, \epsilon) \tag{22}$$

where $\boldsymbol{x}_0(t, \epsilon)$ and $z_0(t, \epsilon)$ are called outer solutions, and $\boldsymbol{x}_c(\tau, \epsilon) = \bar{\boldsymbol{x}}(\tau, \epsilon) - \underline{\boldsymbol{x}}(\tau, \epsilon)$ and $z_c(\tau, \epsilon) = \bar{z}(\tau, \epsilon) - \underline{z}(\tau, \epsilon)$ are often called boundary-layer corrections, which are obtained as series solutions from[257,289]

$$\frac{\mathrm{d}\boldsymbol{x}_c(\tau)}{\mathrm{d}\tau} = \epsilon \boldsymbol{f}[\boldsymbol{x}_0(\epsilon\tau, \epsilon) + \boldsymbol{x}_c(\tau, \epsilon), z_0(\epsilon\tau, \epsilon) + z_c(\tau, \epsilon), \epsilon, \epsilon\tau] - \epsilon \boldsymbol{f}[\boldsymbol{x}_0(\epsilon\tau, \epsilon), z_0(\epsilon\tau, \epsilon), \epsilon, \epsilon\tau]$$

$$\frac{\mathrm{d}z_c(\tau)}{\mathrm{d}\tau} = \boldsymbol{g}[\boldsymbol{x}_0(\epsilon\tau, \epsilon) + \boldsymbol{x}_c(\tau, \epsilon), z_0(\epsilon\tau, \epsilon) + z_c(\tau, \epsilon), \epsilon, \epsilon\tau] - \boldsymbol{g}[\boldsymbol{x}_0(\epsilon\tau, \epsilon), z_0(\epsilon\tau, \epsilon), \epsilon, \epsilon\tau] \tag{23}$$

In the case of a singularly perturbed linear, time-invariant system given by Eq. (4), the preceding two theorems imply that stability conditions require that

$$Re\{\lambda_i[\boldsymbol{A}_{22}]\} < 0, \qquad i = 1, \ldots, m \tag{24}$$

In other words, if the matrix $\boldsymbol{A}_{22}$ is stable, then the asymptotic expansions can be carried out to arbitrary order.[66,206]

In the case of a general boundary value problem, it is expected to have initial and final boundary layers, and, hence, the asymptotic solution is obtained as an outer solution in terms of the original independent variable t, initial layer correction in terms of an initial stretched variable $\tau = t/\epsilon$, and final layer correction in terms of a final stretched variable[290] $\sigma = (t_f - t)/\epsilon$.

Again, to illustrate the earlier given points, reconsider the simple initial value problem given by Eq. (7) with $\epsilon = 0.1$, $x(0) = 2$, and $z(0) = 3$. For the state variable $z(t, \epsilon)$, the various zeroth-order solutions shown in Fig. 2 are as follows.

1) The outer solution, $z^{(0)}(t)$ is obtained from Eq. (8).

2) The inner solution, $\bar{z}^{(0)}(\tau)$ is obtained as $\bar{z}^{(0)}(\tau) = -x(0) + [x(0) + z(0)]e^{-\tau}$ by solving

$$\frac{\mathrm{d}\bar{x}^{(0)}(\tau)}{\mathrm{d}\tau} = 0, \qquad \bar{x}^{(0)}(\tau = 0) = x(0)$$

$$\frac{\mathrm{d}\bar{z}^{(0)}(\tau)}{\mathrm{d}\tau} = -\bar{x}^{(0)}(\tau) - \bar{z}^{(0)}(\tau), \qquad \bar{z}^{(0)}(\tau = 0) = z(0) \tag{25}$$

3) The intermediate solution, $\underline{z}^{(0)}(\tau)$ is obtained as $\underline{z}^{(0)}(\tau) = -x(0)$ by solving

$$\frac{\mathrm{d}\underline{x}^{(0)}(\tau)}{\mathrm{d}\tau} = 0, \qquad \underline{x}^{(0)}(\tau = 0) = x^{(0)}(0) = x(0)$$

$$\frac{\mathrm{d}\underline{z}^{(0)}(\tau)}{\mathrm{d}\tau} = -\underline{x}^{(0)}(\tau) - \underline{z}^{(0)}(\tau), \qquad \underline{z}^{(0)}(\tau = 0) = z^{(0)}(0) \tag{26}$$

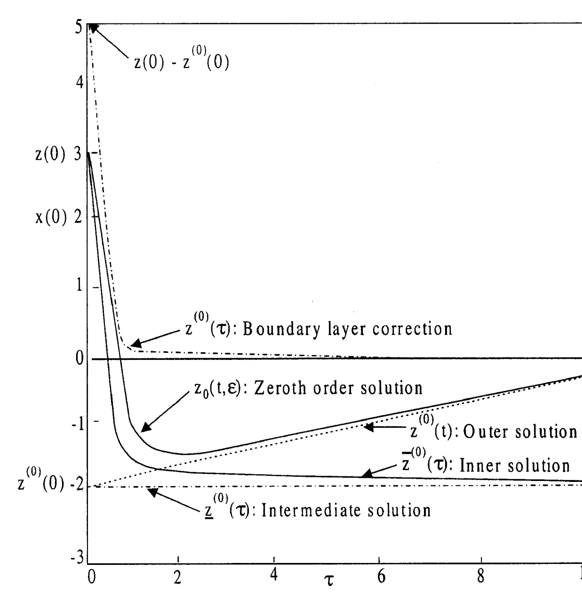

Fig. 2 Outer, inner, intermediate, and boundary-layer correction solutions.

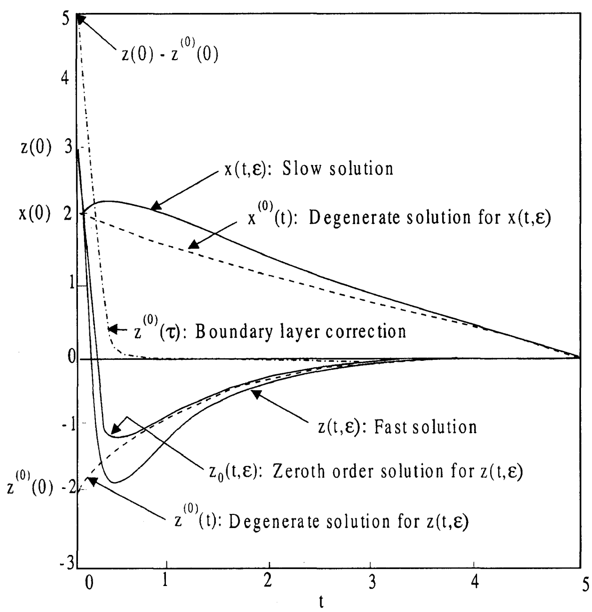

Fig. 3 Basic solutions of singular perturbation technique.

4) The boundary layer correction is $z_c^{(0)}(\tau) = \bar{z}^{(0)}(\tau) - \underline{z}^{(0)}(\tau) = [x(0) + z(0)]e^{-\tau}$.

5) The zeroth-order solution is $z_0(t, \epsilon) = z^{(0)}(t) + \bar{z}^{(0)}(\tau) - \underline{z}^{(0)}(\tau)$.

The various zeroth-order solutions for both $x(t, \epsilon)$ and $z(t, \epsilon)$ are shown in Fig. 3.

1) The boundary-layer corrections are $x_c(\tau, \epsilon)$ and $z_c(\tau, \epsilon)$.

2) The zeroth-order solutions are

$$x_0(t, \epsilon) = x^{(0)}(t) + x_c(\tau, \epsilon), \qquad z_0(t, \epsilon) = z^{(0)}(t) + z_c(\tau, \epsilon) \tag{27}$$

3) However, the zeroth-order boundary-layer correction $x_c(\tau, \epsilon)$ for the slow solution turns out to be zero in the present formulation.

B. Method of Matched Asymptotic Expansions

A method closely related to Vasiléva's called the method of MAE, has been used extensively in fluid mechanics.[90,102,115–117] Briefly, in this method for the singularly perturbed, initial value problem given by Eq. (13), the approximate composite solution is expressed in three parts, outer, inner, and common solutions. The outer solution is valid in the region outside the boundary layer, whereas the inner solution is valid inside the boundary layer. Because these two regions are bound to overlap, a matching process is required to identify the common solution. A composite solution, valid in the entire region, is constructed as the sum of the outer solution and inner solution from which we need to subtract the common solution.

Thus matching is accomplished by extending the outer solution into the inner region by transforming the outer variable t to that of the inner variable τ and taking the limit as $\epsilon \to 0$. This is called the inner limit of the outer solution or expansion. Similarly, the outer limit of the inner solution or expansion is obtained by extending the inner solution into the outer region by transforming the inner variable τ to that of the outer variable t and taking the limit as $\epsilon \to 0$. By equating the inner limit of outer expansion with the outer limit of inner expansion, we can determine the common solution.

Thus, the composite solution is composed of

$$\boldsymbol{x}_c = \boldsymbol{x}^o + \boldsymbol{x}^i - (\boldsymbol{x}^o)^i = \boldsymbol{x}^o + \boldsymbol{x}^i - (\boldsymbol{x}^i)^o \tag{28}$$

where $\boldsymbol{x}^o$ is the outer solution, $\boldsymbol{x}^i$ is the inner solution, $(\boldsymbol{x}^o)^i$ is the inner expansion of the outer solution, and $(\boldsymbol{x}^i)^o$ is the outer expansion of the inner solution. A critical step in the method of MAE is the evaluation of initial values for the outer solution. This is evaluated by using the matching principle, which is simply stated as

$$\text{inner expansion of outer solution, } (\boldsymbol{x}^o)^i = \text{outer expansion of inner solution, } (\boldsymbol{x}^i)^o \tag{29}$$

Similar expressions are easily written down for the fast variable z. It was further shown in Ref. 273 that the common solution $(\boldsymbol{x}^o)^i = (\boldsymbol{x}^i)^o$ is equivalent to the intermediate solution used in the singular perturbation method. Further details of the method of MAE are given in Refs. 184, 185, 260, 273, 383.

C. Time Scale Analysis

In general, the time scale system need not be in the singularly perturbed structure with a small parameter multiplying the highest derivative or some of the state variables of the state equation describing the system as given in Eq. (4) or (6). In other words, a singularly perturbed structure is only one form of two-time scale systems. In time scale analysis of a linear system, a block diagonalization transformation is used to decouple the original two time-scale system into two low-order slow and fast subsystems. Let us consider a general two-time scale, linear system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_{11}\boldsymbol{x}(t) + \boldsymbol{A}_{12}\boldsymbol{z}(t) + \boldsymbol{B}_1\boldsymbol{u}(t) \\ \dot{\boldsymbol{z}}(t) &= \boldsymbol{A}_{21}\boldsymbol{x}(t) + \boldsymbol{A}_{22}\boldsymbol{z}(t) + \boldsymbol{B}_2\boldsymbol{u}(t) \end{aligned} \tag{30}$$

possessing two widely separated groups of eigenvalues. Thus, we assume that n eigenvalues of the system given by Eq. (30) are small and that the remaining m eigenvalues are large, giving rise to slow and fast responses, respectively. We now use a two-stage linear transformation,[73,201]

$$\boldsymbol{x}_s(t) = (\boldsymbol{I}_s - \boldsymbol{ML})\boldsymbol{x}(t) - \boldsymbol{Mz}(t), \qquad \boldsymbol{z}_f(t) = \boldsymbol{Lx}(t) + \boldsymbol{I}_f\boldsymbol{z}(t) \tag{31}$$

to decouple the original system described by Eq. (30) into two slow and fast subsystems,

$$\dot{\boldsymbol{x}}_s(t) = \boldsymbol{A}_s\boldsymbol{x}_s(t) + \boldsymbol{B}_s\boldsymbol{u}(t), \qquad \dot{\boldsymbol{z}}_f(t) = \boldsymbol{A}_f\boldsymbol{z}_f(t) + \boldsymbol{B}_f\boldsymbol{u}(t) \tag{32}$$

where

$$\begin{aligned} \boldsymbol{A}_s &= \boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{L}, \qquad & \boldsymbol{A}_f &= \boldsymbol{A}_{22} + \boldsymbol{LA}_{12} \\ \boldsymbol{B}_s &= \boldsymbol{B}_1 - \boldsymbol{MLB}_1 - \boldsymbol{MB}, \qquad & \boldsymbol{B}_f &= \boldsymbol{B}_2 + \boldsymbol{LB}_1 \end{aligned} \tag{33}$$

$\boldsymbol{L}$ and $\boldsymbol{M}$ are the solutions of the nonlinear algebraic Riccati-type equations

$$\begin{aligned} \boldsymbol{LA}_{11} + \boldsymbol{A}_{12} - \boldsymbol{LA}_{12}\boldsymbol{L} - \boldsymbol{A}_{22}\boldsymbol{L} &= 0 \\ \boldsymbol{A}_{11}\boldsymbol{M} - \boldsymbol{A}_{12}\boldsymbol{LM} - \boldsymbol{MA}_{22} - \boldsymbol{MLA}_{12} + \boldsymbol{A}_{12} &= 0 \end{aligned} \tag{34}$$

and $\boldsymbol{I}$ is unity matrix.

Similar analysis exists for two-time scale discrete-time systems.[235,257,278]

D. Open-Loop Optimal Control

From the guidance and control point of view, we focus on the SPaTS in optimal control systems and the related area of differential games. The need for order reduction associated with singular perturbation methodology is most acutely felt in optimal control design that demands the solution of state and costate equations with initial and final conditions. For the singularly perturbed continuous-time, nonlinear system given by Eq. (6), the performance index in a simplified form is usually taken as

$$J = S[\boldsymbol{x}(t_f), \boldsymbol{z}(t_f), t_f, \epsilon)] + \int_0^{t_f} V[\boldsymbol{x}(t, \epsilon), \boldsymbol{z}(t, \epsilon), \boldsymbol{u}(t, \epsilon), t, \epsilon]\, \mathrm{d}t \tag{35}$$

When the well-known theory of optimal control[55] is used, the optimization of the performance index given by Eq. (35), subject to the plant equation given by Eq. (6) and the boundary conditions [with fixed initial conditions $\boldsymbol{x}(t=0) = x_0, z(t=0) = z_0$ and free final conditions $\boldsymbol{x}(t_f) = x_f, z(t_f) = z_f$], leads us to (for unconstrained control)

$$0 = \frac{\partial \mathcal{H}}{\partial \boldsymbol{u}} = -\mathcal{H}_u$$

$$\frac{d\boldsymbol{\lambda}_x}{dt} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} = -\mathcal{H}_x, \qquad \boldsymbol{\lambda}_x(t_f) = \left.\frac{\partial S}{\partial \boldsymbol{x}}\right|_{t_f}$$

$$\epsilon\frac{d\boldsymbol{\lambda}_z}{dt} = -\frac{\partial \mathcal{H}}{\partial z} = -\mathcal{H}_z, \qquad \epsilon\boldsymbol{\lambda}_z(t_f) = \left.\frac{\partial S}{\partial z}\right|_{t_f} \tag{36}$$

where $\boldsymbol{\lambda}_x$ and $\epsilon\boldsymbol{\lambda}_z$ are the costates or adjoints corresponding to the states $\boldsymbol{x}(t)$ and $z(t)$, respectively, and $\mathcal{H}$ is the Hamiltonian given by

$$\mathcal{H} = V[\boldsymbol{x}(t), z(t), \boldsymbol{u}(t), t, \epsilon] + \boldsymbol{\lambda}_x'\boldsymbol{f}[\boldsymbol{x}(t), z(t), \boldsymbol{u}(t), t, \epsilon] + \boldsymbol{\lambda}_z'\boldsymbol{g}[\boldsymbol{x}(t), z(t), \boldsymbol{u}(t), t, \epsilon] \tag{37}$$

where the prime denotes transpose. For constrained control, $\boldsymbol{u}(t) = \arg\min_{\boldsymbol{u}\in U} \mathcal{H}$, where $\boldsymbol{U}$ is a set of admissible controls. The state and costate systems given by Eqs. (6) and (36), respectively, constitute a singularly perturbed, two-point boundary value problem (TPBVP) as[119,120]

$$\frac{d\boldsymbol{x}(t,\epsilon)}{dt} = \boldsymbol{f}[\boldsymbol{x}(t,\epsilon), z(t,\epsilon), \boldsymbol{u}(t,\epsilon), \epsilon, t]$$

$$\frac{d\boldsymbol{\lambda}_x(t,\epsilon)}{dt} = -\mathcal{H}_x[\boldsymbol{x}(t,\epsilon), z(t,\epsilon), \boldsymbol{\lambda}_x(t,\epsilon), \boldsymbol{\lambda}_z(t,\epsilon), \boldsymbol{u}(t,\epsilon), \epsilon, t]$$

$$\epsilon\frac{dz(t,\epsilon)}{dt} = \boldsymbol{g}[\boldsymbol{x}(t,\epsilon), z(t,\epsilon), \boldsymbol{u}(t,\epsilon), \epsilon, t]$$

$$\epsilon\frac{d\boldsymbol{\lambda}_z(t,\epsilon)}{dt} = -\mathcal{H}_z[\boldsymbol{x}(t,\epsilon), z(t,\epsilon), \boldsymbol{\lambda}_x(t,\epsilon), \boldsymbol{\lambda}_z(t,\epsilon), \boldsymbol{u}(t,\epsilon), \epsilon, t]$$

$$0 = \mathcal{H}_u[\boldsymbol{x}(t,\epsilon), z(t,\epsilon), \boldsymbol{\lambda}_x(t,\epsilon), \boldsymbol{\lambda}_z(t,\epsilon), \boldsymbol{u}(t,\epsilon), \epsilon, t] \tag{38}$$

with boundary conditions $\boldsymbol{x}_0$, z_0, $\boldsymbol{\lambda}_x(t_f)$, and $\boldsymbol{\lambda}_z(t_f)$. Using the boundary-layer method, the solution to the preceding full problem is obtained as

$$\boldsymbol{x}(t,\epsilon) = \boldsymbol{x}_o(t,\epsilon) + \boldsymbol{x}_i(\tau_i,\epsilon) + \boldsymbol{x}_f(\tau_f,\epsilon)$$
$$\cdots$$
$$\boldsymbol{u}(t,\epsilon) = \boldsymbol{u}_o(t,\epsilon) + \boldsymbol{u}_i(\tau_i,\epsilon) + \boldsymbol{u}_f(\tau_f,\epsilon) \tag{39}$$

where $\boldsymbol{x}_o(t,\epsilon)$, $\boldsymbol{x}_i(\tau_i,\epsilon)$, and $\boldsymbol{x}_f(\tau_f,\epsilon)$ are outer, initial boundary-layer correction and final boundary-layer correction solutions, respectively, having asymptotic expansions in power of ϵ, and $\tau_i = t/\epsilon$ and $\tau_f = (t_f - t)/\epsilon$ are initial and final stretching transformations, respectively. Further details are found in Refs. 119 and 120.

Note that the final boundary-layer system needs to be asymptotically stable in backward time, that is, inherently unstable in forward time. This situation can create difficulties in trying to satisfy the given boundary conditions, and Kelley[181] and Cliff et al.[89] suggested a proper selection of boundary conditions to suppress any unstable component of the boundary-layer solution.

The Mayer problem in which the performance index in Eq. (35) contains only the terminal cost function (see Ref. 54) is an important special case. The optimal control problem has been studied by many workers.[10,21,100,101,119,120,171,288,289,326,350,396]

Dynamic programming has also been used for singularly perturbed optimal control problems.[37,214] Similar results exist for singularly perturbed, discrete-time optimal control systems.[170,228,229,257,278,307]

E. Closed-Loop Optimal Control

The closed-loop optimal control problem has some very elegant results for linear systems leading to a matrix Riccati equation. For the singularly perturbed, linear continuous-time system given by Eq. (4), consider a quadratic performance index[257]

$$J = \frac{1}{2}\boldsymbol{y}'(t_f)S\boldsymbol{y}(t_f) + \frac{1}{2}\int_0^{t_f} [\boldsymbol{y}'(t)\boldsymbol{Q}\boldsymbol{y}(t) + \boldsymbol{u}'(t)\boldsymbol{R}\boldsymbol{u}(t)]\, dt \tag{40}$$

where $\boldsymbol{y} = [\boldsymbol{x}, \epsilon z]'$, $\boldsymbol{S}$ and $\boldsymbol{Q}$ are positive semidefinite $(n+m)\times(n+m)$-dimensional matrices, and $\boldsymbol{R}$ is a positive definite $r\times r$ matrix. We arrive at the closed-loop optimal control as

$$\boldsymbol{u}(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}'\boldsymbol{P}\boldsymbol{y}(t) \tag{41}$$

where $\boldsymbol{P}$ is an $(n+m)\times(n+m)$-dimensional, positive-definite, symmetric matrix satisfying the singularly perturbed matrix Riccati differential equation

$$\dot{\boldsymbol{P}}(t) = -\boldsymbol{P}(t)\boldsymbol{A} - \boldsymbol{A}'\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}'\boldsymbol{P}(t) - \boldsymbol{Q}, \qquad \boldsymbol{P}(t_f) = \boldsymbol{S} \tag{42}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \dfrac{\boldsymbol{A}_{21}}{\epsilon} & \dfrac{\boldsymbol{A}_{22}}{\epsilon} \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 \\ \dfrac{\boldsymbol{B}_2}{\epsilon} \end{bmatrix} \tag{43}$$

$$\boldsymbol{P}(t) = \begin{bmatrix} \boldsymbol{P}_{11}(t) & \epsilon\boldsymbol{P}_{12}(t) \\ \epsilon\boldsymbol{P}_{12}'(t) & \epsilon\boldsymbol{P}_{22}(t) \end{bmatrix} \tag{44}$$

Note that the matrix $\boldsymbol{P}$ in the preceding Riccati equation is dependent on the small parameter ϵ and is in the singularly perturbed form. Assuming series expansions for $\boldsymbol{P}$, we can get their asymptotic series solutions. There are several studies on closed-loop optimal control of singularly perturbed continuous-time systems.[4,70,101,170,202,203,230,255,396] The linear quadratic regulator problem with three-time scale behavior has been investigated in Ref. 329. Related work is the closed-loop optimal control of linear and nonlinear systems decomposed into slow and fast subsystems.[121,134,144,171,174,286,287,289,317,371]

A two-time scale approximation to the linear quadratic optimal output regulator problem was examined in Refs. 253–255. Also, see Ref. 1 for optimal control of bilinear systems. Time-optimal control of singularly perturbed systems was studied in Refs. 15, 16, 101, 140, 141, and 336.

Near-optimal control of a class of singularly perturbed systems nonlinear in fast states and linear in slow states was investigated by Sannuti,[326] Chow and Kokotovic,[84] Kokotovic and Chow,[204] Saberi and Khalil,[316] Kokotovic and Khalil,[205] Kokotovic et al.,[206] and Khalil and Hu.[192]

Another class of problems occurring in optimal control, where the condition on the Hamiltonian $\mathcal{H}_{uu}$ becomes singular, is called singular optimal control problems.[54] If $\mathcal{H}_{uu} = 0$ during a finite interval of time, the corresponding trajectories are called singular arcs. Such problems were treated in Refs. 9 and 66. Some singular problems can be treated as limiting cases of cheap control problems as in Refs. 291 and 292. A related problem arises when the full problem contains a state-constrained arc, which was treated in Ref. 68.

Similar results exist for closed-loop optimal control of singularly perturbed, discrete-time linear systems, leading to matrix Riccati difference equation (see Refs. 169, 170, 199, 228, 229, 257, 277, 278, 294, and 302). Time scale analysis of optimal regulator problems in discrete-time control systems was considered in Refs. 171, 172, 270, 271, and 294.

F. Differential Games

Another type of problem that often arise in aerospace systems is differential games. In the design of multi-input control problems, the objective in the optimal policy may be met by formulating the control problem as a differential game. In a general differential game, there are several players, each trying to minimize their individual cost functional. Each player controls a different set of inputs to a single system. The strategies usually considered are Pareto optimal, Nash equilibrium, or Stackelberg (see Refs. 128 and 196). In the case of two-player Nash game, we have

$$\dot{x}(t,\epsilon) = f[x(t,\epsilon), z(t,\epsilon), u_1(t), u_2(t), t], \qquad x(t=0) = x_0$$
$$\epsilon\dot{z}(t,\epsilon) = g[x(t,\epsilon), z(t,\epsilon), u_1(t), u_2(t), t], \qquad z(t=0) = z_0 \tag{45}$$

and the performance index

$$J = \int_0^{t_f} V_i[x(t,\epsilon), z(t,\epsilon), u_1(t), u_2(t), t]\, dt, \qquad i = 1,2 \tag{46}$$

The main question investigated has been one of well posedness whereby the limit of performance using the exact strategies as $\epsilon \to 0$ is compared to the performance using simplified strategies with $\epsilon = 0$. The problem is said to be well posed if the two limits are equal and the singularly perturbed zero-sum games are always well posed. Also, note that the structure of a well-posed singularly perturbed (two-player) Nash game is composed of a reduced-order Nash game and two independent optimal control problems of the players (see Ref. 320). These and other aspects of differential games have been studied in Refs. 124, 187, 188, 194–196, 211, 319, 321–323, and 324.

Application of SPaTS to other control topics (only typical publications are referenced) such as observers,[293] high-gain feedback,[407] multimodeling,[97,132,193] stochastic control,[191] averaging and differential inclusion,[133] invariant manifolds,[210,361] H_∞ control,[30,190,373] robust control,[85,386] sliding mode control,[3] parameter identification,[74] uncertain systems,[41,96] distributed parameter systems,[22,32] and adaptive control[161,312] are not discussed in detail here.

IV. Applications of SPaTS to Aerospace Guidance and Control

Singular perturbation and time scale problems arise in a natural way in many fields of applied mathematics, engineering, and biological sciences such as fluid dynamics, electrical and electronic circuits and systems, electrical power systems, aerospace systems, nuclear reactors, biology, and ecology.[106,384] In this section, we present various problems in aerospace systems that possess SPaTS character.[66] A brief historical development of SPaTS follows first.

A. Brief History of SPaTS in Aerospace Systems

An excellent account of the "historical development of techniques for flight path optimization of high performance aircraft" is found in a NASA report by Mehra et al.[247] The report starts with the work of Kaiser[166] in 1944 on the vertical-plane minimum-time problem and reviews other works due to Miele[251] (1950), Kelley[179] (1959), and so on. In the horizontal-plane, minimum-time problem, the report reviews the works of Connor[92] (1967), Bryson and Lele[52] (1969), and others. In the three-dimensional, minimum-time problem, important contributions were made by Kelley and Edelbaum[183] (1970), Hedrick and Bryson[142] (1971), and others.

Singular perturbation analysis in flight mechanics is intimately connected with the concept of energy-state approximation, first introduced by Kaiser[166] in 1944. Kaiser introduced the notion of resultant height, which is today called energy height or specific energy, as the sum of an aircraft's potential and kinetic energy per unit weight.

An excellent account of the connection of Kaiser's[166] early work and that of singular perturbation analysis of aircraft energy climbs can be found in Ref. 250. The use of energy-state approximation in both two- and three-dimensional optimal trajectory analysis persisted until the late 1960s. Excellent examples of such analyses can be found in the work of Rutowski[315] in 1954 and later by Bryson et al.[51] and Hedrick and Bryson[143] in the late 1960s and early 1970s.

Specific investigation on the application of the theory of SPaTS to aerospace systems began in the early 1970s by Kelley[176,181] and Kelley and Edelbaum.[183] Kelley in particular was the first to suggest the use of an artificial small parameter to provide a singular perturbation structure. This analysis was later called forced singular perturbation analysis.[352] However, we note the article by Ashley,[23] who first suggested the use of multiple time scales in vehicle dynamic analysis.

According to Mehra et al.,[247] Kelley and his associates,[176,181,183] in the early 1970s, were the first to apply the theory of singular perturbations to aircraft trajectory optimization problems. In the first paper, Kelley and Edelbaum[183] addressed three-dimensional maneuvers, both energy climbs and energy turns. Subsequently, some general theoretical problems for a two-state system[176] and horizontal plane control[175] of a rocket were studied. Other problems considered by Kelley were energy state models with turns[177] and three-dimensional maneuvers with variable mass.[178,181] In Ref. 181, Kelley gave a detailed account of singular perturbations in aircraft optimization.

Ardema[7] applied the method of MAE to the vertical plane minimum time-to-climb problem and further gave an excellent general treatment of aircraft problems via singular perturbations.[8] Breakwell[45,46] considered the vertical plane minimum-time problem where drag D is much less than lift L, thus defining a natural singular perturbation parameter $\epsilon = D/L$.

The works so far applied the theory of SPaTS to obtain open-loop optimal controls. Calise, in a series of papers, focused on complete time scale separation and obtained closed-loop (feedback) controls. In particular, Calise considered the vertical plane minimum-time problem in Refs. 60 and 61. An excellent study devoted entirely to the application of singular perturbation theory to a variety of aerospace problems with special emphasis on real-time computation of nonlinear feedback controls for optimal three-dimensional aircraft maneuvers is given by Mehra et al.[247]

Thereafter, there was a steady interest in this area of the application of SPaTS to aerospace problems. Among others are Ardema,[11] Ardema and Rajan,[17] Calise,[66] Kelley et al.,[182] Naidu and Price,[272] and Shinar and Farber.[352]

Problems in flight mechanics are by their very nature nonlinear, particularly in formulations that are appropriate for aircraft performance analysis and development of guidance and control strategies. The nonlinear equations of motion are further complicated by the presence of aerodynamic and propulsive forces that are dependent on flight conditions, often given in the form of tabular data. Consequently, from the very beginning, simplified analysis models based on quasi-steady approximations were employed in studies of aircraft performance analysis and design. These approximations were invariably introduced to achieve an order reduction and, thus, simplification in the equations of motion, permitting an approximate analysis of an otherwise complicated optimization problem, thus leading naturally to an interest in singular perturbation methods in flight dynamics. These methods of approximation were essential before the advent of high-speed digital computation and the present-day availability of powerful numerical optimization algorithms based on either the calculus of variations or nonlinear programming. However, the development of simplified models, order reduction, and perturbation methods of analysis continue to play an important role mainly because these methods lead to the development of near-optimal, closed-loop (often simple) solutions, which enhance our physical insight into the problem, and in most cases these solutions are useful for onboard implementation.

The importance of singular perturbations in flight mechanics is that it represents a mathematical realization of the intuitive approach to simplified models obtained via order reduction. More important, the theory of SPaTS provides a mechanism for correcting the solutions for the neglected dynamics that is essential to the development of guidance and control strategies for many aerospace systems. For example, a slow phugoid mode and a fast short-period mode are well-known time-scale characteristics of the longitudinal motion of an airplane.

In many aerospace problems, no singular perturbation parameter appears explicitly on physical considerations. In such cases, a parameter may be artificially inserted to suppress the variables in the equations that are expected to have relatively negligible effects. For example, in a flight dynamics problem for a crewed vehicle, a complete set of equations of motion would consist of the coupled system of the six equations of rigid-body motion of the vehicle as a whole, the equations describing the dynamics of the control systems, the pilot's arm and foot, etc. It is obvious that many of these effects can be neglected if, for example, the vehicle trajectory is the only thing of interest. In particular, in the minimum time-to-climb (MTC) problem, it has been found in practical problems for supersonic aircraft that the flight-path angle is capable of relatively rapid change as compared with the altitude, which, in turn, is fast compared to specific energy. It is this separation of the speed of the variables that motivates the formulation of singular perturbation problems by the artificial (forced) insertion of the singular perturbation parameter. This is often referred to as the forced singular perturbation technique. A good account of the applications of SPaTS theory to a variety of aerospace problems, such as piloting a missile by controlling the transverse acceleration while keeping a constant role angle and a pursuit problem, are described by Fossard et al.[114]

B. Selection of Time Scales

Singular perturbation and, hence, the (slow–fast) time scale character is often associated with a small parameter multiplying the highest derivative of the differential equation or multiplying some of the state variables of the state equations describing a physical system. However, often the small parameter does not appear in the desired form, or the small parameter may not be identifiable at all. Only by physical insight and past experience does one know that a particular system has slow and fast modes. For cases where it is not possible to identify the small parameter ϵ, one can artificially introduce the small parameter ϵ to be associated with the fast dynamics. Thus, the selection of time scales is an important aspect of the theory of SPaTS[17,18,69,247] and can be categorized into three approaches: 1) direct identification of small parameters such as small time constants, moments of inertia, high Reynolds number, and so on; 2) transformation of state equations; and 3) linearization of the state equations. Although it is possible to identify the small parameters in some simple cases,[45,63] it is quite tedious in the case of more complex problems of interest. Kelley[180,181] considered transformations of state variables for nonlinear systems that reduce dynamic coupling and expose time scale separation, but obtaining the transformations requires solving partial differential equations. In the third approach, the standard linearization of a nonlinear system around an operating point is performed, and the eigenvalues of the linearized dynamics are examined for time scale separation.[370]

Ardema and Rajan[17,18] proposed "a rational method of identifying time-scale separation that is based on the concept of the speed of state variables and requires the knowledge only of the state equation.[17]" They chose the F-4C aircraft to illustrate their method. Furthermore, it is noted that in the case of supersonic aircraft, the state variables altitude h and velocity V are approximately of the same speed and are, therefore, not time scale separable for singular perturbation analysis. Consider the three-dimensional dynamics of an aircraft center of mass,

$$\dot{x} = V\cos\gamma\cos\psi, \qquad \dot{y} = V\cos\gamma\sin\psi$$
$$\dot{h} = V\sin\gamma, \qquad \dot{V} = g(T - D - \sin\gamma)$$
$$\dot{\psi} = gL\sin\sigma/V\cos\gamma, \qquad \dot{\gamma} = g(L\cos\sigma - \cos\gamma)/V \tag{47}$$

where x and y are the horizontal position coordinates, ψ is the heading angle, γ is the flight-path angle, and σ is the bank angle. T, D, and L are the thrust, drag, and lift per unit weight, respectively, given as

$$T = \beta T_M(h, V), \qquad D = D_0(h, V) + D_I(h, V)L^2$$
$$L_m \le L \le L_M = \min[n, \hat{L}(h, V)]$$
$$\hat{L}(h, V) = C_{L_\alpha}(h, V)\alpha_M\rho(h)V^2/2W \tag{48}$$

where the control variables are the throttle angle β, the bank angle γ, and the lift per unit weight L. The preceding relations are obtained under the normal assumptions[8,388] of 1) flat Earth, 2) constant weight, 3) thrust being independent of angle of attack, and so on. Although one can assume[17] that x, y, and ψ are the slowest state variables and that γ is the fastest variable, a transformation of state variables[305] is needed to provide a better time scale separation for the intermediate variables. Thus, we transform h and V to a new set of variables E and f resulting in the state equations[18,182]

$$\dot{x} = V\cos\gamma\sin\psi, \qquad \dot{y} = V\cos\gamma\cos\psi$$
$$\dot{\psi} = gL\sin\mu/V\sin\gamma, \qquad \epsilon\dot{E} = P$$
$$\epsilon^2\dot{f} = f_h V\sin\gamma + f_V g(P - V\sin\gamma)/V$$
$$\epsilon^2\dot{\gamma} = g(L\cos\mu - \cos\gamma)/V \tag{49}$$

where $E = V^2/2 + gh$ is specific energy, P is specific excess power, and f is a variable that is constant along the reduced solution. For zoom climbs and dives that provide rapid variations in h and V by the interchange of kinetic and potential energy, the changes in E are relatively slow.

Another approach to the identification of time scales in dynamic systems proposed by Ardema[12] is called the computational singular perturbation (CSP) method, the development of which was based on analysis of complex chemical reactions.[222–225] The CSP method produces time scale information in the course of numerical computation. Also, see recent work by Rao and Mease[308,309] for further use of CSP in solving hypersensitive optimal control problems.

Recently, a systematic approach was developed for identifying the singular perturbation parameter via nondimensionalization of the problem variables arising in airbreathing vehicles with hypersonic cruise and orbital capabilities.[69] For example, the flight dynamics of the center of mass of an aircraft in flight in a vertical plane are governed by

$$\dot{E} = V(T - D)/m, \qquad \dot{\gamma} = L/mV - (g\cos\gamma)/V$$
$$\dot{h} = V\sin\gamma \tag{50}$$

In Refs. 7, 60, 64, and 183, based on experience, an artificial parameter ϵ (whose nominal value is equal to 1) is inserted to make γ and h fast variables:

$$\epsilon\dot{\gamma} = L/mV - (g\cos\gamma)/V, \qquad \epsilon\dot{h} = V\sin\gamma \tag{51}$$

On the other hand, in Ref. 69, the authors define a set $S \equiv \{t_0, E_0, h_0, V_0, T_0, D_0, L_0\}$ of nondimensional quantities as

$$\tau = t/t_0, \qquad E_n = E/E_0, \qquad h_n = h/h_0, \qquad V_n = V/V_0$$
$$T_n = T/T_0, \qquad D_n = D/D_0, \qquad L_n = L/L_0 \tag{52}$$

and imposing the following conditions $T_0 = D_0$, $T_0 t_0 V_0/E_0 m = 1$, and $L_0 h_0/mV_0^2 = 1$, it was possible to put Eq. (50) as

$$\frac{\mathrm{d}E_n}{\mathrm{d}\tau} = V_n(T_n - D_n), \qquad \epsilon\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = \frac{(L_n - \cos\gamma)}{V_n}$$
$$\epsilon\frac{\mathrm{d}h_n}{\mathrm{d}\tau} = V_n\sin\gamma \tag{53}$$

where now the singular perturbation parameter $\epsilon = h_0/V_0 t_0$. Thus, it is possible to identify a parameter ϵ naturally instead of introducing the same artificially.

Further consideration is given to choice of state variables suitable for singular perturbation analysis in Ref. 182 in connection with the MTC problem. See Ref. 360 for separation of time scales for the use of nonlinear dynamic inversion if the design of a flight control system for a supermaneuverable aircraft where angle of attack, sideslip angle, and bank angle are identified as the slow variables and the fast variables are the three angular rates: body-axis roll and pitch and yaw rates. Use of time scale separation technique for inverse simulation, where control inputs are to be determined for a prescribed flight maneuver, is found in Ref. 27 with an application to F-16 fighter aircraft and in Ref. 25 with an application to helicopter model. Other related material is found in Ref. 159.

Before proceeding, it is better to have some typical solutions of aircraft motion showing slow and fast solutions. In Ref. 26 for a typical aircraft, it was shown, for an F-16 fighter aircraft, that the slow variables are the inertial position and velocity components and that the fast variables are the Euler angles and the angular velocity components. One of the slow variables, the velocity V, and two of the fast variables, the angular velocity p and the Euler angle θ, are shown in Fig. 4.

C. Atmospheric Flight

The MTC problem is solved by Ardema[7] using the technique of MAE. In the MTC problem, we wish to minimize the final time t_f subject to the equations of motion [a rearranged form of Eq. (50)]

$$\dot{h} = V\sin\gamma, \qquad \dot{E} = V(F - D_L)$$
$$\dot{\gamma} = (1/V)(L - \cos\gamma) \tag{54}$$

and the boundary conditions $h(0) = h_0$, $h(t_f) = h_f$, $E(0) = E_0$, $E(t_f) = E_f$, $\gamma(0)$, and $\gamma(t_f)$ are either free or fixed. Here, h is altitude divided by the acceleration due to gravity at sea level g, v is velocity divided by acceleration due to gravity at sea level, L is the lift divided by weight, D_L is the drag due to lift divided by weight, and F is the thrust less zero lift drag, divided by weight. Experience

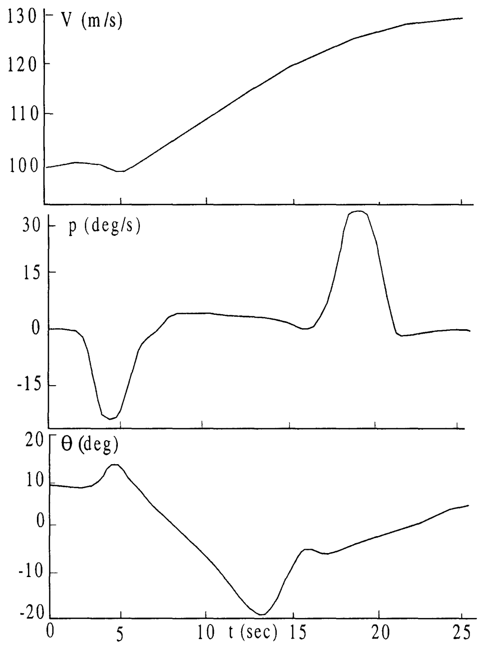

Fig. 4 Slow and fast solutions for an F-16 aircraft.[26]

"indicates that among the state variables, E is *slow* relative to h, and h is *slow* relative to γ. It is this separation of the *speed* of the variables that motivates the selection of E as a state variable instead of V."[8] Thus, a possible formulation for the singular perturbation structure is

$$\dot{E} = V(F - D_L), \qquad \epsilon\dot{h} = V\sin\gamma$$
$$\epsilon\dot{\gamma} = (1/V)(L - \cos\gamma) \tag{55}$$

The application of optimal control theory to the preceding singularly perturbed problem given by Eq. (55) leads to singularly perturbed TPBVP in terms of the state and costate variables. When the method of MAE is used, results are obtained in Ref. 8 up to first-order approximation. Furthermore, in Ref. 11, specific numerical algorithms of Picard, Newton, and averaging types are formally developed for solving the TPBVP arising in nonlinear singularly perturbed optimal control and compared with the computational requirements of the method of MAE. The (approximate) solutions for MTC problem by MAE are shown in Fig. 5 to illustrate the zeroth- and first-order solutions. In particular, note the improvement of the first-order solution over the zeroth-order solution for the flight-path angle γ.

Also, see Ref. 410 for obtaining a feedback solution by modifying the performance index and obtaining the same eigenvalues in the Hamiltonian matrix for the linearized problem obtained by Ardema.[8]

Next, consider three-dimensional flight-path optimization[59] with equations of motion as

$$\dot{V} = (T - D)/m - g\sin\gamma, \qquad \dot{h} = V\sin\gamma$$
$$\dot{\gamma} = (L\cos\sigma/m - g\cos\gamma)/V, \qquad \dot{\psi} = L\sin\sigma/mV\cos\gamma \tag{56}$$

A singularly perturbed structure for Eq. (56) is

$$\dot{E} = (T - D)V/mg, \qquad \dot{\psi} = L\sin\sigma/mV\cos\gamma$$
$$\epsilon\dot{h} = V\sin\gamma, \qquad \epsilon\dot{\gamma} = (L\cos\sigma/m - g\cos\gamma)/V \tag{57}$$

where the singular perturbation parameter ϵ, nominally equal to 1.0, is associated with the fast state variable $V = [2g(E - h)]^{1/2}$. Separation of the $\dot{E}$ and $\dot{\psi}$ dynamics in Eq. (57) is accomplished by introducing a second time scaling parameter. Then the dynamics becomes

$$\epsilon_1\dot{E} = (T - D)V/mg, \qquad \dot{\psi} = L\sin\sigma/mV\cos\gamma$$
$$\epsilon_2\dot{h} = V\sin\gamma, \qquad \epsilon_2\dot{\gamma} = (L\cos\sigma/m - g\cos\gamma)/V \tag{58}$$

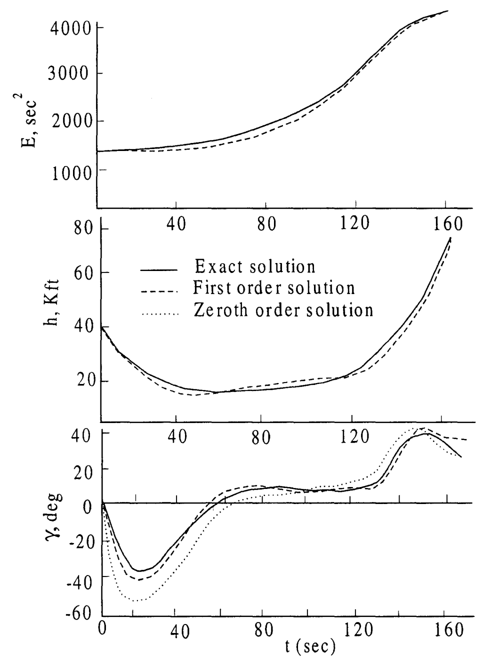

Fig. 5 Solutions for MTC problem.[8]

where ϵ_1 and ϵ_2 are taken such that

$$\lim_{\epsilon_1 \to 0,\, \epsilon_2 \to 0} \{\epsilon_2/\epsilon_1\} = 0 \tag{59}$$

Alternatively, in Eq. (58), we can take $\epsilon_1 = \epsilon$ and $\epsilon_2 = \epsilon^2$. For minimum-time flight, the optimization problem was solved with specific numerical results for an F-106 and an F-4E aircraft.

A different procedure was presented for the MAE to separately analyze state dynamics even when they vary on the same time scale. Several examples dealing with optimal aircraft flight that fall within this class of problems were discussed in Ref. 61. For optimal thrust magnitude control (TMC) and optimal lift control, the singular perturbation model considered for horizontal plane dynamics in Ref. 62 was

$$\dot{x} = V\cos\psi, \qquad \dot{y} = V\sin\psi$$
$$\epsilon\dot{\phi} = gL_n/WV, \qquad \epsilon^2\dot{V} = g(T - D)/W \tag{60}$$

The thrust T is constrained as $T_{\min} \leq T \leq T_{\max}$. The performance index to minimize the time-of-flight and fuel was chosen as

$$J = \int_{t_0}^{t_f} [ncT + (1 - n)]\,dt \tag{61}$$

where t_f is free, $n = 1$ corresponds to minimum fuel, and $n = 0$ corresponds to minimum time. The parameter c is the fuel flow per pound of thrust or a suitable scaling parameter.

Linearized models of longitudinal dynamics of airplanes are cast in the singularly perturbed form, and an output-feedback design method for linear, two-time scale systems was used to analyze specifically a stable (F8 aircraft) and an unstable (Boeing transport plane) airplane.[75] In this method, a fast compensator is designed first using the fast model; then a slow compensator is designed using a modified slow model.

An interesting problem dealing with synthesis of nonlinear flight-test trajectory controllers using results of prelinearizing transformation theory and singular perturbation theory was presented by Menon et al.[248] The equations of motion for aircraft flight are written in a compact form as

$$\dot{x} = A_1(x) + B_1(x) + C_1(x)u_1$$
$$\epsilon\dot{z} = A_2(x, u_1, z) + B_2(z) + C_2(x)u_2 \tag{62}$$

where $x = [V, \gamma, \beta, \theta, \phi, h, H]'$, $z = [p, q, r]'$, $u_1 = T$, $u_2 = [\delta_e, \delta_a, \delta_r]'$, β is angle of sideslip, θ is pitch attitude, ϕ is roll attitude, h is altitude, H is altitude rate, p is the total aerodynamic and thrust moment, q is pitch body rate, r is yaw body rate, T is engine thrust, δ_e is elevator deflection, δ_a is aileron deflection, δ_r is rudder deflection, and ϵ is the artificial small parameter forced into the system dynamics such that the system exhibits the singular perturbation (time scale) character. The controller synthesis was carried out for an operational, fixed-wing, high-performance fighter/military aircraft.

Also, viewing high-gain feedback systems as a class of singular perturbation problems, decoupling of linear multivariable systems were discussed in Ref. 311 with applications to fighter/military aircraft (an experimental vertical/standard takeoff and landing aircraft) performing a number of maneuvers.

Another interesting application of the singular perturbation method in time-controlled optimal flight trajectory involving a military aircraft was provided in Ref. 385. The analysis included the effects of risk from a threat environment. Assuming that the risk can be quantified in terms of risk index per unit time, the cost due to risk is minimized in the optimization process. Here, in considering the horizontal plane aircraft motion using lateral equations, the slow variables identified are downrange position and aircraft mass, whereas cross-track position, energy height, and heading angle are identified as fast variables.

In the analysis of onboard, real-time, near-optimal guidance for the climb–dash mission involving a high-performance aircraft,[400] some of the boundary-layer structure and hierarchical ideas of singular perturbations were used. Here, the singularly perturbed model used was

$$\epsilon^2 \dot{h} = V \sin\gamma, \qquad \epsilon^2 \dot{\gamma} = (L - W\cos\gamma)/mV$$
$$\epsilon \dot{E} = V(\eta T - D)/W, \qquad \dot{x} = V\cos\gamma \tag{63}$$

where η is the throttle coefficient.

The equations of motion for longitudinal dynamic stability and response of an aircraft to small disturbances in terms of short- and long-term periods were analyzed using singular perturbation theory in Ref. 404. By the use of the theory of decoupling of input–output maps of nonlinear systems, Singh[355] considered a scheme that gives rise to a singularly perturbed system describing the fast dynamics of the control vector for a nonlinear model of an aircraft.

An interesting application of SPaTS to atmospheric flight was proposed in Ref. 113 where "the objective was to optimize a direct operating cost over the whole trajectory, with a weighting for the price per minute of flight and the consumption."

Also, see Ref. 88 for a simple model that includes attitude dynamics in booster optimization and Ref. 87 for a simple model that includes thrust-vector control in aircraft optimization, where for certain boundary conditions there are two families of extremal solutions giving rise to a Darbout locus.

Robust control of a high-performance aircraft (model of the NASA high-angle-of-attack research vehicle) using feedback linearization coupled with structured singular value μ synthesis was studied by Reiner et al.,[310] where feedback linearization uses natural time scale separation between pitch rate and angle of attack.

In a typical singularly perturbed optimal control problem, the approximate solution consists of an outer solution, initial boundary-layer solution (correction) and terminal boundary-layer solution (correction). These solutions are of reduced order, and it is assumed that these are continuous functions of time for getting the asymptotic series solutions.[20] Many trajectory optimization problems, however, have discontinuous reduced-order solutions. Typical situations are the vertical-plane optimal climb problem when posed so that energy E is the single slow variable,[7,8] and altitude h and velocity V are modeled as slow variables.[45] For supersonic aircraft, the outer solution, that is, the energy climb path, is typically discontinuous in the transonic region.[305,399] These discontinuities, which occur at interior points, give rise to instantaneous jumps called interior transition layers and have the nature of boundary (initial and final) layers. To get uniformly valid approximate solutions, these interior transition layers are treated as the composition of two boundary layers, a forward layer and a backward layer, both beginning at the time, for example, $\bar{t}$, where the discontinuity occurs. In particular, the vertical-plane, point-mass, two-dimensional energy state model[20] that is in the singularly perturbed form and gives rise to interior transition layers is

$$\dot{E} = P, \qquad \epsilon\dot{h} = V\sin\gamma, \qquad \epsilon\dot{\gamma} = g(n - \cos\gamma)/V \tag{64}$$

where $P(E, H, n) = V(T - D)/W, V(E, h) = \sqrt{[2g(E - h)]}, n = L/W$, and $D(E, h, n) = A + Bn^2$.

The examples just given all suggest that altitude and flight path angle dynamics should be analyzed on the same time scale. This is due to the fact that complex eigenvalues appear in the Hamiltonian matrix associated with the necessary conditions for optimality associated with the boundary layer analysis of these dynamics.[7,8] However, when h and γ are analyzed on the same time scale, it is not possible to obtain a feedback solution. A highly accurate method for obtaining a feedback which permits the analysis of h and γ dynamics on separate time scales can be found in Ref. 410.

D. Pursuit–Evasion and Target Interception

Pursuit–evasion problems, having their origin in differential games, were first discussed thoroughly by Shinar[349] in all aspects of modeling and analysis by using (forced) singular perturbation technique (SPT). Subsequently, Shinar,[350] Mishe and Speyer,[352] and Shinar[351] solved a class of pursuit–evasion problems using forced SPT.

The interception problem in the horizontal plane is described by Breakwell et al.,[50]

$$\dot{R} = V_E\cos\psi - V_p\cos(\psi - \theta)$$
$$\dot{\psi} = -[V_e\sin\psi - V_p\sin(\psi - \theta)]/R, \qquad \dot{\theta} = (V_p/r_p)u \tag{65}$$

where V_E and V_p are the constant velocities of the evader (target) and the pursuer (interceptor), r_p is the minimum turn radius of the pursuer, R is the range, and $u(t) \leq 1$ is the normalized control function. In the preceding problem, if the initial range R_0 is much larger than r_p, then $\dot{\psi}$ varies much more slowly than $\dot{\theta}$, and hence, ψ is much slower than θ. This time scale separation is expressed mathematically by first defining a small parameter as $\epsilon = r_p/R_0$ and rewriting Eq. (65) as

$$\dot{R} = V_E\cos\psi - V_p\cos(\psi - \theta)$$
$$\dot{\psi} = -[V_e\sin\psi - V_p\sin(\psi - \theta)]/R, \qquad \epsilon\dot{\theta} = (V_p/r_p)u \tag{66}$$

which is now in the singularly perturbed structure. The performance index to be minimized here is

$$J = \int_{t_0}^{t_f} (1 + ku^2)\,\mathrm{d}t \tag{67}$$

where k is a constant. Here, the authors[50] developed a method to construct a feedback control law for this class of singularly perturbed nonlinear optimization problems[119] by assuming that the optimal cost functional J^* can be expanded in an asymptotic power series in the small parameter ϵ.

In Ref. 305, the problem of minimum-time interception of a target flying in three-dimensional space was analyzed using an energy-state approximation. A sixth-order model considered was approximated by assuming that there is a time scale separation between the faster (x, y, E, ψ) and the slower (h, γ) variables. The aircraft considered was an F4-C.

An approach based on a composite control as the sum of a reduced control and two boundary-layer controls was developed for the problem of steering the state of a nonlinear singularly perturbed system (whose fast dynamics are weakly nonlinear in the fast variables and control inputs) from a given initial state to a given final state, while minimizing a cost functional. The problem was that of planar pursuit in which a pursuer of constant speed attempts to intercept a constant speed target in a given direction.[192]

A singular perturbation method was used to develop computer algorithms for online control of the minimum-time intercept problem using an F-4C aircraft.[64] Furthermore, the optimization of aircraft altitude and flight-path angle dynamics in a form suitable for online computation and control was addressed in Ref. 11. In Ref. 19, one finds an algorithm for real-time, near-optimal, three-dimensional energy-state guidance for high-performance aircraft (the F-15 was used as an example) in pursiut–evasion and target-interception missions. The work in Ref. 395 obtained a neighboring optimal (minimum-time) guidance scheme for a long-range, air-to-air intercept, three-dimension problem. Here, the resulting sixth-order nonlinear differential equation was simplified to a fourth-order problem using the energy-state approximation, that is, by neglecting the speed (derivative terms) of the two fast variables: velocity and flight-path angle.

In another work in Ref. 342, a feedback control law was developed for three-dimensional minimum-time interception. The dynamics considered, which was slightly different from others, was

$$\dot{x} = V\cos\gamma\cos\psi - u_T, \qquad \dot{y} = V\cos\gamma\sin\psi - v_T$$
$$\dot{z} = -V\sin\gamma - w_T, \qquad \dot{E} = [(T-D)V]/W$$
$$\dot{\gamma} = (g/V)(n_c - \cos\gamma), \qquad \dot{\psi} = (g/V)(n_s/\cos\gamma) \quad (68)$$

Here x, y, and z are the components of vector $\boldsymbol{R} = \boldsymbol{R}_I - \boldsymbol{R}_T$; $\boldsymbol{R}_I$ and $\boldsymbol{R}_T$ are the interceptor's and target's position vectors, respectively, in an inertial frame; u_T, v_T, and w_T are the target velocity components; n, n_c, and n_s are the aerodynamic load factors, and $n\cos\sigma$ and $n\sin\sigma$. In this work, the relative position (x, y, z) and the specific energy E are considered as slow variables and the heading ψ and flight-path angle γ as fast variables for singular perturbation analysis.

E. Digital Flight Control Systems

The first applications of digital technology to flight control was a digital implementation of basic analog autopilot functions.[38,139] A digital control system uses a digital computer to implement its logic and the development of reliable, faster and inexpensive microcomputers made possible for many military and civilian aircraft to have digital control systems or digital-fly-by-wire systems.[53]

In Sec. II.C., singular perturbations in discrete-time systems were briefly described. In this section, we focus on the applications of SPaTS developed for digital (discrete-time) systems described by ordinary difference equations as opposed to those developed for continuous-time systems. The theory and applications of SPaTS in digital control systems is of relatively recent origin.[24,42,91,234,257,276,278,299,304] Some attempts have been made to apply the SPaTS technique to digital flight control systems, limited to a class of digital control of continuous systems.[44,367,368] In Ref. 270, a composite, discrete-time, feedback control was obtained in terms of the lower order slow and fast controls for a microcomputer-controlled aircraft flight control system[318,359] where the original fifth-order model has pitch angle, velocity, and altitude as slow variables and angle of attack and pitch rate as fast variables.

A good account of the applications of SPaTS to digital flight control systems may be found in Refs. 271 and 272. Also see Ref. 397 for near-optimal observer-based controller design for a twin-engine aircraft model.

F. Atmospheric Entry

Some of the earlier work using perturbation procedures for the atmospheric entry problem was done by Shen,[340,341] Shi,[343] and Shi and associates.[344–347] Basically, in these works, the equations of motion (for both planar and nonplanar entry) are obtained for a vehicle entering an atmosphere; the small parameter ϵ is identified in most of these cases as the ratio of the atmospheric scale height to the radius of the Earth and a perturbation method such as the method of MAE is used to obtain approximate solutions. Furthermore, Shi[343] use the method of MAE to solve the problem of optimal lift control of a hypersonic lifting body entering the atmosphere from the Keplerian region as well as from low altitudes. Separate expansions were introduced for the outer Keplerian region, where gravitational forces are dominant, and the inner atmospheric region, where the aerodynamic forces are dominant.

In a typical three-dimensional atmospheric entry problem, the equations of motion are given by[387,389] (assuming a nonrotating spherical Earth)

$$\dot{r} = V\sin\gamma, \qquad \dot{\theta} = (V\cos\gamma\cos\psi)/(r\cos\phi)$$
$$\dot{\phi} = (V\cos\gamma\sin\psi)/r, \qquad \dot{V} = F_T/m - g\sin\gamma$$
$$V\dot{\gamma} = (F_N/m)\cos\sigma - (g - V^2/r)\cos\gamma$$
$$V\dot{\psi} = F_N\sin\sigma/(m\cos\gamma) - (V^2/r)\cos\gamma\cos\psi\tan\phi \quad (69)$$

where $F_T = T\cos\alpha - D$, $F_N = T\sin\alpha + L$, α is the thrust angle of attack, F_T is the component of the combined aerodynamic and propulsive forces along the velocity vector, and F_N is its component orthogonal to the velocity in the lift–drag plane. In applying the method of MAE to the atmospheric entry problem, the small parameter is identified as $\epsilon = 1/\beta r$, where the constant βr, the reciprocal of the scale height, is large, for example, for Earth, $\beta r = 900$.

When optimization was introduced into the atmospheric entry problem using the method of MAE, Frostic and Vinh[123] used a dimensionless altitude as the independent variable, whereas other approaches were taken by using Chapman variables (see Ref. 58) and using radial distance r as the independent variable instead of the time t (Refs. 258 and 260).

The work of Willis et al.[401] contributed to the usefulness of the method of MAE as an analytical tool for problems in hypervelocity mechanics with "significantly different dynamic structures of entry trajectories into Mars and Titan as opposed to Earth and Venus or Jupiter and Saturn."

In a recent work, using singular perturbation theory, Sero-Guillaume et al.[337,338] solved an optimal control problem to find the thrust that must be applied to a vehicle during an extra-atmospheric flight such that the vehicle reaches a minimum time at given point on the surface of the Earth. The singular perturbation parameter was based on the small ratio of thrust time to the rotation time for the vehicle.

Also, see Ref. 393 for improved MAE solutions for evaluating the maximum deceleration during atmospheric entry of space vehicles. The improvement was obtained by extending the previous work beyond the first-order composite solutions by artificially extending the endpoint boundaries to strengthen the physical assumptions on the outer and inner expansions for the matching procedure.

G. Satellite and Interplanetary Trajectories

First to be discussed is the trajectories of satellites. In the study of asymptotic stability of steady spins of satellites,[137] a singular perturbation formulation was obtained for attitude maneuvers of a torque-free rigid gyrostat with a discrete damper. The model consisted of a rigid body with rigid axisymmetric rotors and a mass particle constrained to move along a line fixed in the rigid body and the small parameter was represented as the ratio of the particle mass and the system mass.

The limiting case of the restricted three-body (Earth, moon, and a particle of negligible mass) problem, in which the mass of one of the bodies (Earth) is much larger than the mass of the second body (moon), is analyzed for finite time intervals by perturbation methods in Refs. 220 and 221. However, the straightforward first-order perturbation solution is not uniformly valid because it has a logarithmic singularity at the position of the moon, and higher approximations are increasingly more singular in the region of nonuniformity. Hence, this three-body problem is of the singular perturbation type.[281] An interesting comparison analysis of this singularly perturbed three-body problem was done in Ref. 281 using three methods: method of MAE, the method of strained coordinates (Poincaré–Lighthill–Kuo method) (see Ref. 376), and the generalized method of treating singular perturbation problems,[280] where "the intermediate region is treated equally with the outer and inner regions" unlike the method of MAE. An interplanetary trajectory transfer is, in general, divided into a helioconcentric portion (where the gravitational attraction of the sun is greater than that of the planets) and two planet-centered portions.[49]

The early work by Lagerstrom and Kevorkian[220,221] focused on applying a patched conics idea to obtain an approximate solution to the planar restricted three-body problem by carrying out the asymptotic matching when the normalized initial angular momentum with respect to the larger body was very small (of the order of $\epsilon^{\frac{1}{2}}$, where ϵ is the mass ratio between the smaller and larger bodies). In particular, with ϵ as the ratio of the mass of Earth to the total mass of Earth and moon, for motions of a particle of negligible mass, which pass within a distance of order ϵ of the moon, the gravitational attraction of the moon is not uniformly small during the entire motion, and hence, singular perturbation methods were used. The orbit is decomposed into three parts: approach orbit to moon (outer solution), moon passage (boundary layer and inner solution), and the orbit after moon passage (outer solution).

Further results[47,298] concentrated on obtaining an approximate solution for all three-dimensional trajectories that reach the target planet with finite velocity to include interplanetary trajectories. Here, the perturbation series is expanded in powers of a small parameter $\lambda(=m_1/m_0 \ll 1)$, where, m_0 is the mass of the sun and m_1 is the mass of the planet. In particular, the authors[47,298] considered "fly-by (or swing-by) interplanetary trajectories," such as a trajectory from Earth to Mars via Venus.

The method of MAE was applied to the optimization of a minimum-fuel power-limited interplanetary trajectory in Refs. 48 and 49. The composite solution was obtained in terms of an outer solution, valid between planets, consisting of heliocentric portion of the trajectory perturbed by the planets, and an inner solution, valid in the vicinity of planets, consisting of a large number of revolutions, slightly perturbed by the sun, during which the small acceleration causes the trajectory to spiral away gradually from the planet, and matching of the outer and inner solutions.

Other related works study a special case of the restricted three-body problem by a perturbation technique that leads to an asymptotic representation of the solution valid for long times.[108] Here, the model consists of a primary body (planet) having a mass much smaller than the second body (sun) and a third body (satellite) of negligible mass taken very close to the planet.

Further, the influence of the sun on the motion of a spacecraft traveling from the Earth to the moon was found in Ref. 345 to be substantial, and the problem was formulated for noncoplanar Earth-to-Moon trajectories in the restricted four-body problem and solved by using the method of MAE. Here, the small parameter ϵ used in the expansion procedure was again taken as the ratio of the mass of moon to the mass of Earth.

H. Missiles

Calise[63] investigates the performance improvement due to the use of optimal TMC on a conventional missile that utilizes proportional navigation guidance. The missile state equations are

$$\dot{x} = V\cos\psi, \qquad \dot{y} = V\sin\psi - V_T$$
$$\epsilon\dot{\phi} = L_n/mV = N\dot{\theta}, \qquad \epsilon^2\dot{V} = (T-D)/m \tag{70}$$

where x is cross-range position, y is downrange position, ϕ is missile heading, V is missile velocity, T is thrust, D is drag, V_T is target velocity, L_n is the component of the missile lift $\boldsymbol{L}$ vector in the horizontal plane, N is the navigational gain, and m is missile mass. Here, the singular perturbation parameter ϵ, whose nominal value is 1, is introduced intentionally to extract the time scale character of the missile dynamics. Thus, the downrange and cross-range coordinates x and y are slow variables, ψ is a fast variable, and V is the fastest variable. The controls given by Eq. (70) are the lift L and thrust T. The singular perturbation parameter is nominally set to 1.0 so that the state dynamics are ordered on separate time scales in accordance with relative speeds. Using singular perturbation method, examples were presented for air-launched tactical missiles to show the effect of TMC on increasing the missile launch envelope and in reducing the track crossing angle at intercept.

Another formulation of the missile problem was considered by Chichka et al.[82] Here the dynamic system considered for optimal range–fuel–time trajectories for a scramjet missile is in the singularly perturbed form as

$$\epsilon^2\dot{h} = V\sin\gamma, \qquad \epsilon^2\dot{\gamma} = (g/V)[(L/W) - \cos\gamma]$$
$$\epsilon\dot{E} = [(T-D)/W]V, \qquad \dot{R} = V\cos\gamma, \qquad \dot{W}_F = Q \tag{71}$$

where W is the weight, W_F is the specific amount of fuel to be used, and Q is the fuel rate.

The application of singular perturbation techniques for missile guidance has been discussed by many workers.[64,77,82,301,366] In particular, Cheng et al.[78] studied the pulse ignition problem for a generic medium-range air-to-air missile from an optimal control point of view.

Another interesting application of time scale analysis to missile problems is given by Hepner and Geering,[147] who considered the time scale separation inherent in the tracking dynamics and developed a method that is a combination of tracking filter (based on extended Kalman filter) and guidance law. In particular, the tracking dynamics considered consists of a slow part of bearing rate, range rate, bearing angle, and range as slow variables and a fast part of bearing rate, target heading angle, and target heading angle rate as fast variables.

In developing near-optimal midcourse guidance laws for air-to-air missiles using singular perturbation methods,[249] four state variables are treated as slow and two state variables are treated as fast. Thus, the point-mass equations of motion for a missile flying over a flat, nonrotation Earth with a quiescent atmosphere are formulated as

$$\dot{E} = VT_D/mg, \qquad \dot{\phi} = gn_h/V\cos\gamma$$
$$\dot{x} = V\cos\gamma\cos\phi, \qquad \dot{y} = V\cos\gamma\sin\phi$$
$$\epsilon\dot{h} = V\sin\gamma, \qquad \epsilon\dot{\gamma} = (g/V)(n_v - \cos\gamma) \tag{72}$$

where n_h and n_v are the control variables representing horizontal and vertical components of the load factor, respectively. The performance index considered for the optimal guidance problem is

$$\min_{n_h,n_v}\left[-\zeta E(t_f) + (1-\zeta)\int_0^{t_f} dt\right] \tag{73}$$

where ζ is the weighting factor enabling the tradeoff between flight time and terminal energy.

In Ref. 366, based on the time constants of the missile dynamics, the intercept problem was divided into six parts: missile velocity (very slow), relative position (slow), missile flight path angle and heading angle (fast), and acceleration and its direction (very fast) to pave the way for singular perturbation analysis and to obtain optimal guidance laws.

Visser and Shinar,[396] using first-order correction terms, developed a new method based on the classical method of MAE to obtain uniformly valid feedback control laws for a class of singularly perturbed nonlinear optimal control problems frequently arising in aerospace applications. The new technique, based on the explicit solution of the integrals arising from the first-order matching conditions, was applied to a constant speed planar pursuit problem.

For a bank-to-turn, air-to-air missile, the closed-loop stability was examined in Ref. 335 with a dynamic inversion controller using two-time scale separation of inner-loop dynamics consisting of the fast variables roll, pitch, and yaw rates and the outer-loop dynamics consisting of the slow variables angle of attack, sideslip angle, and bank angle. The two nonlinear controllers based on gain-scheduled H_∞ design and nonlinear dynamic inversion design were presented in Ref. 334.

In Ref. 226, a new approach to acceleration control of skid-to-turn missiles was proposed that can handle effectively the nonminimum phase property as well as nonlinearities of the missile dynamics by incorporating the singular perturbation technique into the functional inversion technique. The singular perturbation parameter was associated with a design parameter in a linear controller.

I. Launch Vehicles and Hypersonic Flight

Space Shuttle

In one of the earliest and interesting works[306] on the application of singular perturbation theory to aerospace problems, the longitudinal dynamics of a space shuttle during entry into the Earth's atmosphere was investigated. Under the usual assumptions, the equations of motion are formulated as

$$\dot{V} = -\rho SC_D V^2/(2m) - g\sin\gamma$$
$$\dot{\gamma} = \rho SC_L V/2m - (g/V - V/r)\cos\gamma$$
$$\dot{q} = \rho SLC_m V/2I_y - (3g/2Vr)[(I_x - I_z)/I_y]\sin 2\theta$$
$$\dot{\theta} = q + (V/r)\cos\gamma, \qquad \dot{r} = V\sin\gamma \tag{74}$$

where $\theta = \gamma + \bar{\alpha}$; $\bar{\alpha}$ is the angle of attack; q is angular velocity in pitch relative to Earth; I_x, I_y, and I_z are principle moments of inertia; and θ is the pitch angle. The interesting feature of the analysis is that by the elimination of θ and V from the preceding equations, linearizing the aerodynamic coefficients, and changing the independent variable from t to κ, the preceding equations of motion are transformed into a second-order equation in a perturbation of the angle of attack ($\alpha = \bar{\alpha} - \alpha_0$) as

$$\ddot{\alpha} + \omega_1(\kappa)\dot{\alpha} + \omega_0(\kappa)\alpha = f(\kappa) \tag{75}$$

where the prime denotes differentiation with respect to κ and the coefficients ω_1 and ω_0 are functions of the various parameters in Eq. (74). Further, experience with entry trajectories of missile and space shuttle suggests that the coefficients of Eq. (75) can be realistically considered to be slowly varying compared to the time constant of the motion of the vehicle. Thus, the coefficients of Eq. (75) vary on a new slow variable $\bar{\kappa} = \epsilon\kappa$, where $= \epsilon$ is a small positive parameter; this allows Eq. (75) to be cast in the singularly perturbed form as

$$\epsilon^2\ddot{\alpha} + \epsilon\omega_1(\bar{\kappa})\dot{\alpha} + \omega_0(\bar{\kappa})\alpha = f(\bar{\kappa}) \tag{76}$$

In particular, the longitudinal dynamics of the space shuttle vehicle 049 about a prescribed optimal trajectory was discussed.

In Ref. 367, the effects of deterministic and stochastic parameter variations on the lateral directional stability of an aircraft, using space shuttle dynamic model, were studied. Here, a fourth-order linear model was used with the Dutch roll motions as slow variables and roll and spiral motions as fast variables. Also, see Ref. 353 for the use of multitimescale continuous sliding-mode control during the descent portion of a reusable launch vehicle.

Hypersonic Vehicles

This section is adapted from a recent status survey by the first author on guidance and control issues for hypersonic vehicles conducted at the U.S. Air Force Research Laboratory (see Refs. 262, 264–266, and 269). The U.S. Air Force has recognized hypersonics as one of the key technologies to be developed for the 21st century. Another study, by the Committee on Hypersonic Technology for Military Applications of the Air Force Studies Board,[158] concluded that hypersonic technology and ramjet/supersonic combustion ramjet propulsion offer potentially large increases in speed, altitude, and range with flexible recall, en route redirection, and return to base for military aircraft.

Singular perturbation techniques have been very effective in addressing problems associated with onboard trajectory optimization, propulsion system cycle selection, and the synthesis of guidance laws for ascent to low Earth orbit of an airbreathing, single-stage-to-orbit (SSTO) vehicle as given by Corban[93] and Corban et al.[94] The governing equations of flight in a vertical plane are

$$\dot{E} = V(F_C - D)/m, \qquad \dot{m} = -f(r, E, \pi, \alpha)$$

$$\epsilon\dot{\gamma} = (F_S + L)\cos\sigma/mV - \mu\cos\gamma/Vr^2 + V\cos\gamma/r$$

$$\epsilon\dot{r} = V\sin\gamma \tag{77}$$

where the specific energy $E = V^2/2 - \mu/r$ and mass m are found to be slow variables and the flight-path angle γ and radial distance r (or altitude) are considered as fast variables. It can be shown, however, that ϵ is a small parameter depending on physical constants of the system.[69,93,258] Also, it was recently rediscovered by Hermann and Schmidt[145] and Schmidt and Hermann[333] that the energy-state approach to the system dynamics during the scramjet-powered phase of the hypervelocity vehicle does exhibit a two- (or multi-)-time scale character, which was verified by actual simulation of the dynamics using a nonlinear programming routine and a multiple shooting algorithm.[39]

In Eq. (77), the control variables are angle of attack α, bank angle σ, fuel equivalence ratios ϕ_i for engine types 1–n and engine throttle settings η_j for engine types $n+1$–p. Using the performance index $J = -m(t_f)$ for maximum payload to orbit (or minimum-fuel consumption), an algorithm for generating fuel-optimal climb profiles was obtained using singular perturbation theory and the Pontryagin minimum principle. In addition, switching conditions, under appropriate assumptions, are derived for transition from one propulsion mode to another (turbojet, ramjet, scramjet, and rocket engine). The problem of state-variable inequality constraints was discussed by Calise and Corban[68] and Markopoulos and Calise,[238] where it was shown that the state constraint of the full problem is transformed into state and control constraints in the boundary-layer problem. Also, see a similar treatment by Ardema et al.[14] for using the theory of SPaTS to investigate the optimal throttle switching of airbreathing and rocket engine modes; it was found that the airbreathing engine is always at full throttle and that the rocket is on full at takeoff and at very high Mach numbers, but off otherwise.

An interesting problem for an aerospace plane (horizontal takeoff, SSTO vehicle) guidance was investigated by Van Buren and Mease[57] using the theories of singular perturbations and feedback linearization. Here, the minimum-fuel problem is formulated for the vehicle along the super- and hypersonic segments of the trajectory, and feedback guidance logic was obtained, and the effects of dynamic pressure, acceleration, and heating constraints are studied. Further, it was shown that by Kremer[212] and Kremer and Mease[213] that for the cases where the slow solution lies on the state constraint boundary, the constraint may be modeled in the initial boundary-layer solution using an appropriate penalty function (soft constraint). Also, see Ref. 369 for a four-dimensional guidance scheme for atmospheric vehicles using model predictive control, nonlinear inverse control and singular perturbation theory.

In a study of hypersonic flight trajectories under a class of path constraints, Lu[232] obtained explicit analytical solutions to flight-path angle and altitude using a natural singular perturbation parameter ϵ (inverse of atmospheric scale height). However, only outer solutions are obtained without any corrections to the boundary layer. Calise and Bae[67] used singular perturbation theory for obtaining optimal heading changes with minimum energy loss for a hypersonic gliding vehicle.

Other investigations by Ardema et al.[13] focused on using singular perturbation methods for examining the occurrence of instantaneous transitions in altitude and velocity in the energy-state formulation of optimal trajectories by modeling the transition as two boundary layers back-to-back, one in backward time and the other in forward time, and by matching the two boundary layers at the transition energy to obtain the location of the transition.

Also, see recent work by Kuo and Vinh[216] for an improved MAE method for a three-dimensional atmospheric entry trajectory by considering discrepancies between the exact solutions and uniformly valid first-order solutions and generating and solving the second-order solutions.

Feedback linearization is an elegant technique for control of a nonlinear system, in which a nonlinear coordinate transformation converts the original nonlinear system into an equivalent linear system.[162] This technique along with singular perturbation theory was effectively used for hypersonic vehicles by Corban et al.,[94] Van Buren and Mease,[57] and Mease and Van Buren.[245] In particular, the feedback linearization technique was used for the fast dynamics under certain conditions and a variable structure (sliding-mode) control obtained to drive the linear state to the origin by Mease and Van Buren.[245]

Also, in Ref. 243 matched asymptotic expansion solutions were developed for trajectories of a direct launch system projectiles during atmospheric ascent, where the small parameter was taken as the ratio of atmospheric scale height to the mean equatorial radius of the Earth.

An interesting application of SPaTS to supersonic transportation has been given in Refs. 402 and 403 for the first time.

J. Orbital Transfer

Here, we include both aeroassisted and nonaeroassisted orbital transfers. The problem of ascent or descent from an initial Keplerian orbit by a constant low-thrust force was examined by using a two-variable expansion procedure in Ref. 344. In particular, the planar motion of a satellite accelerated by low thrust in a central force field is governed by

$$\frac{d^2r}{dt^2} - r\left(\frac{dr}{dt}\right)^2 = -\frac{1}{r^2} + \epsilon\cos\alpha, \qquad \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = r\epsilon\sin\alpha \tag{78}$$

where r and θ are the polar coordinates, α is the angle between the thrust vector and the center of attraction, and the small dimensionless parameter ϵ is the ratio of the magnitude of thrust vector to the initial weight of the satellite at its initial distance.

In analyzing the problem of departure from a circular orbit by a small thrust in the circumferential direction, described by a third-order nonlinear differential equation, an approximate solution is obtained by neglecting the radial acceleration compared to the centrifugal acceleration.[375] In this approximation, the original third-order differential equation degenerates to a first-order differential equation, thus leading to a singular perturbation situation. The small parameter ϵ was chosen as the ratio of thrust to the initial gravitational force. In particular, the general case of thrust vector at a nonzero angle to the radius vector, instead of circumferential thrust only, was considered and the method of MAE was applied for obtaining higher-order solutions uniformly valid in the entire time interval.

Moss[256] was one of the first to use a perturbation method for orbit analysis. Here, an approximate solution to the problem of orbit expansion by constant circumferential low-thrust, including the case of constant acceleration, was presented using the two variable expansion technique. The two time scales considered are the normal time and a slow time characteristic of the gradual evolution of the orbit.

In using perturbation methods for problems arising in orbital transfer, Mease and McCreary[246] developed guidance laws for coplanar skip trajectories based on a method of matched asymptotic expansions. The small parameter was identified as the ratio of atmospheric scale height to a reference radius.

In Ref. 252, an approximate suboptimal feedback control law was developed by an asymptotic expansion about a zeroth-order solution obtained by assuming that inertial forces are negligible compared to the aerodynamic forces. The small parameter used in the expansion essentially represents the ratio of inertial forces to the atmospheric forces.

Anthony[6] developed approximate analytical solutions of the problem of transfer between coplanar circular orbits using very small tangential thrust, where the thrust acceleration is constant. For both ascending and descending motions, a two-variable expansion method, based on the work of Moss,[256] was developed. Here, the small parameter is proportional to the thrust acceleration, and the orbit eccentricity changes slowly with one time variable t_1 and oscillates in the other time variable t_2, where t_1 and t_2 are the two time variables used in describing the motion. The approximate analytical results obtained using the two-variable expansion method compare remarkably well with the numerical results obtained by integrating the actual equations using a fourth-order Runga–Kutta procedure.

With a typical aeroassisted orbital transfer vehicle (AOTV), the transfer from a high Earth orbit to a low Earth orbit with plane change is achieved by three impulses: a deorbit impulse, a boost impulse, and a reorbit impulse. The objective of the optimal orbital plane change problem is to minimize the fuel required for the three impulsive maneuver. Regarding energy as a slow variable and altitude and flight-path angle as fast variables, a three- state model that is suitable for singular perturbation analysis is[65]

$$\dot{\phi} = C_L^* \rho S V \lambda \sin\mu / 2m\cos\gamma, \qquad \epsilon\dot{h} = V\sin\gamma$$
$$\epsilon\dot{\gamma} = C_L^* \rho S V(\lambda\cos\mu + M\cos\gamma)/2m \tag{79}$$

where h is the altitude, V is the velocity, ϕ is the cross-range angle, $\lambda = C_L/C_L^*$ is the normalized lift coefficient, C_L is the lift coefficient, μ is the bank angle, ρ is the density, and the asterisk indicates the maximum lift-to-drag ratio.

In developing analytical methods for optimal guidance of AOTV problems using singular perturbations, the resulting TPBVP was solved in terms of reduced-order and boundary-layer solutions and compared to the numerical optimal solutions obtained using multiple shooting methods.[56] When alternative approximations were considered to solve the boundary-layer problem, three guidance laws in feedback form were obtained.[67]

Also, see an excellent survey on optimal strategies in aeroassisted orbital transfer by Mease,[244] a research monograph by Naidu,[260] and important contributions by Calise and Melamed,[411,412] Vinh and Hanson,[390] Vinh and Johannesen,[244] and Vinh et al.[392,394]

V. Other Aerospace Related Applications

A. Structures and Other Mechanical Systems

Another interesting area of the application of SPaTS is structural dynamics and control.

In Ref. 332, the deformed state of a thin, inextensible beam, which is under the action of axial and transverse loading and which also rests on an elastic foundation, is governed by[112]

$$\epsilon\dot{\kappa} = -Q, \qquad \epsilon\dot{Q} = -\kappa\sec\theta - \epsilon\kappa\tan\theta + (\lambda^2 y - p)\cos\theta$$
$$\dot{x} = \cos\theta, \qquad \dot{\theta} = \kappa \tag{80}$$

Here the constants and variables are dimensionless and proportional to arc length t, curvature $\kappa(t)$, normal component of the inner force $Q(t)$, horizontal and vertical displacement $x(t)$ and $y(t)$, gradient angle $\theta(t)$, transversal loading $p(t)$, resistance of the foundation λ^2, and bending stiffness ϵ. For thin beams the bending stiffness ϵ is small, and hence, the system given by Eq. (80) is a singularly perturbed system of ordinary differential equations. Formal approximations of the solutions to Eq. (80) are obtained in the form of MAE.

The asymptotic solution of a time-optimal, soft-constrained, cheap control problem was obtained using a new approach solely based on expanding the controllability gramian without resorting to the method of MAE. The method was applied to the time-optimal single-axis rotation problem for a system consisting of a rigid hub with an elastic appendage due to an external torque applied at the hub.[40]

Other works dealing with singular perturbations in structures are found in Refs. 31, 112, 164, 242, and 377. Mechanical systems involving flexible dual rudder are considered in Ref. 95.

Singular perturbation concepts are exploited to develop a procedure for designing a constant gain, output feedback control system with application to a large space structure.[70] In this system, the third and fourth modes are approximately five time faster than the first and second modes, thus leading to the small parameter value as $\epsilon = \frac{1}{5}$. A singular perturbation analysis that relaxes the requirement on boundary-layer system stability (but not necessarily asymptotic stability, as required in the normal case) was provided by an application to a flexible dual-rudder steering mechanism in Ref. 95.

Recently, an analysis of the underlying geometric structure of two-time scale, nonlinear optimal control systems was developed by Rao and Mease[308,309] without requiring a priori knowledge of the singular perturbation structure. The methodology is based on splitting the Hamiltonian boundary-value problem into stable and unstable components using a dichotomic basis. An illustration of a mass connected to a nonlinear spring was given.

B. Space Robotics

This is a new area where the theory of SPaTS has an important application. In robotics, the singular perturbation parameter is usually identified as the inverse of a stiffness parameter associated with a flexible mode. For example, in a typical flexible slewing arm with a rigid-body rotation and flexible clamped mass modes, one can select the quantity $\epsilon = (1/k_2)^{1/2}$ as the singular perturbation parameter, where k_2 is the stiffness parameter associated with the second flexible mode. Thus, the slow subsystem states are the joint angle, the first flexible modal displacement, and their respective rates, whereas the fast subsystem states are the second flexible modal displacement and its rate.[354]

In particular, work has been done in space robotics[405] and teleoperation,[356] intelligent robotics systems for space exploration,[98] and perturbation techniques for flexible manipulators in Ref. 118 and robotics in Refs. 34, 71, 79, 80, 130, 197, 198, 237, 354, and 363–365.

VI. Other Applications

There are a number of other interesting and challenging applications of singular perturbation and time scale methodologies in a variety of fields.[106,257,358,384] Some typical applications include Markov chains,[406] electrical circuits,[327] electrostatics,[81] semiconductor modeling,[239] computer disk drives,[16] electrical machines[279,408] and power systems,[83,160] chemical reactions,[138] chemical reactors,[76,86] nuclear reactor,[168] soil mechanics,[99] celestial mechanics,[281,330,357,384] quantum mechanics,[110] thermodynamics,[149,236,314] plates and shells,[295,384] elasticity,[109,241] lubrication,[72] vibration,[357] renewal processes,[150] compressors,[227] magnetohydrodynamics,[348] oceanography,[146] welding,[5] queuing theory,[200] production inventory systems[43] and manufacturing,[163,339,362] wave propagation,[33] ionization of gases,[148] lasers,[107] automobiles and biped locomotion,[136,227] agricultural engineering,[372] reliability,[215] two-dimensional image modeling and processing,[28,29,167,409] ecology,[151,275] and biology.[285]

VII. Conclusions

This paper focused on a survey of the applications of the theory and techniques of singular perturbations and time scales in guidance and control of aerospace systems such as aircraft, missiles, spacecraft, transatmospheric vehicles, and aeroassisted orbital transfer vehicles. In particular, emphasis was placed on problem formulation and solution approaches that were useful in applying the theory for various types of problems arising in aerospace systems. A unique feature of this survey is that it assumes no prior knowledge in the subject and hence provides a brief introduction to the subject. Furthermore, the survey included related fields such as fluid dynamics, space structures, and space robotics.

Besides seeking new applications for the theory of SPaTS in aerospace systems, there remain numerous theoretical issues that require further investigation. These include the development of a systematic methodology for (slow and fast) state-variable selection in general nonlinear optimal control problems and further work on the application of SPaTS to state- and/or control-constrained optimal control problems.

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