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SINGULAR PERTURBATIONS AND TIME SCALES IN CONTROL THEORY AND APPLICATIONS: AN OVERVIEW[1]

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Abstract. This paper presents an overview of singular perturbations and time scales (SPaTS) in control theory and applications during the period 1984-2001 (the last such overviews were provided by [231, 371]). Due to the limitations on space, this is in way intended to be an exhaustive survey on the topic.
Key Words: Singular perturbations, time scales, control systems, order reduction, control theory and applications.

1 Introduction

A basic problem in the control system theory is the mathematical modeling of a physical system. The modeling of many systems calls for high-order dynamic equations. The presence of some "parasitic" parameters such as small time constants, resistances, inductances, capacitances, moments of inertia, and Reynolds number, is often the source for the increased order and "stiffness" of these systems. The stiffness, attributed to the simultaneous occurrence of "slow" and "fast" phenomena, gives rise to time scales. The systems in which the suppression of a small parameter is responsible for the degeneration (or reduction) of dimension (or order) of the system are labeled as "singularly perturbed" systems, which are a special representation of the general class of time-scale systems. The "curse" of dimensionality coupled with stiffness poses formidable computational complexities for the analysis and design of multiple time-scale systems.

From the perspective of systems and control, Kokotovic and Sannuti [372, 233, 373] were the first to explore the application of the theory of singular perturbations to continuous-time optimal control, both open-loop formulation leading to two-point boundary value problem [233] and closed-loop formulation leading to the matrix Riccati equation [373]. The methodology of

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singular perturbations and time-scales (SPaTS), "gifted" with the remedial features of both dimensional reduction and stiffness relief, is considered as a "boon" to systems and control engineers. The technique has now attained a high level of maturity in the theory of continuous-time and discrete-time control systems described by ordinary differential and difference equations respectively. The growth of research activity in the field of SPaTS resulted in the publication of excellent survey papers [439, 336, 50, 250, 149, 231, 440, 226, 371, 227, 321, 315, 192, 244, 313, 51, 441, 316, 314, 317], reports and proceedings of special conferences [232, 105, 15]. Also, see research monographs and books (including the general area of singular perturbation theory) [110, 102, 458, 198, 80, 103, 327, 442, 337, 104, 296, 299, 203, 328, 64, 183, 57, 325, 329, 411, 229, 230, 161, 228, 32, 311, 118, 141, 245, 174, 309, 338, 49, 452, 143, 9, 443, 204, 205, 31, 376, 10, 138], encyclopedia [403] and control handbook [206].

In this paper we present an overview of SPaTS in control theory and applications during the period 1985-present (the last such overviews or surveys were provided by [231, 371, 226, 227]). This overview is in no way meant to be an exhaustive survey due to limitations on space. A brief introductory material is provided to those readers who do not have prior knowledge in SPaTS.

2 Modeling

2.1 Singularly Perturbed Systems

Here, we present some basic definitions and mathematical preliminaries of SPaTS. Consider a system described by a linear, second order, boundary value problem

$$\epsilon\ddot{x}(t) + \dot{x}(t) + x(t) = 0; \quad x(t=0) = x_i, \quad x(t=1) = x_f \tag{1}$$

where the small parameter ϵ multiplies the highest derivative. Here and in the rest of this paper, *dot* (.) and *double dot* (..) indicate first and second derivatives, respectively, with respect to t. As ϵ tends to zero either from positive or negative values, we have

$$\begin{aligned} \lim_{\epsilon\to 0_+}\{x(t,\epsilon)\} &= x_f exp(1-t), \quad 0 < t \leq 1 \\ \lim_{\epsilon\to 0_-}\{x(t,\epsilon)\} &= x_i exp(-t), \quad 0 \leq t < 1. \end{aligned} \tag{2}$$

The degenerate (outer, or reduced order) problem,

$$\dot{x}^{(0)}(t) + x^{(0)}(t) = 0 \tag{3}$$

obtained by suppressing the small parameter ϵ in (1), has the boundary condition $x^{(0)}(t=1) = x_f$ if ϵ tends to 0_+ and $x^{(0)}(t=0) = x_i$ if ϵ tends to 0_-. In either case, one boundary condition has to be "sacrificed" in the process of degeneration. The important features of singular perturbations are summarized as follows.

1. The problem (1) where the small parameter ϵ is multiplying the highest derivative is called a "singularly perturbed" (singular perturbation) problem if the order of the problem becomes lower for $\epsilon = 0$ than for $\epsilon \neq 0$ [458].
2. There exists a boundary layer where the solution changes rapidly.
3. The degenerate problem, also called the "unperturbed" problem, is of reduced order and cannot satisfy all the given boundary conditions of the original (full, or perturbed) problem.
4. The singularly perturbed problem (1) has two widely separated characteristic roots giving rise to "slow" and "fast" components (modes) in its solution. Thus, the singularly perturbed problem possesses a "two-time-scale" property. The simultaneous presence of "slow" and "fast" phenomena makes the problem "stiff" from the numerical solution point of view.

2.2 Continuous-Time Control Systems

Let us now introduce the idea of singular perturbations from the systems and control perspective. Using the state variable representation for a general case of (1), a linear time-invariant system becomes

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}_{11}\mathbf{x}(t) + \mathbf{A}_{12}\mathbf{z}(t) + \mathbf{B}_1\mathbf{u}(t), \qquad \mathbf{x}(t=0) = \mathbf{x}_0, \\ \epsilon\dot{\mathbf{z}}(t) &= \mathbf{A}_{21}\mathbf{x}(t) + \mathbf{A}_{22}\mathbf{z}(t) + \mathbf{B}_2\mathbf{u}(t), \qquad \mathbf{z}(t=0) = \mathbf{z}_0, \end{aligned} \tag{4}$$

where, $\mathbf{x}(t)$ and $\mathbf{z}(t)$ are $n-$ and $m-$dimensional state vectors, respectively, $\mathbf{u}(t)$ is an $r-$dimensional control vector and ϵ is a small, scalar parameter. The matrices $\mathbf{A}$ and $\mathbf{B}$ are of appropriate dimensions. The system (4) is said to be in the singularly perturbed form in the sense that by making $\epsilon = 0$ in (4) the degenerate system

$$\begin{aligned} \dot{\mathbf{x}}^{(0)}(t) &= \mathbf{A}_{11}\mathbf{x}^{(0)}(t) + \mathbf{A}_{12}\mathbf{z}^{(0)}(t) + \mathbf{B}_1\mathbf{u}(t), \qquad \mathbf{x}^{(0)}(t=0) = \mathbf{x}_0 \\ 0 &= \mathbf{A}_{21}\mathbf{x}^{(0)}(t) + \mathbf{A}_{22}\mathbf{z}^{(0)}(t) + \mathbf{B}_2\mathbf{u}(t), \qquad \mathbf{z}^{(0)}(t=0) \neq \mathbf{z}_0 \end{aligned} \tag{5}$$

is a combination of "differential" system in $\mathbf{x}^{(0)}(t)$ of order n and "algebraic" system in $\mathbf{z}^{(0)}(t)$ of order m. The effect of degeneration is not only to "cripple" the order of the system from $(n+m)$ to n by "dethroning" $\mathbf{z}(t)$ from its original state variable status, but also to "desert" its initial conditions $\mathbf{z}_0$. This is a harsh "punishment" on $\mathbf{z}(t)$ for having a close association (multiplication) with the singular perturbation parameter ϵ. We assume that the matrix $\mathbf{A}_{22}$ is nonsingular and hence we have a *standard* singular perturbation problem. However, [209, 295, 237, 201] deal with the situation where

A_{22} may be singular in which case it is called a *nonstandard* singular perturbation problem. We can also view the degeneration as equivalent to letting the forward gain of the system go to infinity.

In the case of a nonlinear singularly perturbed system, we have

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t), \epsilon, t), \qquad \mathbf{x}(t=0) = \mathbf{x}_0 \\ \epsilon \dot{\mathbf{z}}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t), \epsilon, t), \qquad \mathbf{z}(t=0) = \mathbf{z}_0. \end{aligned} \tag{6}$$

In the above discussion, we assumed an "initial" value problem. As a "boundary" value problem, we have the conditions as $\mathbf{x}(t=0) = \mathbf{x}_0$, and $\mathbf{z}(t=t_f) = \mathbf{z}_f$ or other sets of boundary conditions.

Also, see [11] for a new method for modeling a two-time scale system in the singularly perturbed form using an "ordered" real Schur decomposition which can be easily computed. In [397], relative gain array, used in the design of multivariable control systems to determine the best pairings of the input and output variables, was applied to singular perturbations illustrating via a three-stage chemical distillation system.

2.3 Discrete-Time Control Systems

As in the case of continuous-time systems, there are singularly perturbed, discrete-time control systems. Basically, there are two sources of modeling the discrete-time systems [325, 311].

Source 1: Pure Difference Equations: For a general linear, time-invariant discrete-time system,

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}_{11}\mathbf{x}(k) + \epsilon^{1-j}\mathbf{A}_{12}\mathbf{z}(k) + \mathbf{B}_1\mathbf{u}(k) \\ \epsilon^{2i}\mathbf{z}(k+1) &= \epsilon^{i}\mathbf{A}_{21}\mathbf{x}(k) + \epsilon\mathbf{A}_{22}\mathbf{z}(k) + \epsilon^{j}\mathbf{B}_2\mathbf{u}(k) \end{aligned} \tag{7}$$

$i \in \{0,1\}$; $j \in \{0,1\}$, where, $\mathbf{x}(k)$ and $\mathbf{z}(k)$ are n- and m-dimensional state vectors respectively, and $\mathbf{u}(k)$ is an r-dimensional control vector. Depending on the values for i and j, the three limiting cases of (7) are (i) C-model ($i=0; j=0$), where the small parameter ϵ appears in the "column" of the system matrix, (ii) R-model ($i=0; j=1$), where we see the small parameter ϵ in the "row" of the system matrix, and (iii) D-model ($i=1; j=1$), where ϵ is positioned in an identical fashion to that of the continuous-time system (4) described by "differential" equations. See Refs. [325, 311] for further details.
Source 2: Discrete-Time Modeling of Continuous-Time Systems: Under this case, either numerical solution or sampling of singularly perturbed continuous-time systems results in discrete-time models. Consider the continuous-time system (4). Applying a block diagonalization transformation [280], the original state variables $\mathbf{x}(t)$ and $\mathbf{z}(t)$ can be expressed in terms of the decoupled system consisting of slow and fast variables $\mathbf{x}_s(t)$ and $\mathbf{z}_f(t)$ respectively. Using a sampling device with the decoupled continuous-time system, we get a discrete-time model which critically depends on the sampling interval T [195].

Depending on the sampling interval, we get a "fast"(subscripted by f) or "slow" (subscripted by s) sampling model. In a particular case, when $T_f = \epsilon$, we get the *fast sampling model* as

$$\begin{aligned}\mathbf{x}(n+1) &= (\mathbf{I}_s + \epsilon\mathbf{D}_{11})\mathbf{x}(n) + \epsilon\mathbf{D}_{12}\mathbf{z}(n) + \epsilon\mathbf{E}_1\mathbf{u}(n) \\ \mathbf{z}(n+1) &= \mathbf{D}_{21}\mathbf{x}(n) + \mathbf{D}_{22}\mathbf{z}(n) + \mathbf{E}_2\mathbf{u}(n) \end{aligned} \tag{8}$$

where n denotes with the fast sampling instant (not to be confused with the system order described previously). Similarly, if $T_s = 1$, we obtain the *slow sampling model* as

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{E}_{11}\mathbf{x}(k) + \epsilon\mathbf{E}_{12}\mathbf{z}(k) + \mathbf{E}_1\mathbf{u}(k) \\ \mathbf{z}(k+1) &= \mathbf{E}_{21}\mathbf{x}(k) + \epsilon\mathbf{E}_{22}\mathbf{z}(k) + \mathbf{E}_2\mathbf{u}(k) \end{aligned} \tag{9}$$

where k represents the slow sampling instant, and $n = k[1/\epsilon]$. Also, the $\mathbf{D}$ and $\mathbf{E}$ matrices are related to the matrices $\mathbf{A}$, $\mathbf{B}$, and transformation matrices [195]. Note that the fast sampling model (8) can be viewed as the discrete analog (either by exact calculation using the exponential matrix or by using the Euler approximation) of the continuous-time system

$$\begin{aligned}\frac{d\mathbf{x}}{d\tau} &= \epsilon\mathbf{A}_{11}\mathbf{x}(\tau) + \epsilon\mathbf{A}_{12}\mathbf{z}(\tau) + \epsilon\mathbf{B}_1\mathbf{u}(\tau) \\ \frac{d\mathbf{z}}{d\tau} &= \mathbf{A}_{21}\mathbf{x}(\tau) + \mathbf{A}_{22}\mathbf{z}(\tau) + \mathbf{B}_2\mathbf{u}(\tau) \end{aligned} \tag{10}$$

which itself is obtained from the continuous-time system (4) using the stretching transformation $\tau = t/\epsilon$. It is usually said that the singularly perturbed continuous-time systems (4) and (10) are the *slow* time scale (t) and *fast* time scale (τ) versions, respectively. Also, note that the slow sampling model (9) is the same as the state space C-model.

See [28, 41] for some more results regarding slow and fast sampling models for singularly perturbed systems.

3 Singular Perturbation Technique

3.1 Basic Theorems

For simplicity, consider the nonlinear initial value problem (6) without input function $\mathbf{u}$ as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{z}(t), \epsilon, t), \qquad \mathbf{x}(t=0) = \mathbf{x}_0 \\ \epsilon\dot{\mathbf{z}}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{z}(t), \epsilon, t), \qquad \mathbf{z}(t=0) = \mathbf{z}_0. \end{aligned} \tag{11}$$

Here, we follow the seminal works of Tikhonov [430] Vasileva [439]. By setting the small parameter $\epsilon = 0$ in (11), the degenerate problem is given by

$$\dot{\mathbf{x}}^{(0)}(t) = \mathbf{f}(\mathbf{x}^{(0)}(t), \mathbf{z}^{(0)}(t), 0, t) \tag{12}$$

$$0 = \mathbf{g}(\mathbf{x}^{(0)}(t), \mathbf{z}^{(0)}(t), 0, t). \tag{13}$$

Assuming that we are able to solve the above algebraic equation (13), we have

$$\mathbf{z}^{(0)}(t) \quad = \quad \mathbf{h}(\mathbf{x}^{(0)}(t), t). \tag{14}$$

Using (14) in (12), the reduced order problem becomes

$$\dot{\mathbf{x}}^{(0)}(t) \quad = \quad \mathbf{f}^0(\mathbf{x}^{(0)}(t), t), \qquad \mathbf{x}^{(0)}(t=0) = \mathbf{x}_0. \tag{15}$$

From (14), it is evident that $\mathbf{z}^{(0)}(0)$ is, not in general, equal to $\mathbf{z}_0$. The two important features of singular perturbation theory are *degeneration* and *asymptotic expansion*. An important assumption is given below.

Assumption 1 *The solution $\mathbf{z}^{(0)}(t)$ of (12) is an asymptotically stable equilibrium point of the boundary layer equation*

$$\frac{d\mathbf{z}(\tau)}{d\tau} = \mathbf{g}(\mathbf{x}^{(0)}, \mathbf{z}(\tau), 0, 0) \tag{16}$$

as $\tau \rightarrow \infty$. This means that the Jacobian matrix $\mathbf{g_z}$ of (16) has all eigenvalues with negative real parts.

In degeneration, our interest is to find the conditions under which the solution of the full problem (11) tends to the solution of the degenerate problem (15). A theorem due to Tikhonov [430] concerning degeneration is given below without listing all the assumptions (for details see [458] and [311]).

Theorem 3.1 *The exact solutions $\mathbf{x}(t, \epsilon)$ and $\mathbf{z}(t, \epsilon)$ of the full problem (11) are related to the solutions $\mathbf{x}^{(0)}(t)$ and $\mathbf{z}^{(0)}(t)$ of the degenerate problem (12) and (13) as*

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} [\mathbf{x}(t, \epsilon)] &= \mathbf{x}^{(0)}(t), \qquad 0 \leq t \leq \mathcal{T} \\ \lim_{\epsilon \rightarrow 0} [\mathbf{z}(t, \epsilon)] &= \mathbf{z}^{(0)}(t), \qquad 0 < t \leq \mathcal{T} \end{aligned} \tag{17}$$

under certain assumptions [458, 311]. Here, $\mathcal{T}$ is any number such that $\mathbf{z}^{(0)}(t) = \mathbf{h}(\mathbf{x}^{(0)}(t), t)$ is an isolated stable root of (13) for $0 \leq t \leq \mathcal{T}$. The convergence is uniform in $0 \leq t \leq \mathcal{T}$ for $\mathbf{x}(t, \epsilon)$, and in any interval $0 < t_1 \leq t \leq \mathcal{T}$ for $\mathbf{z}(t, \epsilon)$ and the convergence of $\mathbf{z}(t, \epsilon)$ will usually be nonuniform at $t = 0$.

Also, see the result in [20, 451] where the reduced order system is not the standard limit as $\epsilon \rightarrow 0$ but invariant measures of parameterized fast flow are employed to describe the limit behavior.

The second feature in singular perturbation theory is the asymptotic expansion for the solutions. The main result was given by Vasileva [439, 458, 311] in the form of the following theorem.

Theorem 3.2 *Under the same assumptions required for Tikhonov's theorem [430, 439, 458], there exist an* $\epsilon_0 > 0, 0 \leq \epsilon \leq \epsilon_0$*, and* $R(t,\epsilon)$ *and* $S(t,\epsilon)$ *uniformly bounded in the interval considered, such that*

$$\begin{aligned} \mathbf{x}(t,\epsilon) &= \sum_{i=0}^{j} \left[\mathbf{x}^{(i)}(t) + \bar{\mathbf{x}}^{(i)}(\tau) - \underline{\mathbf{x}}^{(i)}(\tau) \right] \epsilon^i + R(t,\epsilon)\epsilon^{j+1} \\ \mathbf{z}(t,\epsilon) &= \sum_{i=0}^{j} \left[\mathbf{z}^{(i)}(t) + \bar{\mathbf{z}}^{(i)}(\tau) - \underline{\mathbf{z}}^{(i)}(\tau) \right] \epsilon^i + S(t,\epsilon)\epsilon^{j+1} \end{aligned} \tag{18}$$

where, $\tau = t/\epsilon$, $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{z}^{(i)}(t)$ *are the* **outer** *or degenerate series solutions (so-called because these solutions are valid* outside *the boundary layer,* $\bar{\mathbf{x}}^{(i)}(\tau)$ *and* $\bar{\mathbf{z}}^{(i)}(\tau)$ *are the* **inner** *solutions (so-called because these solutions are valid* inside *the boundary layer, and* $\underline{\mathbf{x}}^{(i)}(\tau)$ *and* $\underline{\mathbf{z}}^{(i)}(\tau)$ *are the* **intermediate** *solutions (so-called because of the common part of the outer and inner solutions).*

The details of obtaining these various series solutions are given in [458, 311]. The inner and intermediate series solutions are obtained from the "stretched" system

$$\begin{aligned} \frac{d\mathbf{x}(\tau)}{d\tau} &= \epsilon\mathbf{f}(\mathbf{x}(\tau), \mathbf{z}(\tau), \epsilon, \epsilon\tau) \\ \frac{d\mathbf{z}(\tau)}{d\tau} &= \mathbf{g}(\mathbf{x}(\tau), \mathbf{z}(\tau), \epsilon, \epsilon\tau) \end{aligned} \tag{19}$$

obtained by using the stretching transformation $\tau = t/\epsilon$ in (11).

Alternatively, the solution is obtained as

$$\begin{aligned} \mathbf{x}(t,\epsilon) &= \mathbf{x}_0(t,\epsilon) + \mathbf{x}_c(\tau,\epsilon) \\ \mathbf{z}(t,\epsilon) &= \mathbf{z}_0(t,\epsilon) + \mathbf{z}_c(\tau,\epsilon) \end{aligned} \tag{20}$$

where, $\mathbf{x}_c(\tau) = \bar{\mathbf{x}}(\tau) - \underline{\mathbf{x}}(\tau)$ and $\mathbf{z}_c(\tau) = \bar{\mathbf{z}}(\tau) - \underline{\mathbf{z}}(\tau)$ are often called "boundary layer corrections" which are obtained as series solutions from [337, 311]

$$\begin{aligned} \frac{d\mathbf{x}_c(\tau)}{d\tau} &= \epsilon\mathbf{f}(\mathbf{x}_0(\epsilon\tau,\epsilon) + \mathbf{x}_c(\tau,\epsilon), \mathbf{z}_0(\epsilon\tau,\epsilon) + \mathbf{z}_c(\tau,\epsilon), \epsilon, \epsilon\tau) - \\ &\quad \epsilon\mathbf{f}(\mathbf{x}_0(\epsilon\tau,\epsilon), \mathbf{z}_0(\epsilon\tau,\epsilon), \epsilon, \epsilon\tau) \\ \frac{d\mathbf{z}(\tau)}{d\tau} &= \mathbf{g}(\mathbf{x}_0(\epsilon\tau,\epsilon) + \mathbf{x}_c(\tau,\epsilon), \mathbf{z}_0(\epsilon\tau,\epsilon) + \mathbf{z}_c(\tau,\epsilon), \epsilon, \epsilon\tau) - \\ &\quad \mathbf{g}(\mathbf{x}_0(\epsilon\tau,\epsilon), \mathbf{z}_0(\epsilon\tau,\epsilon), \epsilon, \epsilon\tau) \end{aligned} \tag{21}$$

In the case of a singularly perturbed *linear* system (4), the above two theorems imply that stability conditions require that

$$Re\{\lambda_i[\mathbf{A}_{22}]\} < 0, \quad i = 1, ..., m. \tag{22}$$

In other words, if the matrix $\mathbf{A}_{22}$ is stable, then the asymptotic expansions can be carried out to arbitrary order [230, 51].

In the case of a general *boundary value problem*, it is expected to have initial and final boundary layers and hence the asymptotic solution is obtained as an *outer* solution in terms of the original independent variable t, *initial* layer correction in terms of an initial stretched variable $\tau = t/\epsilon$, and *final* layer correction in terms of a final stretched variable $\sigma = (t_f - t)/\epsilon$ [338].

In [92], a Tikhonov-type theorem was developed for singularly perturbed differential inclusions. Also, see [162] for an exposition of singularly perturbed initial value problems, and some recent results in [44, 42, 43] for modeling and analysis of two-time scale, discrete nonlinear systems including a discrete version of the well-known Tikhonov's theorem for continuous-time system.

4 Time Scale Analysis

In general, a time-scale system need not be in the singularly perturbed form with a small parameter multiplying the highest derivative or some of the state variables of the state equation as given in (4) or (6). In other words, a singularly perturbed structure is only one form of the two-time scale systems. In time-scale analysis of a linear system, a block diagonalization transformation [412] is used to decouple the original two-time scale system into two low-order slow and fast subsystems. Let us consider a general two-time scale, linear system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}_{11}\mathbf{x}(t) + \mathbf{A}_{12}\mathbf{z}(t) + \mathbf{B}_1\mathbf{u}(t) \\ \dot{\mathbf{z}}(t) &= \mathbf{A}_{21}\mathbf{x}(t) + \mathbf{A}_{22}\mathbf{z}(t) + \mathbf{B}_2\mathbf{u}(t) \end{aligned} \tag{23}$$

possessing two widely-separated groups of eigenvalues. Thus, we assume that the n eigenvalues of the system (23) are "small" and the remaining m eigenvalues are "large", giving rise to slow and fast responses respectively. We now use a two-stage linear transformation [56, 225],

$$\begin{aligned} \mathbf{x}_s(t) &= (\mathbf{I}_s - \mathbf{ML})\mathbf{x}(t) - \mathbf{Mz}(t) \\ \mathbf{z}_f(t) &= \mathbf{Lx}(t) + \mathbf{I}_f\mathbf{z}(t) \end{aligned} \tag{24}$$

to decouple the original system (23) into two slow and fast subsystems,

$$\begin{aligned} \dot{\mathbf{x}}_s(t) &= \mathbf{A}_s\mathbf{x}_s(t) + \mathbf{B}_s\mathbf{u}(t) \\ \dot{\mathbf{z}}_f(t) &= \mathbf{A}_f\mathbf{z}_f(t) + \mathbf{B}_f\mathbf{u}(t) \end{aligned} \tag{25}$$

where,

$$\begin{aligned} \mathbf{A}_s &= \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{L}; \qquad \mathbf{A}_f = \mathbf{A}_{22} + \mathbf{LA}_{12} \\ \mathbf{B}_s &= \mathbf{B}_1 - \mathbf{MLB}_1 - \mathbf{MB}; \qquad \mathbf{B}_f = \mathbf{B}_2 + \mathbf{LB}_1 \end{aligned} \tag{26}$$

$\mathbf{L}$ and $\mathbf{M}$ are the solutions of the nonlinear algebraic Riccati type equations

$$\begin{aligned}\mathbf{L}\mathbf{A}_{11} + \mathbf{A}_{12} - \mathbf{L}\mathbf{A}_{12}\mathbf{L} - \mathbf{A}_{22}\mathbf{L} &= 0 \\ \mathbf{A}_{11}\mathbf{M} - \mathbf{A}_{12}\mathbf{L}\mathbf{M} - \mathbf{M}\mathbf{A}_{22} - \mathbf{M}\mathbf{L}\mathbf{A}_{12} + \mathbf{A}_{12} &= 0\end{aligned} \tag{27}$$

and $\mathbf{I}$ is unity matrix.

Further results on control of two-time scale systems are found in [207] where parameterization of the set of all stabilizing compensators was achieved preserving the two-time scale character. Also, see [269] for an algebraic approach to time-scale analysis and [286] for more results on decoupling ideas where fast subsystem is not influenced by slow subsystem and [287] for a similar treatment for a nonlinear system.

Similar analysis exists for two-time scale discrete-time systems [325, 278, 218, 311] and see [109] for an overview of singularly perturbed discrete-time system.

An interesting work [87] on the representation of a system by bond graph model allows greater simplification of order reduction and decoupling of the system and the results are very close to those obtained by singular perturbation method.

Time scale analysis of input-output systems was studied in [400, 401, 402] including the study of positive realness for both linear and nonlinear systems.

See [78] for a coordinate-free or geometric setting for decomposing a linear singularly perturbed system using ideas of analytic manifolds.

In [466], a new approach to singular systems which includes singular perturbations was presented with applications to proportional and derivative control and state feedback.

Using delta operators [297], it was shown in [429] with a correction in [459] that, under sampling of a continuous linear single-input, single output (SISO) system of relative degree ≤ 2, the resulting delta model can be regarded as a regular perturbation if the sampling interval is considered as a parameter but that the zero dynamics is singularly perturbed.

In [76], a new general bilinear relationship was found between the continuous and discrete generalized singular perturbation (GSP) reduced order models with applications to balanced systems. Also, see related results in [268, 350, 332] and [310] for a further generalization of GSP by introducing several parameters that can be tuned to provide a specific performance.

In [445], continuity properties of the solutions to a differential inclusion subject to a singular perturbation were studied. Also, asymptotic properties of nonlinear, discrete-time control systems were discussed in [154, 155, 156] using averaging method and differential inclusions.

Various schemes were proposed for both state feedback and output feedback control of discrete-time systems in [274, 281, 260, 258, 259, 262].

5 SPaTS in Optimal Control

5.1 Open-Loop Optimal Control

The need for order reduction associated with singular perturbation methodology is most acutely felt in optimal control design which demands the solution of *state* and *costate* equations with initial and final conditions. For the singularly perturbed continuous-time, nonlinear system (6), the performance index in a simplified form is usually taken as

$$J = S\left(\mathbf{x}(t_f), \mathbf{z}(t_f), t_f, \epsilon)\right) + \int_0^{t_f} V\left(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t), t, \epsilon\right) dt. \tag{28}$$

Using the well-known theory of optimal control [48], the optimization of the performance index (28) subject to the plant equation (6) and the boundary conditions (with *fixed* initial conditions $\mathbf{x}(t=0) = \mathbf{x}_0$, $\mathbf{z}(t=0) = \mathbf{z}_0$ and *free* final conditions $\mathbf{x}(t_f) = \mathbf{x}_f$, $\mathbf{z}(t_f) = \mathbf{z}_f$), leads us to (for *unconstrained* control).

$$\begin{aligned} 0 &= +\frac{\partial \mathcal{H}}{\partial \mathbf{u}} \\ \frac{d\lambda_{\mathbf{x}}}{dt} &= -\frac{\partial \mathcal{H}}{\partial \mathbf{x}}; \quad \lambda_{\mathbf{x}}(t_f) = \left.\frac{\partial S}{\partial \mathbf{x}}\right|_{t_f} \\ \epsilon\frac{d\lambda_{\mathbf{z}}}{dt} &= -\frac{\partial \mathcal{H}}{\partial \mathbf{z}}; \quad \epsilon\lambda_{\mathbf{z}}(t_f) = \left.\frac{\partial S}{\partial \mathbf{z}}\right|_{t_f} \end{aligned} \tag{29}$$

where, $\lambda_{\mathbf{x}}$ and $\epsilon\lambda_{\mathbf{z}}$ are the *costates* or *adjoints* corresponding to the states $\mathbf{x}(t)$ and $\mathbf{z}(t)$ respectively, and $\mathcal{H}$ is the Hamiltonian given by

$$\begin{aligned} \mathcal{H} = {} & V\left(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t), t, \epsilon\right) + \\ & \lambda'_{\mathbf{x}}\mathbf{f}(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t), t, \epsilon) + \lambda'_{\mathbf{z}}\mathbf{g}(\mathbf{x}(t), \mathbf{z}(t), \mathbf{u}(t), t, \epsilon) \end{aligned} \tag{30}$$

where, $'$ denotes transpose. Note that $\mathbf{u}(t) = \arg\min_{\mathbf{u}\in\mathbf{U}}\{\mathcal{H}\}$ for *constrained* control, where $\mathbf{U}$ is a set of admissible controls. The system of state equations (6) and costate equations (29) constitute a singularly perturbed, two-point boundary value problem (TPBVP) leading to the formulation of the solution in terms of an *outer* solution, *initial* layer correction and *final* layer correction. It is to be noted that the final boundary layer system needs to be asymptotically stable in *backward* time, i.e., inherently unstable in *forward* time. This situation can create difficulties in trying to satisfy the given boundary conditions and Kelley [202] and Cliff [77] suggested that a proper selection of boundary conditions to suppress any unstable component of the boundary layer solution.

Also, note that there is another important Mayer problem in optimal control where the performance index (28) contains the terminal cost function only [48]. The optimal control and the related tracking problems were studied by many workers [94, 453, 187, 188, 19].

See [131, 132, 133] for a new methodology on solving singularly perturbed optimal control problems where the traditional boundary layer method [230, 443] cannot be used when the optimal control contains fast oscillations. The new method is based on approximation of a family of low-order solutions using differential inclusions.

In particular, optimal control problem with control constraints where the fast system contains a stable part and an unstable part which are weakly coupled to each other was studied in [94]. Further results including on state-constrained optimal control problems are found in [193, 213, 52, 134, 37, 27, 38, 39].

See [125] for a study on Hamilton-Jacobi equation resulting from the optimal control of a *nonlinear* singularly perturbed system and a method of exact decomposition of the full-order slow-fast manifold into a low-order slow submanifold of the Hamiltonian system and a fast submanifold of an auxiliary system. See similar result [126] for a *linearized* system. Also, see [157] for developing a maximum principle of Pontryagin type for the time optimal control of a hybrid system described by coupled ordinary differential equations and partial differential equations.

Also, see [298] for solving optimal control problems using averaging method and [352] for studying the convergence of value-functions for a nonlinear optimal control problem with singular perturbation. Dynamic programming has also been used for singularly perturbed optimal control problems in [236, 33].

Similar results exist for singularly perturbed, discrete-time optimal control systems [266, 325, 311].

5.2 Closed-Loop Optimal Control

The closed-loop optimal control has some very elegant results for *linear* systems leading to a matrix Riccati differential or algebraic equations. For the singularly perturbed, linear continuous-time system (4), consider a quadratic performance index [311]

$$J = \frac{1}{2}\mathbf{y}'(t_f)S\mathbf{y}(t_f) + \frac{1}{2}\int_0^{t_f} \left(\mathbf{y}'(t)\mathbf{Q}\mathbf{y}(t) + \mathbf{u}'(t)\mathbf{R}\mathbf{u}(t)\right) dt \qquad (31)$$

where, $\mathbf{y} = [\mathbf{x}, \epsilon\mathbf{z}]'$, $\mathbf{S}$ and $\mathbf{Q}$ are positive semidefinite $(n+m)$x$(n+m)$ dimensional matrices, and $\mathbf{R}$ is a positive definite rxr matrix, we arrive at the closed-loop optimal control as

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}'\mathbf{P}\mathbf{y}(t) \qquad (32)$$

Here, $\mathbf{P}$ is an $(n+m)$x$(n+m)$-dimensional, positive-definite, symmetric matrix satisfying the singularly perturbed matrix Riccati differential equation

$$\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A} - \mathbf{A}'\mathbf{P}(t) + \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{P}(t) - \mathbf{Q}, \quad \mathbf{P}(t_f) = \mathbf{S} \qquad (33)$$

where,

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \frac{\mathbf{A}_{21}}{\epsilon} & \frac{\mathbf{A}_{22}}{\epsilon} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \frac{\mathbf{B}_2}{\epsilon} \end{bmatrix}; \\
\mathbf{P}(t) &= \begin{bmatrix} \mathbf{P}_{11}(t) & \epsilon\mathbf{P}_{12}(t) \\ \epsilon\mathbf{P}'_{12}(t) & \epsilon\mathbf{P}_{22}(t) \end{bmatrix} \qquad (34)
\end{aligned}$$

Note that the matrix $\mathbf{P}$ in the above Riccati equation, is dependent on the small parameter ϵ and is in the singularly perturbed form. Assuming series expansions for $\mathbf{P}$'s, we can get their asymptotic series solutions. There are several studies on closed-loop optimal control of singularly perturbed continuous-time systems [226, 227, 53, 94, 453, 303]. The linear quadratic regulator problem with three-time-scale behavior has been investigated in [374]. Related work is the closed-loop optimal control including tracking linear and nonlinear systems decomposed into slow and fast subsystems [152, 425, 365, 160, 200, 126].

A two-time scale approximation to the linear quadratic optimal *output* regulator problem was examined in Refs. [302, 301, 303]. Also, see [9] for optimal control of *bilinear* systems. Also see [308] for design of a high gain regulator for linear multivariable system in terms of decoupling and the singular perturbation theory. Also, see [455, 454] for more details on complete decomposition of sub-optimal regulator for singularly perturbed systems.

Similar results exist for closed-loop optimal control of singularly perturbed, discrete-time linear systems, leading to matrix Riccati difference equation [311, 265, 266, 351, 323, 325, 341, 257, 258, 259]. Time scale analysis of optimal regulator problems in discrete-time control systems was considered in [341, 195, 196, 319, 197, 320, 275, 276]. Time-optimal control of singularly perturbed continuous systems was studied in [94, 385] and discrete systems in [384]. Also, see [331] for heuristic approach to reducing the order of a two-time scale discrete, linear, time-varying system subjected to white noise representation of a fast state vector. In [169] the effect of quantization of control signal was studied and [170] analyzed a piece-wise linear system.

Another type of problem that arises more often in aerospace systems is differential games. In the design of multi-input control problems, the objective in the optimal policy may be met by formulating the control problem as a differential game problem. In a general differential game, there are several players, each trying to minimize his or her individual cost functional. Each player controls a different set of inputs to a single system. The strategies usually considered are Pareto optimal, Nash equilibrium or Stackelberg [214]. In the case of two-player Nash game, we have

$$\begin{aligned}
\dot{x}(t) &= f(x(t), z(t), u_1(t), u_2(t), t), \qquad x(t=0) = x_0 \\
\epsilon\dot{z}(t) &= g(x(t), z(t), u_1(t), u_2(t), t), \qquad z(t=0) = z_0 \qquad (35)
\end{aligned}$$

and the performance index

$$J = \int_0^{t_f} V_i(x(t), z(t), u_1(t), u_2(t), t)dt, \qquad i = 1, 2. \tag{36}$$

The main question investigated has been one of well-posedness whereby the limit of performance using the exact strategies as $\epsilon \to 0$ is compared to the performance using simplified strategies with $\epsilon = 0$. The problem is said to be well posed if the two limits are equal and the singularly perturbed zero-sum games are always well posed. Also, it is important to note that the structure of a well-posed singularly perturbed (two-player) Nash game is composed of a reduced-order Nash game and two independent optimal control problems of the players [368]. Further results on well-posedness are available in [136] for stochastic systems. These and other aspects of differential games have been studied in [369, 370, 367, 371, 133, 424, 460].

Time-optimal control of singularly perturbed systems with an application to disk-drive actuator was discussed in [18, 17]. In [75], an optimal control problem exhibiting chattering phenomena was transformed into a problem similar to a relaxation oscillator, having slow, almost equilibrium motions connected by fast, jump type transitions and an asymptotic expansion constructed. A new design method for regulator problems for singularly perturbed systems with and without constrained controls was given in [435, 436].

Matrix Riccati Equations

Gajić and his associates [135, 159, 140, 165, 141, 390, 391, 142, 353, 423, 143, 392, 393, 139, 85, 138] have developed systematic methodologies for exact decomposition of algebraic Riccati equations (including numerical algorithms) arising in optimal control of singularly perturbed deterministic and stochastic and continuous-time and discrete-time systems. See their recent book [138] for complete details. In particular, recursive algorithm for solving the cross-coupled algebraic Riccati equations arising in singularly perturbed systems have been developed in [300, 306, 307]. Further, the Kleinman algorithm was used in [304] to solve the algebraic Riccati equation where the quadratic terms may be indefinite.

A new method for composite optimal control was proposed in [457, 461, 460] based on a generalized algebraic Riccati equation arising in descriptor systems and valid for both standard and nonstandard singularly perturbed systems. Also, see [456] for further details on this topic.

A method is developed in [464] for asymptotic solution of the optimal periodic control of linear systems

A "cheap" control problem where the control cost is associated with a small parameter was investigated by [379, 374, 254, 365, 36, 349, 386].

6 Other Control Problems

6.1 Nonlinear Systems and Stability

Here, we review the SPaTS methodology as applicable to nonlinear systems and the related stability problems.

In [364], a composite control is designed for stabilization and regulation of a class of nonlinear singularly perturbed systems establishing well-posedness of the full regulator problem. Also see related works by the same authors [363, 362].

See [222, 220, 215, 217] for addressing feedback linearization of full order nonlinear system via that of reduced-order systems. Also, see [389, 388] for design of composite control for a particular class of nonlinear systems using integral manifold approach. Hopf bifurcation in the simplest way in which periodic solutions can emerge from an equilibrium point of an autonomous systems described by ordinary differential equations containing a small parameter. See relevant results in [5]

In [89] one finds a study on asymptotic stability of a three-time scale nonlinear system and [93] for studying Lipschitz properties of linear singularly perturbed systems. Also see [419] for the stability analysis using relaxed method for nonlinear systems with periodic input. It was shown in [377] that when a linear or nonlinear system with relative degree equal or higher than 2, then their zero dynamics may be singularly perturbed.

For further results in nonlinear systems, see [333, 22, 23] where the stabilization of a class of nonlinear systems was presented using singular perturbation principles and high-gain observer designs; [60] for proposing a globally exponentially stabilizing composite feedback control for a general class of nonlinear systems by choosing two appropriate Lyapunov functions one for the reduced-order system and the other for boundary layer system and then forming a composite Lyapunov function to investigate the stability for the full-order nonlinear system; [284] for analysis of nonlinear problems using geometric approach.

Also, a system F is said to be D-stable if the eigenvalues of DF have negative real parts for any diagonal matrix D with positive diagonal elements. The issues of D stability for singularly perturbed systems were discussed in [8]. Regarding stability issues, see some results in [4, 161, 291, 115, 260, 383, 259, 267, 292, 151, 180, 59, 256, 179] for stability bounds on the singular perturbation parameter for both continuous-time and discrete-time systems. For *discrete-time* systems in particular, stability bounds on the small parameter were obtained in [61] for a composite controlled system in terms of the bounds for subsystems. Also, see [264, 256] for more on this topic. A related problem of utilizing guardian map theory was studied in [378] to investigate stability issues of linear time-invariant systems containing singular perturbation parameter ϵ and another uncertain parameter μ.

In [71], analysis of a singularly perturbed system having a reduced-order system that is input-to-state stable with respect to disturbances was performed. Also see [68] to design a well-conditioned state feedback controllers using a combination of singular perturbation and geometric methods for a two-time scale nonlinear system and [69] for developing a feedforward/feedback control methodology for a nonlinear system with time-varying disturbances. Also, see [84] for showing that the full-order system is exponential stable if the reduced order slow and fast subsystems are exponentially stable.

In [98] find a detailed study of the stability radius of a singularly perturbed system based on generalized Popov-Yakubovich theory and work in [158] investigates on the problem of asymptotics of Lyapunov exponents for a class of nonlinear singularly perturbed systems using an averaging technique.

See a recent result [437] for a multivariable circle criteria for multiparameter singularly perturbed systems obtained directly without using the Kalman-Yukubovich-Popov lemma based on the results of [435, 436].

Integral Manifold Theory

For some results on the topic of manifolds theory, see [413, 234, 14] and [167, 167] for an integral manifold approach to a class of single-input, single-output non-minimum phase linear systems with time-scale character.

Further, the decomposition of singularly perturbed systems using integral manifold approach [129, 422, 230] was discussed for linear and nonlinear optimal control theory by Fridman and associates [121, 120, 122, 124, 123, 125, 126, 128]. In particular, the exact pure-slow and pure-fast decomposition of the linear quadratic optimal control problems was addressed based on slow-fast integral manifold theory in [129, 413, 422, 121, 122, 124] for both finite horizon and infinite horizon cases similar to the Hamilton approach [423]. The approach to H_∞ optimal control of time-delay systems was discussed in [128, 128] and to some classes of nonlinear optimal control problems in [413, 125, 126].

6.2 Robustness

Here we briefly present the robustness of singularly perturbed linear and nonlinear systems. See [216] for feedback stabilization of a nonlinear system subject to two sources of uncertainty due uncertain elements and parameters and the unmodeled high frequency dynamics and [395] for an investigation of the problem of robust stability and robust disturbance attenuation with norm-bounded parameter uncertainties in both state and output relations.

In [62] we find two types of robust controllers for stabilizing singularly perturbed discrete-time bilinear systems, one is ϵ-dependent and the other ϵ-independent. See [72, 70] for developing robust controllers for a multi-input and multi-output (MIMO) two-time scale systems. See [111] for a brief study on the robustness of a discrete-time system with unmodeled high-frequency dynamics showing that the frequency of unmodeled dynamics needs to be much higher than the sampling frequency. See related results in [450, 340, 261, 434].

For synthesizing a robust output feedback controller that ensures boundedness and robust asymptotic output tracking for a system with time-varying uncertain variables using a high-gain observer, see [67]. Also, see [79] for designing robust controllers for systems with order uncertainty.

Recently [210] addresses the design of a robust output feedback controller for minimum phase nonlinear systems to achieve asymptotic tracking and disturbance rejection.

6.3 High-Gain Feedback

The high-gain feedback is a source for singular perturbation behavior of any physical system. In [361], a stabilizing high-gain dynamic output feedback controller with almost-disturbance-decoupling property is designed for a class of square-invertible and minimum phase systems and see [99] for stabilizing a linear system by using high-gain feedback using procedures similar to the stabilization of singularly perturbed systems. Also, see [283] for investigating high-gain state and output feedback for nonlinear systems using a control-dependent fast equilibrium manifold and change of coordinates and [366] for developing a systematic general theory of assigning desired time-scale structure to a multivariable system via either state or output feedback.

6.4 Observers

For singular perturbation analysis using observer principles, see [95] for designing mutli-time scale observers for linear systems with several time scales and [130] for placement and observer designs, [334] for designing a stabilizing controller in terms of the controllers for the slow and fast subsystems of the original shift-invariant discrete-time singularly perturbed systems with inaccessible states. Also see [432, 433, 431] for this and related topics in discrete-time systems.

An interesting approach to the design of discrete-time observers for nonlinear singularly perturbed continuous-time systems based on inversion of state-to-measurement maps was given in [398].

Also, see [22, 23, 86] where a separation principle was advocated for stabilization of a class of nonlinear systems using fast high-gain observer designs.

6.5 Multi-Time-Scale and Stochastic Systems

In multimodeling or multiparameter (multi-time-scale) deterministic and stochastic systems, we decompose a full-order system with several small parameters into one low-order slow subsystem and several low-order fast subsystems. The corresponding results are found in [3, 221, 247, 6, 7, 249, 164, 208, 199, 248, 153, 85].

Further, see [273, 270, 271, 272] for treatment of multitime scale systems from the frequency domain point of view. Further results on this are found in [335] for a study of the asymptotic behavior of the zeros of a two-frequency scale transfer function in terms of the zeros of its slow and fast transfer functions.

For discrete-time control systems with multi-time scale character, see [318, 242, 238, 255, 239, 240, 343, 241]

In **stochastic systems**, where the system is described by a set of stochastic differential equations containing small parameters, there are several results [212, 33, 246, 32, 245, 467, 1, 2, 191, 46, 201, 414] (to mention a few). In addition, see [116] for optimal control of a hybrid system having a fast deterministic system and a slow stochastic jump process.

See [108] for a highlight of the interplay of two asymptotic phenomena in singularly perturbed systems one arising in deterministic systems and the other arising in systems driven by wide-band noise and [137] for an LQG problem for multi-time scale systems. Also, see [177] for a study of singularly perturbed systems in the presence of noise under two aspects of quasi-steady state and averaging principles and [101] for a study on a singularly perturbed stochastic system such as phase locked loop system.

6.6 H_∞ and Other Control Problems

Some of the first works to examine the H_∞ control of singularly perturbed systems are [272, 211, 449]. Further, Başar *et. al.*, using differential game theoretic approach, studied the H_∞ optimal control of singularly perturbed systems for both finite and infinite horizons, under perfect state measurements [344] and under imperfect state measurements [345], robust controller design for nonlinear systems [26, 347] and the optimal control of a class of stochastic singularly perturbed systems with perfect and noisy state measurements with positively and negatively exponential quadratic cost [346]. Further, see [305] for near-optimal H_{inf} control problems using iterative algorithms.

Also see [428] for showing that any state/output feedback robust H_∞ controller of the linearized singularly perturbed system yields a local solution of the nonlinear H_∞ control system.

In [122, 124], one finds exact decomposition of the full-order matrix Riccati equations into reduced-order pure-slow and pure-fast equations arising in H_∞ optimal control and see [128] for an investigation of achieving minimum entropy by static output feedback ϵ-independent H_∞ controller for nonstandard singularly perturbed systems. Also, see [127] for designing state-feedback H_∞ controllers for nonlinear singularly perturbed systems.

One comes across in [97] a note on H_∞-norm of the transfer function matrix function of a singularly perturbed system tends to be the largest of the H_∞-norms for the boundary layer system and the reduced slow system. Next, in [96], asymptotic expansions were obtained for solving Riccati equations resulting in H_∞ control using game-theoretic approach and a composite controller was constructed based on the slow and fast subsystems and in [394] the H_∞ control of uncertain systems. Also, see [100] for a study on H_∞ control problem with Markovian jump parameters and for Markov chains and processes [35, 34, 2, 463].

For *parameter identification* see [58] where a new identification procedure using two-time structure of the system was proposed to provide a more accurate parameter estimation and for *uncertain or imperfectly known systems* see [144]. The H_2 guaranteed cost control problem for singularly perturbed norm bounded uncertain systems was addressed using quadratic stabilizability. Also, see [146, 147, 148, 82, 83, 38, 39].

Regarding **Adaptive Control**, see [183, 184, 185, 357, 186] for investigations on the effects of unmodeled high frequency dynamics and bounded disturbances on stability and performance of adaptive control systems including multiparameter models. Also, see [279] for a study on the performance and stability of discrete adaptive systems in the presence of fast parasitics.

See [168, 12] for addressing the robust asymptotic stability of a class of nonlinear singularly perturbed systems using **sliding-mode control** techniques.

The **numerical issues** associated with singularly perturbed systems are addressed in [251, 235, 192, 21]. Another interesting study was found in [171], which investigated the application of non-commutative computer algebra using Gröbner basis algorithms in analyzing the messy sets of non-commutative polynomial equations arising in singular perturbation analysis.

A computational singular perturbation (CSP) analysis was developed by [253, 252, 252, 16, 253], where the analysis does not depend upon the prior knowledge of the system behavior, but the time-scale character is evolved during a numerical simulation by neglecting the effect of a small parameter when its contribution is negligibly small during a numerical computation rather than simply neglecting the small parameter in the first place.

7 Applications of SPaTS

7.1 Aerospace

The theory of SPaTS has its roots in fluid dynamics and naturally found its wide applications in the area of aerospace systems besides electrical circuits and systems, machines and power systems. Here, in order to save space, we merely mention recent surveys [51, 316, 314, 317] and in particular the recent survey by the author [317] for a detailed exposition of applications of SPaTS in aerospace systems under the categories of

1. singular perturbations in mathematics and fluid dynamics,

2. brief history of SPaTS in aerospace systems,

3. method of matched asymptotic expansions,
4. selection of time-scales,
5. atmospheric flight,
6. pursuit-evasion and target interception,
7. digital flight control systems,
8. atmospheric entry,
9. satellite and interplanetary trajectories,
10. missiles,
11. launch vehicles and hypersonic flight, and
12. orbital transfer.

Also, see the research monograph [312] on optimal strategies in aeroassisted orbital transfer.

There are a number of other interesting and challenging applications of singular perturbation and time-scale methodologies in a variety of fields [448, 105, 411, 311]. Hence, some typical applications are briefly listed here.

7.2 Electrical and Electronics

For electrical circuits and systems see [73, 74] for a new approach based on the calculation of *infinitesimal deformations* to singular perturbation problems with application to bifurcation problems. Also, see [107] for a transformation of the original singularly perturbed system into a form where the normal constraint can be solved for the fast state and the resulting reduced system emerges without further manipulations and application to a set three electrical network problems. In [181], we find a study on jump behavior in an electrical network using geometric singular perturbation theory.

Regarding semiconductor device modeling and simulation, see [290, 289, 288, 420, 358, 359, 47, 421, 426, 339, 381]. For an interesting application on computer disk drive system [18].

For electrical machines and electric power systems see
[64, 182] for an overview on this area,
[66] for a time-scale approach to the decomposition and aggregation of dynamic networks with dense and sparse connections with an illustration of a 2000-node power network,
[324] the application to the discrete model of a fifth-order steam power system with reheat and [348] for a design of a periodic output feedback controller,
[164] for a multi-time-scale analysis of automatic gain control (AGC) of an interconnected power system and [444] for further analysis on interconnected systems,

[404, 405, 406, 407, 408, 447] for analysis of synchronous machines using time scale techniques,
[234] for a slow manifold concept as a tool for decomposition of a nonlinear system with an illustration of a synchronous machine,
[65] for designing a high-frequency filter for highly damped modes of a two-mass turbine-generator model. Next, in [88], a three-time scale method was presented in two stages of order reduction with an application to doubly fed synchronous machines,
[63] for application of a two-time-scale discrete-time model of a seventh-order synchronous machine,
[446] for a rigorous formulation of a time-varying phasor representation for the balanced three-phase large power system leading to singular perturbation behavior of the resulting dynamics. Also, see [356, 277, 223] for further results on this topic.

7.3 Structures and Mechanics

Another interesting area of the application of SPaTS is structural dynamics and control. In [380], "the deformed state of a thin, inextensible beam, which is under the action of axial and transverse loading and which also rests on an elastic foundation was considered by [117]. The asymptotic solution of a time-optimal, soft-constrained "cheap" control problem was obtained using a new approach solely based on expanding the controllability Grammian without resorting to the method of matched asymptotic expansion and the method was applied to the time-optimal single-axis rotation problem for a system consisting of a rigid hub with an elastic appendage due to an external torque applied at the hub [36].

Other works dealing with singular perturbations in structures are [118] for analysis of a singularly perturbed eigenvalue problem describing an elastic rod at the equilibrium state in the presence of a large-pulling out force with one end clamped and the other being free, where the singular perturbation parameter is identified as the inverse of the square root of the dimensionless parameter describing the pulling-out longitudinal force. Also see [438, 294, 190] for more works on this topic. For mechanical systems involving flexible dual rudder [81].

Singular perturbation concepts are exploited to develop a procedure for designing a constant gain, output feedback control system with application to a large space structure [53]. In this system, the third and fourth modes are approximately five time faster than the first and second modes, thus leading to the small parameter value as $\epsilon = 1/5$.

A singular perturbation analysis which relaxes the requirement on the boundary-layer system to stability (but not necessarily asymptotic stability, as required in the normal case) was provided with an application to a flexible dual rudder steering mechanism [81].

Recently, an interesting contribution to determine the underlying geometric structure of two-time scale, nonlinear optimal control systems was developed by Rao and Mease [354, 355] without requiring a priori knowledge of the singular perturbation structure. The methodology is based on splitting the Hamiltonian boundary-value problem (HBVP) into stable and unstable components using a dichotomic basis with an illustration via a mass connected to a nonlinear spring.

7.4 Robotics

In robotics, the singular perturbation parameter is usually identified as the inverse of a stiffness parameter associated with a flexible mode. For example, in a typical flexible slewing arm with a rigid body rotation and flexible "clamped mass" modes, one can select the quantity $\epsilon = (1/k_2)^{1/2}$ as the singular perturbation parameter, where k_2 is the stiffness parameter associated with the second flexible mode. Thus, the slow subsystem states are the joint angle, the first flexible modal displacement, and their respective rates, whereas the fast subsystem states are the second flexible modal displacement and its rate [399]. In particular we mention about space robotics [462] and tele-operation and space robotics [409], intelligent robotics systems for space exploration [90] and perturbation techniques for flexible manipulators [119] and robotics [285, 219, 220, 416, 417, 418, 399, 150, 54, 30].

7.5 Chemical Reactors

For singular perturbations in chemical reaction and reaction-diffusion convection and related topics, see [145] for a study stability aspects of a singularly perturbed reaction-diffusion system arising in a predator-prey interaction model and [178] for a study nonlinear degenerate diffusion equation arising in population dynamics in obtaining stationary solutions using the method of matched asymptotic expansion and geometric singular perturbation method. Also see [166, 40, 252, 253] for more works on this topic.

Related topics are *chemical reactors* [68, 69] where a biochemical continuous stirred tank reactor (CSTR) was analyzed, [70] for application to a fluidized catalytic cracker, and *nuclear reactors* [113] in which an estimation algorithm is presented using a joint state and parameter estimation technique akin to extended Kalman filter (EKF) for a sodium-cooled plutonium-uranium (Pu-U) metal-fueled fast-breeder reactor.

Further see [382] for examining quasi-steady-state assumption as a case study of singular perturbation in biochemistry with a biochemical reaction wherein an *enzyme* reacts reversibly with another chemical concentration *substrate* to form an enzyme-substrate complex.

7.6 Other Areas

Other applications are soil mechanics [91], celestial mechanics [326, 448, 410, 375], quantum mechanics [114], thermodynamics [175, 282, 360], thermoelasticity [112] where a uniform asymptotic expansion was given for a one-dimensional linear thermoelasticity in terms of the inertial constant assigned as the singular perturbation parameter, plates and shells [448, 342], elasticity [293, 112], lubrication [55], vibration [410], renewal processes [176], compressors [263], magnetohydrodynamics (MHD) [396], oceanography [172], welding [13], queuing theory [224], production inventory systems [45] and manufacturing [415, 189, 387], wave propagation [29], ionization of gases [173], lasers [106] where a nonlinear diffusion equation which models a laser-sustained plasma was analyzed using the method of matched asymptotic expansion, automobiles and biped locomotion [263, 163], agricultural engineering [427], reliability [243], 2D image modeling and processing [24, 25, 465, 194], ecology [322, 178], and biology [330].

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