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DISCRETE SINGULARLY PERTURBED CONTROL PROBLEMS (A SURVEY)

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This paper is dedicated to A. B. Vasil'eva.

Abstract. The paper presents the review of various types of discrete singularly perturbed control problems and methods for solving them. The bibliography containing 157 titles is included.

Keywords. Discrete optimal control problems, nonlinear and linear-quadratic problems, singular perturbations, weakly controllable systems, asymptotic expansions, motions decomposition, stabilization, game problems, stochastic systems, systems with a small step, descriptor systems.

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1 Introduction

Many problems from applied sciences lead to dynamical systems, where the state space variables have certain components which vary rapidly, and other which vary relatively slowly in time. Usually, such problems are studied within the framework of singular perturbations and integral manifolds. We will consider the singularity in a broad sense as the change of some qualitative characteristics of a perturbed problem if a small parameter equals zero.

The most part of publications devoted to singularly perturbed control problems deals with continuous systems while a lot of problems in economics, sociology, biology is described by discrete models. The another source of appearing discrete models is digital simulation of continuous systems where the differential equations are approximated by the corresponding difference equations. The study of sampled-data control systems and computer-based adaptive control systems leads in a natural way to the third source of discrete-time models. The control theory for discrete-time systems and sampled systems are receiving growing attention since the controllers are implemented by digital computers. If a dynamic system exhibits both continuous and discrete dynamic behavior, then it is called a hybrid system. Such system has the benefit of encompassing a larger class of systems within its structure, allowing for more flexibility in modeling dynamic phenomena.

In many situations, dynamic models consisting of a large number of difference equations can be derived from theoretical considerations. Often such models are so large that they may be impractical for many purposes, including simulation, control system design, and stability properties. Consequently, an approach to derive reduced order dynamic models from high order models is needed. Several methods for simplifying linear large-scale systems suggested in the literature have been listed in [16]. One of the most popular methods for model reduction of large-scale systems is the singular perturbation method. This technique has attracted much attention because of its simplicity and good performance in some experimental situations. The main advantage of the method is that it is applicable to nonlinear systems. Note that many physical systems are two-time scale systems, but do not appear in the form of a system with a small parameter. The major problem of the singular perturbation method is to group the state variables into slow and fast states and to select formally a small parameter.

A. B. Vasil'eva [138], [139] was perhaps the first to study solutions of discrete singularly perturbed dynamical systems using the asymptotic method of boundary-layer functions that has proved to be an effective tool in the analysis of singularly perturbed systems of ordinary differential equations. Note also the works [17], [142], [137], [23], [52], [111], [141] where perturbed discrete-time systems have been analyzed with the help of different methods. In [107], depending on the position of the small parameter, three state space discrete models are formulated and techniques are developed to obtain approximate series solutions. The idea of singular perturbation method for two-time scale discrete nonlinear systems is described, for instance, in [16] and [13].

Apparently, [4] is the first paper dealing with an asymptotic solution of a continuous singularly perturbed optimal control problem. Discrete singularly perturbed control problems have been the topic for the intensive research since the ending of the seventies of the last century (the corresponding references see, for instance, in [115], [140], [120], [107], [102], [8], [78], [29], [30], [155]).

Studying optimal control problems with a small parameter starts from solving the degenerate problem with the zero value of the parameter whenever possible. Sometimes this solution is enough. However, for the most of applied problems such approximation is too rough.

Two approaches are possible for constructing asymptotic solution of optimal control problems with a small parameter. In the first approach boundary value problems following from the control optimality conditions are used. This method is mostly applied. In this connection, either an asymptotics of a solution of boundary value problems is constructed immediately or, beforehand, a transformation that decouples a singularly perturbed system into a pure-slow and fast, reduced-order, systems is produced. The decomposition of systems may allow us to make computations in parallel. The second approach called the direct scheme consists of immediate substituting a postulated asymptotic expansion into a problem conditions and determining problems series for finding asymptotic terms. The variational nature of a original problem is taken into account in this case. The essential significance in the second approach is the possibility to prove, that for every fixed sufficiently small positive parameter, values of a minimized functional do not increase when a new approximation to an optimal control is used. Also, programme packages of solving optimal control problems can be applied for finding terms of asymptotic expansion of a solution. For linear-quadratic optimal control problems, asymptotic solution can also be constructed using the presentation of an optimal control in a feedback form and asymptotic solution of an appropriate discrete Riccati equations.

If in a constructed solution expansion with respect to powers of a small parameter ε we remove all terms of order higher than ε^n, we obtain the n-th order solution approximation. Under some conditions, it is possible to prove (see, for instance, [82]) that for sufficiently small $\varepsilon > 0$ the difference between a solution of the original perturbed problem and an approximation of this solution of the n-th order has the order ε^{n+1} and the difference between the value of the minimized functional evaluated for an approximation of an optimal control of the n-th order and the minimal value of this functional has the order $\varepsilon^{2(n+1)}$. Therefore the sequence of constructed approximations for an optimal control is minimizing for the considered performance index.

When a small parameter in a singularly perturbed state equation is equal to zero, we can sometimes obtain the system which is not resolved with respect to a state variable in a future time moment, so-called descriptor system. Hence, the study of singularly perturbed optimal control problems and control problems by descriptor systems are closely connected.

There are reviews of publications devoted to singularly perturbed control problems (see, for instance, [70], [104], [103], [29], [30], [155]), discrete problems in which occupy a negligible part. We should note here the overviews [106] (1987) and [34] (1995) where singular perturbations in discrete control systems are mainly considered. The focus in [106] is in three areas: modeling, analysis, and control. The authors of [34] consider four major classes of problems, commonly encountered in the control theory of discrete singularly perturbed dynamical systems: stabilization, design of a state identifier (observer) and control using phase state vector estimates, optimal quadratic performance criterion control, and robustness of discrete adaptive control systems towards minor dynamical imperfections.

The present survey deals with discrete singularly perturbed control problems only. Namely, we give the review of the publications where asymptotic solutions of discrete optimal control problems have either the form of expansions of boundary-layer type or regular expansions with respect to non-negative powers of a small parameter. Methods of decoupling motions into pure-slow and fast ones are also considered. Besides, we review publications devoted to the stabilization of discrete systems, control problems by discrete systems with a small step, descriptor and stochastic systems, game problems, and applications in various fields.

The part of this survey was presented at the 13th Viennese Workshop "Optimal Control and Dynamic Games" [77].

Everywhere in this paper, $\varepsilon \geq 0$ is a small parameter, N is a fixed natural number, the prime denotes the transposition, I means an identity matrix of any size, $diag(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, and $E_\varepsilon = diag(I, \varepsilon I)$. In all sections, except the ninth, state and control variables have values in some real finite-dimensional spaces.

The paper is organized as follows.

In the introduction, the role of the small parameter methods and especially methods of singular perturbations in studying of control problems is described. The history of using small parameter methods for studying discrete dynamical systems is also shortly presented.

The second section is devoted to the perturbed discrete control problems, where approximate solutions are found by means of asymptotic expansions, which are analogues to ones used in the method of boundary-layer functions for ordinary differential equations.

In the next section, the results of the publications are presented, where asymptotic solutions in the series form with respect to non-negative integer powers of a small parameter were constructed for nonlinear discrete periodic singularly perturbed and weakly controllable systems, using immediate substituting a postulated asymptotic expansion of a solution into a problem conditions and determining problems series for finding asymptotics terms.

The publications devoted the approximate decomposition methods are discussed in Section 4. The splitting transformations leading to an approximate decomposition into separate subsystems are considered.

The fifth section is devoted to stabilization problems of the discrete perturbed systems. The works on the stability analysis of discrete linear and nonlinear systems, including systems with delays, are considered. Asymptotic stabilizing composite feedback controls are presented.

The stochastic problems are considered in the sixth section. Here the questions of decomposition for singularly perturbed discrete linear-quadratic Gaussian control problems and Kalman's filter are discussed.

The works on the theory of dynamical games, where the discrete dynamic equations have fast and slow movements, are studied in section 7.

The theory of discrete singularly perturbed optimal control problems with a small step is discussed in the eighth section of this review.

The results for descriptor discrete problems are analyzed in the next section of the review. The papers connected with program controls and feedback are considered.

The tenth section deals with some results related to the sampling of continuous nonlinear singularly perturbed optimal control problems.

The overview of various results on singularly perturbed hybrid control systems, robust stabilization for discrete-time fuzzy singularly perturbed systems with parameter uncertainty, nonlinear dynamic systems driven by Markov chains, discrete minimax problems in the presence of regular and singular perturbations in the dynamics, and others is made in the 11-th section.

In the twelfth section, the review of papers, devoted to various applications of the theory of discrete singularly perturbed control problems, is given.

The thirteenth section contains the short conclusion.

2 Expansions of boundary-layer type

For finding asymptotic solutions of discrete singularly perturbed optimal control problems, the analog of Vasil'eva's boundary-layer functions method for continuous systems is very often used in order to construct asymptotic solutions of two-point boundary value problems following from control optimality conditions. Namely, the solution is sought in the form of a sum of three series with respect to powers of a small parameter ε, where the main term of one series, called the regular series, is a solution of a degenerate problem obtained from the original perturbed problem if we set $\varepsilon = 0$. Other two series are initial boundary-layer correction series and final boundary-layer correction series.

We explain the reason of the appearance of a boundary layer with the help of the following system considered, for instance, in [156]

$$\begin{aligned} x(k+1) &= f_1(x(k), \varepsilon y(k)) + g_1(x(k), \varepsilon y(k))u(k),\ x(0) = x^0, \\ y(k+1) &= f_2(x(k), \varepsilon y(k)) + g_2(x(k), \varepsilon y(k))u(k),\ y(0) = y^0. \end{aligned} \tag{1}$$

If $\varepsilon = 0$, we obtain the degenerate problem

$$\overline{x}(k+1) = f_1(\overline{x}(k), 0) + g_1(\overline{x}(k), 0)\overline{u}(k),\ \overline{x}(0) = x^0,$$

$$\overline{y}(k+1) = f_2(\overline{x}(k), 0) + g_2(\overline{x}(k), 0)\overline{u}(k).$$

It is straightforward to see that $\overline{x}$ starts from the same initial position as x, however $\overline{y}$ is not free to start from y^0. Therefore, the approximation of x by

$\overline{x}$ may be uniform for all $k \in N$, that is $x(k) = \overline{x}(k) + O(\varepsilon)$. By contrast the approximation

$$y(k) = \overline{y}(k) + O(\varepsilon) \tag{2}$$

holds only on an interval excluding 0, for $k \in \{k_1, k_1+1, ...\}$, where $k_1 > 0$. Boundary-layer correction terms have to be added to the solution $\overline{y}(k)$ for the approximation (2) to be valid over the entire interval.

The algorithm for constructing an asymptotic expansion of a solution $v(k,\varepsilon)$ of a two-point boundary value problem for the Hamiltonian system derived from control optimality conditions in the problem with time-invariant coefficients

$$\begin{pmatrix} x(k+1) \\ y(k+1) \end{pmatrix} = \begin{pmatrix} A_1 & \varepsilon A_2 \\ A_3 & \varepsilon A_4 \end{pmatrix} \begin{pmatrix} x(k) \\ y(k) \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u(k), \ k = \overline{0, N-1},$$

$$x(0) = x^0, \ y(0) = y^0,$$

$$J_\varepsilon(u) = \tfrac{1}{2} z'(N) F z(N) + \tfrac{1}{2} \sum_{k=0}^{N-1} (z'(k) W z(k) + u'(k) R u(k)) \to \min, \tag{3}$$

where $z'(k) = (x'(k), \varepsilon y'(k)),\ F,\ W \geq 0,\ R > 0$, is presented in [107], [102]. The asymptotics for $v(k,\varepsilon)$ has the form

$$v(k,\varepsilon) = \sum_{i \geq 0} \varepsilon^i (\bar{v}_i(k) + \varepsilon^k \Pi_i v(k) + \varepsilon^{N-k} Q_i v(k)),$$

where $\bar{v}_i(k)$ corresponds to the regular (outer) series, $\Pi_i v(k)$ corresponds to the initial boundary-layer correction series, and $Q_i v(k)$ corresponds to the final boundary-layer correction series.

The algorithm for constructing the asymptotic solution of the discrete Riccati equation of the form

$$P(k) = W + A' P(k+1) (I + B R^{-1} B' P(k+1))^{-1} A \tag{4}$$

with the final condition $P(N) = F$ appearing in describing the feedback for this problem is also given. Here

$$A = \begin{pmatrix} A_1 & \varepsilon A_2 \\ A_3 & \varepsilon A_4 \end{pmatrix}, \ B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

$$W = \begin{pmatrix} W_1 & \varepsilon W_2 \\ \varepsilon W_2' & \varepsilon^2 W_3 \end{pmatrix}, F = \begin{pmatrix} F_1 & \varepsilon F_2 \\ \varepsilon F_2' & \varepsilon^2 F_3 \end{pmatrix}.$$

In [63], a quadratic functional on a finite-time interval with time-varying matrices is minimized on trajectories of a singularly perturbed linear system of the type

$$E_\varepsilon \begin{pmatrix} x(k+1) \\ y(k+1) \end{pmatrix} = A(k,\varepsilon) \begin{pmatrix} x(k) \\ y(k) \end{pmatrix} + B(k,\varepsilon) u(k),$$

$$x(0) = x^0,\ y(0) = y^0.$$

Each of the relations between the discrete Riccati equations, the optimal controls, and the corresponding trajectories of the full system and the associated degenerate system is discussed, when a small positive parameter ε tends to zero.

Asymptotic solution of two-point boundary value problems for three-time-scale systems

$$\begin{pmatrix} x(k+1) \\ y(k+1) \\ \varepsilon z(k+1) \end{pmatrix} = \begin{pmatrix} A_1 & \varepsilon A_2 & A_3 \\ A_4 & \varepsilon A_5 & A_6 \\ A_7 & \varepsilon A_8 & A_9 \end{pmatrix} \begin{pmatrix} x(k) \\ y(k) \\ z(k) \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} u(k),$$

$$k = \overline{0, N-1},\ x(0) = x^0 \ or \ x(N) = x^N,\ y(0) = y^0,\ z(N) = z^N$$

is presented in [102], [107]. It is assumed that A_9 is non-singular.

Paper [68] deals with a class of initial and boundary value problems for multi-parameter linear time-invariant discrete equations. In this case, the approximate solution consists of an outer solution and a number of boundary layer correction solutions equal to the number of initial conditions lost in the process of degeneration.

The three types of boundary value problem arising in singularly perturbed discrete control systems with two small parameters are considered in [69]. Methods are developed to obtain approximate solutions. Suboptimal control of singularly perturbed two parameter discrete control system is also discussed in [65].

3 Regular expansions

Using the direct scheme, an asymptotic solution in a series form with respect to non-negative integer powers of a small parameter has been constructed under some conditions for the following three types of problems.

3.1 - nonlinear discrete periodic optimal control problem with a small parameter [79], [82]:

$$\begin{gathered} J_\varepsilon(u) = \sum_{k=0}^{N-1} F(k, y(k), \varepsilon z(k), u(k)) \to \min, \\ y(k+1) = f(k, y(k), \varepsilon z(k), u(k)), \\ z(k+1) = g(k, y(k), \varepsilon z(k), u(k)),\ k = \overline{0, N-1}, \\ y(0) = y(N), \quad z(0) = z(N). \end{gathered} \tag{5}$$

If $\varepsilon = 0$, then we obtain from (5) the degenerate problem, from which we can obtain the optimal control problem with a state variable of a smaller dimension than that of the original perturbed problem.

3.2 - nonlinear discrete singularly perturbed optimal control problem where slow state variables have a fixed left point and fast state variables satisfy the periodic condition [80]:

$$\begin{gathered} J_\varepsilon(u) = F(N, y(N)) + \sum_{k=0}^{N-1} F(k, y(k), z(k), u(k)) \to \min, \\ y(k+1) = f(k, y(k), z(k), u(k)), \\ \varepsilon z(k+1) = g(k, y(k), z(k), u(k)), \quad k = \overline{0, N-1}, \\ y(0) = y^0, \quad z(0) = z(N). \end{gathered} \tag{6}$$

It is assumed that the system $g(k, y(k), z(k), u(k)) = 0, k = \overline{0, N-1}$, is uniquely solvable with respect to $z(k)$ for any $y(k), u(k)$. If $\varepsilon = 0$ we can obtain from (6) the optimal control problem with the control variable $u(k)$ and the state variable $y(k)$. As in the previous problem, the dimension of the state variable is decreased.

3.3 - nonlinear discrete optimal control problem for a class of weakly controllable systems [81], [83]:

$$\begin{gathered} J_\varepsilon(u) = \sum_{k=0}^{N} F(k, x(k)) + \varepsilon \sum_{k=0}^{N-1} G(k, x(k), u(k)) \to \min, \\ x(k+1) = f(k, x(k)) + \varepsilon^p g(k, x(k), u(k)), \ k = \overline{0, N-1}, \\ x(0) = x^0, \end{gathered} \tag{7}$$

$p \geq 2$ is a natural number. For $p = 1$, the problem (7) has been studied in [74].

Control by small signals is encountered in controlled spacecrafts with low thrust (electronuclear engines, solar sail, etc.), in a variety of correction problems and in economics.

In the considered case, the degenerate problem

$$P_{-p} : x(k+1) = f(k, x(k)), \ k = \overline{0, N-1}, \ x(0) = x^0,$$

obtained from (7) with $\varepsilon = 0$, is uncontrollable.

Periodic conditions in problems (5), (6) and the absence of the additional conditions loss under the passage to the degenerate problem in (7) allow us to construct asymptotic expansions of solutions of these problems in a series form with respect to integer non-negative powers of ε.

In problems (5) and (6), we set $x = (y', z')'$. Asymptotic expansion of problems (5)-(7) solution is sought in the form

$$x(k) = \sum_{j \geq 0} \varepsilon^j x_j(k), \ k = \overline{0, N}, \quad u(k) = \sum_{j \geq 0} \varepsilon^j u_j(k), \ k = \overline{0, N-1}. \tag{8}$$

Substituting expansions (8) into problems (5)-(7) conditions we obtain an expansion of the minimized functional in series form with respect to integer non-negative powers of ε and relations for coefficients of expansions (8), from here, under some conditions, the problems P_j for finding expansions coefficients are explicitly defined.

For problems (5) and (6), the zero-order approximation of a solution is found from the degenerate problem P_0, and the j-th order approximations are found from linear-quadratic problems P_j. For problem (7), the problem P_j, where $j = \overline{-p, -1}$, is an initial problem from which x_{j+p} is found, the problem P_j, where $j = \overline{0, p-2}$, is reduced to an unconditional minimization problem for finding u_j and an initial problem for finding x_{j+p}. For defining the pair of functions (u_j, x_{j+p}), where $j \geq p-1$, linear-quadratic problems P_j are used.

The unique solvability of problems (5)-(7) in a neighborhood of the control u_0 has been proved. Estimates of the proximity of an approximate asymptotic solution to the exact one with respect to the control, trajectory and functional have been obtained. The non-increasing of minimized functional values with each new asymptotic approximation of optimal control has been established, i. e.

$$J_\varepsilon(\sum_{j=0}^{n} \varepsilon^j u_j(k)) \leq J_\varepsilon(\sum_{j=0}^{n-1} \varepsilon^j u_j(k)).$$

4 Asymptotic decomposition method

The essence of the asymptotic decomposition method used for asymptotic solving singularly perturbed optimal control problems consists of decoupling a system following from control optimality conditions into slow and fast subsystems.

At first, we present here a result from [143], [147] concerning a nonlinear discrete system

$$\begin{gathered} x(k+1) = x(k) + \varepsilon f(x(k), y(k), \varepsilon), \\ y(k+1) = g_0(x(k)) + D(x(k), \varepsilon) y(k) + \varepsilon g(x(k), y(k), \varepsilon),\ g(x(k), 0, \varepsilon) = 0, \end{gathered} \tag{9}$$

where $k = 0, 1, 2, ...$, $x(k)$ is a slow variable, $y(k)$ is a fast variable, eigenvalues of the matrix $D(x(k), 0)$ are situated inside of a unit circle. The existence of a splitting transformation of the form

$$x(k) = v(k) + \varepsilon H(v(k), z(k), \varepsilon), \quad y(k) = z(k) + h(x(k), \varepsilon), \tag{10}$$

reducing systems (9) to a "block-triangular" form

$$\begin{aligned} v(k+1) &= v(k) + \varepsilon F(v(k), \varepsilon), \\ z(k+1) &= G(v(k), z(k), \varepsilon), \quad G(v(k), 0, \varepsilon) = 0, \end{aligned} \tag{11}$$

with an independent slow subsystem is established.

Splitting transformation (10) can be found with an arbitrary order of accuracy in the asymptotic expansion form in powers of ε, i. e.

$$H(v(k), z(k), \varepsilon) = \sum_{j \geq 0} \varepsilon^j H_j(v(k), z(k)),\ h(x(k), \varepsilon) = \sum_{j \geq 0} \varepsilon^j h_j(x(k)).$$

Note that the L - transformation in [2] used for the decomposition of linear discrete systems is analogous to transformation (10).

Justification of the asymptotic decomposition method for nonlinear discrete systems with slow and fast variables of the form (9) is presented in [145].

In the papers [149] and [146], similar results are obtained for the discrete nonlinear system

$$\begin{aligned} x(k+1) &= Ax(k) + \varepsilon f(x(k), y(k), \varepsilon), \\ y(k+1) &= \varepsilon g(x(k), y(k), \varepsilon), \end{aligned} \tag{12}$$

where the matrix A is nonsingular. Using a splitting transformation of type (10) system (12) is reduced to a "block-triangular" form

$$v(k+1) = Av(k) + \varepsilon F(v(k), \varepsilon),$$

$$z(k+1) = \varepsilon G(v(k), z(k), \varepsilon), \quad G(v(k), 0, \varepsilon) = 0.$$

A mode-decoupling approach which yields two separate subsystems containing approximations of the zero order for the slow and fast parts is discussed in [14] for a system of the form

$$\begin{aligned} x(k+1) = f(k, x(k), \varepsilon y(k)),\ x(0) = x^0, \\ y(k+1) = g(k, x(k), \varepsilon y(k)),\ y(0) = y^0. \end{aligned} \tag{13}$$

An existence and smoothness result for center-like invariant manifolds of non-autonomous difference equations with slow and fast state-space variables is presented in [116].

The method of nonlinear systems decomposition has been used for splitting matrix discrete Riccati equations of the form (4) appearing under solving linear-quadratic optimal control problems on finite-time interval. See, for instance, [148] for $A = \begin{pmatrix} I + \varepsilon A_1 & \varepsilon A_2 \\ A_3 & A_4 \end{pmatrix}$, $B = \begin{pmatrix} \varepsilon B_1 \\ \varepsilon B_2 \end{pmatrix}$ and [146] for $A = \begin{pmatrix} A_1 & \varepsilon A_2 \\ \varepsilon A_3 & \varepsilon A_4 \end{pmatrix}$, $B = \begin{pmatrix} B_1 \\ \varepsilon B_2 \end{pmatrix}$. The paper [144], where optimal control and estimation problems are studied, also deals with the decomposition of matrix discrete Riccati equations.

Following to [90], in [43], linear-quadratic problems with a state equation, containing slow and fast variables, of the form

$$\begin{aligned} x(k+1) = (I + \varepsilon A_1)x(k) + \varepsilon A_2 y(k) + \varepsilon B_1 u(k),\ x(0) = x^0, \\ y(k+1) = A_3 x(k) + A_4 y(k) + B_2 u(k),\ y(0) = y^0 \end{aligned} \tag{14}$$

are studied with the performance indices (3), where $z'(k) = (x'(k), y'(k))$, $F = \begin{pmatrix} \frac{1}{\varepsilon}F_1 & F_2 \\ F_2' & F_3 \end{pmatrix} \geq 0$, and

$$J_\varepsilon(u) = \frac{\varepsilon}{2}\sum_{k=0}^{\infty}(z'(k)Wz(k) + u'(k)Ru(k)), \tag{15}$$

where $z'(k) = (x'(k), y'(k)), W = W' \geq 0, R = R' > 0$. The problem with infinite-time interval is considered in [89] for the nonstandard case, i.e. in contrast to [43] it is assumed that the matrix $A_4 - I$ is singular.

Under some assumptions, non-degenerate linear transformations are applied to decompose the linear Hamiltonian system obtained from the control optimality condition into reduced-order independent systems. For control on an infinite interval, a solution of the discrete algebraic matrix Riccati equation

$$P = W + A'PA - A'PB(R + B'PB)^{-1}B'PA,$$

where $P = \begin{pmatrix} \frac{1}{\varepsilon}P_1 & P_2 \\ P_2' & P_3 \end{pmatrix}$, $A = \begin{pmatrix} I + \varepsilon A_1 & \varepsilon A_2 \\ A_3 & A_4 \end{pmatrix}$, $B = \begin{pmatrix} \varepsilon B_1 \\ B_2 \end{pmatrix}$, expressing the optimal control as a feedback is found using two reduced order non-symmetric pure-slow and pure-fast algebraic Riccati equations.

Using the exact decomposition and the conversion of discrete algebraic Riccati equations into corresponding continuous-time equations, infinite-time optimal linear regulator problem for a linear discrete-time singularly perturbed system with performance index (15) is also studied in [8]. The research of problem (14), (15) by applying a bilinear transformation to the discrete algebraic Riccati equation is also described in [46]. A controller for system of form (14) is considered in [87].

The paper [150] deals with the application of the invariant manifold method to problem (14), (3), where $B_2 = O(\varepsilon)$ and $F = 0$. The transformation is constructed reducing the boundary value problem for a linear Hamiltonian system, following from the maximum principle, to a boundary value problem for slow variables and two initial value problems for fast variables. Splitting transformation is constructed in the form of asymptotic expansion with respect to non-negative integer powers of ε.

The regulator problem on an infinity interval with the state equation

$$\begin{aligned} x(k+1) &= f(x(k), \varepsilon y(k), u(k)),\ x(0) = x^0, \\ y(k+1) &= g(x(k), \varepsilon y(k), u(k)),\ y(0) = y^0 \end{aligned} \tag{16}$$

and with the performance index

$$J_\varepsilon(u) = \sum_{k=0}^{\infty} F(x(k), y(k), u(k)) \tag{17}$$

is considered in [15] and [13]. Composite feedback controls are proposed which are based on a decomposition of the model into reduced and boundary layer models.

A class of nonlinear discrete-time systems for which the control is designed using slow and fast sliding controllers is considered in [121]. In addition, a discrete-time observer is designed to estimate the nonmeasurable states required in control law.

5 Stability and stabilization

In the last years, the singularly perturbed discrete systems have received much attention for the stability analysis and controller design (see, for instance, [96], [112]).

We note here the paper [14] where the discrete version of well-known Tikhonov's theorem on singular perturbation for continuous-time systems is established for a discrete-time nonlinear system of form (13). Using the Lyapunov function both asymptotic and exponential stability of the equilibrium of system (13) are tested in [14] on the base of the analysis of a lower order slow and fast subsystems. See also [15].

In [114], the exponential stability of singularly perturbed discrete systems with time delay is investigated via Lyapunov's direct method. In term of the LMI (Linear Matrix Inequality), the sufficient condition for the exponential stability of linear systems is presented. Based on the linear result, the exponential stability of nonlinear systems with time-delay is also considered.

It is well-known that the accurate knowledge of the stability bound ε^* of a singularly perturbed system (i.e. the system is stable for $\varepsilon \in [0, \varepsilon^*]$) is very important for applications. The information of the upper bounds of the small parameter ε is obtained in [14] and [19].

In the paper [21], the exact stability bound for discrete multiple time-delay singularly perturbed system with constant coefficients

$$x(k+1) = \sum_{i=0}^{n} A_{1i} x(k-i) + \varepsilon \sum_{i=0}^{n} A_{2i} y(k-i),$$

$$y(k+1) = \sum_{i=0}^{n} A_{3i} x(k-i) + \varepsilon \sum_{i=0}^{n} A_{4i} y(k-i)$$

is examined. Instead of directly proving the relationship of stability between the original systems and the reduced systems, it is proved that for their equivalent models. Slow and fast subsystems are derived using the methods of the singular perturbation approach from [106], [102].

Robust stability bounds of linear discrete two-time scale systems are obtained in [94]. The possibility of presenting robust properties through the critical values of the parameter of singular perturbations is also discussed in [33] for linear continuous and discrete systems. Robustness problem of discrete multiple time-delay singularly perturbed systems is investigated in [136]). The paper [113] is devoted to the robust stability of nonstandard nonlinear singularly perturbed discrete systems with uncertainties. The Lyapunov function method is used. The stability analysis and robust controller design for uncertain discrete-time singularly perturbed systems are investigated via a matrix inequality approach in [131].

The paper [86] addresses the static output-feedback stabilization problem for a singularly perturbed discrete-time system. Three issues for this kind of problems are investigated in details. State feedback stabilization of a linear singularly perturbed system with several independent small parameters and discrete time is studied in [95]. The solution algorithm is based on decomposition of the problem into subproblems for slow and fast subsystems. The value domain of parameters within which the closed-loop system is asymptotically stable is estimated.

The conditions are derived in [67] under which a discrete singularly perturbed nonlinear dynamic system is exponentially stable uniformly with respect to a small parameter if its isolated subsystems defining fast and slow motions have the same property. Conditions are stated that guarantee stabilization in the class of feedbacks with respect to the part of the state variables corresponding slow motions.

The design of a stabilizing feedback control for linear singularly perturbed discrete-time systems is decomposed in [91] into the design of slow and fast controllers which are combined to form the composite control. Stabilizing problems for discrete systems via singular perturbation approach are also investigated in [60]. A composite observer-based control is presented in [53].

It is shown in [15] that the asymptotic stabilizing composite feedback control, proposed for problem (16), (17), produces a finite cost which tends to the optimal cost of a slow problem as the singular perturbation parameter tends to zero.

The paper [156] deals with nonlinear singularly perturbed discrete-time systems of the form (1), which decoupled into reduced order slow and fast boundary-layer subsystems. Considering these subsystems on infinite interval together with quadratic performance indices, where weight matrices depend on the states, nonlinear suboptimal controllers $u_s(k)$ and $u_f(k)$ are designed separately for the slow and fast subsystems using discrete-time matrix Riccati equation with state-dependent coefficients. Under some assumptions the local stability of the closed-loop system with composite controller $u_s(k) + \varepsilon^{k+1}u_f(k)$ as input is proved.

The stabilization of discrete singularly perturbed systems of the form

$$x(k+1) = \sum_{i=0}^{n} A_{1i}x(k-i) + \varepsilon \sum_{i=0}^{n} A_{2i}y(k-i) + B_1u(k),$$

$$y(k+1) = \sum_{i=0}^{n} A_{3i}x(k-i) + \varepsilon \sum_{i=0}^{n} A_{4i}y(k-i) + B_2u(k),$$

$$z(k) = C_1x(k) + C_2y(k)$$

is considered in [110]. The corresponding slow and fast subsystems of the original system are first derived. Then the state feedback controller for the slow and fast subsystems are separately designed and a composite stable feedback controller for the original system is after synthesized.

The D-stability problem is studied for discrete time-delay singularly perturbed systems in [54] where a system is called D-stable if the poles of the system are within the specific disk $D(a, r)$ centered at $(a, 0)$ with radius r, in which $|a| + r < 1$. Robust D-stability analysis of discrete uncertain time-delay systems by time-scale separation is presented in [109]. Note here the previous papers [84], [130], [56], and [55] connected with D-stability analysis.

6 Stochastic systems

An approach to the decomposition and approximation of linear-quadratic Gaussian control problems for singularly perturbed discrete systems at steady state is presented in [45]. The global Kalman filter is decomposed into separate reduced-order local filters through the use of a decoupling transformation. A near-optimal control law is derived by approximating coefficients of the optimal control law. The proposed method allows parallel processing of information.

Note also the papers [118] and [58] dealing with singularly perturbed discrete-time stochastic systems, in particular, with the Kalman filter.

We consider here the described in [43] singularly perturbed discrete linear stochastic system

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} I+\varepsilon A_1 & \varepsilon A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} \varepsilon B_1 \\ B_2 \end{pmatrix} u(k) +$$

$$+ \begin{pmatrix} \varepsilon G_1 \\ G_2 \end{pmatrix} w_1(k), \ y(k) = C_1 x_1(k) + C_2 x_2(k) + w_2(k)$$

with the performance index

$$J_\varepsilon(u) = \frac{\varepsilon}{2} E\{\sum_{k=0}^{\infty} (z'(k)z(k) + u'(k)Ru(k))\}, \ R > 0,$$

where $x_i(k), i = 1, 2,$ are slow and fast state vectors respectively, $u(t)$ is the control input, $y(k)$ is the observed output, $w_1(k)$ and $w_2(k)$ are independent zero-mean stationary Gaussian mutually uncorrelated white noise processes with intensities $W_1 > 0$ and $W_2 > 0$, respectively,

$$z(k) = D_1 x_1(k) + D_2 x_2(k)$$

is the controlled output. All matrices are of appropriate dimensions and assumed to be constant.

The optimal control law is given by

$$u(k) = -F\widehat{x}(k)$$

with the time-invariant filter

$$\widehat{x}(k+1) = A\widehat{x}(k) + Bu(k) + K(y(k) - C\widehat{x}(k)),$$

where $A = \begin{pmatrix} I + \varepsilon A_1 & \varepsilon A_2 \\ A_3 & A_4 \end{pmatrix}$, $B = \begin{pmatrix} \varepsilon B_1 \\ B_2 \end{pmatrix}$, $C = (C_1 \ C_2)$, $K = \begin{pmatrix} \varepsilon K_1 \\ K_2 \end{pmatrix}$.

The regulator gain F and filter gain K are obtained from

$$F = (R + B'P_RB)^{-1}B'P_RA,\ K = AP_FC'(W_2 + CP_FC')^{-1},$$

where P_R and P_F are, respectively, the positive semidefinite stabilizing solutions of the discrete-time algebraic regulator and filter Riccati equations, respectively, given by

$$P_R = D'D + A'P_RA - A'P_RB(R + B'P_RB)^{-1}B'P_RA,$$

$$P_F = AP_FA' - A'P_FC'(W_2 + CP_FC')^{-1}CP_FA' + GW_1G'$$

with $D = (D_1\ D_2),\ G = \begin{pmatrix} \varepsilon G_1 \\ G_2 \end{pmatrix}$.

Under some conditions, the exact decomposition method of the discrete-time algebraic regulator and filter Riccati equations presented in [43] (see also [88]) produces two sets of two reduced-order continuous-time nonsymmetric pure-slow and pure-fast algebraic Riccati equations. In addition, the optimal global Kalman filter is decomposed into pure-slow and pure-fast local optimal filters. It is shown that these two filters can be implemented independently in parallel in the different time scales. As a result, the optimal linear-quadratic Gaussian control problem for singularly perturbed linear discrete systems takes the complete decomposition and parallelism between optimal pure-slow and pure-fast filters and controllers.

The similar problem has been considered in [144] for finite time interval.

The same problem as in [43] is studied in [61], only in contrast to [43] the problem is nonstandard.

The recursive approach for obtaining parallel reduced-order controllers for stochastic linear singularly perturbed systems, based on the fixed-point iterations, has been developed in [44]. No analyticity requirements are imposed on the system coefficients, which is the standard assumption in the power series expansion method.

In [124], a singular perturbation approach is presented to study discrete systems with stochastic jump parameters. The feedback controller design is decomposed into the design of slow and fast controllers which are combined to form the composite control. Conditions for complete separation of slow and fast regulator designs are given. It is shown that the composite feedback control is $O(\varepsilon)$ close to the optimal one, which yields an $O(\varepsilon^2)$ approximation of optimal performance.

7 Game problems

In [135], a multi-step control system

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} I+\varepsilon A_1 & \varepsilon A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} \varepsilon B_1 & \varepsilon B_2 \\ B_3 & B_4 \end{pmatrix} \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix},$$

$$y(k) = C_1 x_1(k) + C_2 x_2(k),$$

where $x_1(k)$ is the slow variable, $x_2(k)$ is the fast variable, $u_1(k)$ and $u_2(k)$ are the controls of the first and second players, respectively, and all matrices in the preceding system are constant, is studied. The payoff function of the i-th player (i = 1, 2) is of the form

$$J_1 = -J_2 = \frac{1}{2}\sum_{k=0}^{\infty}(y'(k)y(k) + u_1'(k)R_1u_1(k) + u_2'(k)R_2u_2(k)).$$

The given system is split into a fast and a slow subsystems. Let $\{u_{1f}, u_{2f}\}$ and $\{u_{1s}, u_{2s}\}$ be the Nash equilibrium strategy profile for the fast and slow subsystems, respectively. The strategy profile $\{u_{1c}, u_{2c}\} = \{u_{1f} + u_{1s}, u_{2f} + u_{2s}\}$ is compared with the Nash equilibrium strategy profile of the given game $\{u_1^*, u_2^*\}$. It is proved that $u_{ic} = u_i^* + O(\varepsilon)$ and $J_{ic} = J_i^* + O(\varepsilon^2)$ $(i = 1, 2)$, where J_i^* is the payoff of the i-th player in the Nash equilibrium strategy profile of the initial game and J_{ic} is the payoff of the i-th player in the strategy profile $\{u_{1c}, u_{2c}\}$ if motions are split into fast and slow motions.

When the system contains slow and fast modes, the Stackelberg game is numerically stiff and ill-conditioned. It is necessary to find some methods to alleviate this ill-conditioning and to reduce the amount of computation. In [119], linear closed-loop Stackelberg strategies in sequential decision-making for linear time-invariant discrete systems with slow and fast modes are investigated. Near-optimal strategies that do not require any knowledge of the small singular perturbation parameter are derived. Linear-quadratic infinite-time Stackelberg game for discrete-time systems with slow and fast modes is studied in [100], [129]. A team-near optimal incentive strategy is constructed by solving a series of well-conditioned and reduced-order problems.

In [125], a singularly perturbed zero-sum dynamic game with full information has been considered. The upper (lower) value function of the dynamic game, in which the minimizer (maximizer) can be guaranteed if at the beginning of each interval his move (the choice of decision) precedes the move of the maximizer (minimizer), was introduced. It was shown that when the singular perturbations parameter tends to zero, the upper (lower) value function of the dynamic game has a limit which coincides with a viscosity solution of a Hamilton-Jacobi-Isaacs-type equation.

The paper [51] deals with a class of hybrid stochastic games with the piecewise open-loop information structure. These games are indexed over a parameter which represents the time-scale ratio between the stochastic (jump process) and the deterministic (differential state equation) parts of the dynamical system. The limit behavior of Nash equilibrium solutions of the hybrid stochastic games when the time-scale ratio tends to 0, is studied. It is also established that an approximate equilibrium can be obtained for the hybrid stochastic games using a Nash equilibrium solution of a reduced order sequential discrete-state stochastic game and a family of local deterministic infinite horizon open-loop differential games defined in the stretched out time-scale. A numerical illustration of this approximation scheme is also developed.

8 Discrete systems with a small step

For solving discrete problems with a small step, it is necessary a large number of calculations. Therefore the use of asymptotic methods in this case is very efficiency. After the first paper [138], devoted to discrete systems with a small step, asymptotic methods for the analysis of such systems have been applied under different conditions in [139], [17], [142], [111], [141]. In particular, the paper [17] deals with the system

$$x(t+\varepsilon) = A(t)x(t) + \varepsilon f(x(t), t)$$

where $t = 0, \varepsilon, 2\varepsilon, ...$ and the part of eigenvalues of the matrix $A(t)$ is equal to one. The system of the last form appears, for instance, in the theory of linear accelerators and discrete Markov chains.

Asymptotic solutions of two discrete linear-quadratic optimal control problems with a small step were first constructed in [50] by applying the technique of boundary-layer functions.

The first problem consists of minimization of the functional

$$c'x(1) + \frac{1}{2}\sum_{k=0}^{N-1}(x'(k\varepsilon)D(k\varepsilon)x(k\varepsilon) + u'(k\varepsilon)R(k\varepsilon)u(k\varepsilon))$$

on trajectories of the system

$$x(t+\varepsilon) = A(t)x(t) + B(t)u(t),\ t = k\varepsilon,\ x(0) = x_0,$$

where c is a constant vector, $k = \overline{0, N-1},\ N = 1/\varepsilon$. Two-point boundary value problem derived from the control optimality condition is used. An asymptotic solution of the considered problem has the form

$$z(t,\varepsilon) = \sum_{i\geq 0}\varepsilon^i(\bar{z}_i(t) + \Pi_i z(k) + Q_i z(k-N)),\ z = \begin{pmatrix} x \\ u \end{pmatrix},\ t = k\varepsilon,$$

where boundary-layer functions have the estimates

$$\| \Pi_i z(k) \| \leq a exp(-\alpha k/\varepsilon), \| Q_i z(k-N) \| \leq a exp(\alpha(k-N)/\varepsilon),\ a,\ \alpha > 0.$$

This problem is also considered in [27] under other conditions. In this connection, discrete matrix Riccati equation is used.

In the second problem, the linear term outside of the sum in the performance index is changed by a quadratic term $\frac{1}{2}x'(1)Fx(1)$ and a feedback control is studied.

For a discrete linear-quadratic optimal control problem with a small step and small coefficient in front of the sum in the criterion, an asymptotic expansion with respect to a small step for the solution of the corresponding discrete Riccati equation and quasi-optimal control are constructed in [48].

In [49], an asymptotic solution for a minimization problem of a quadratic quality criterion on trajectories of a linear discrete system of equations with a small step, fixed left and right points, belonging to some domain defined by linear inequalities, is constructed.

For the discrete nonlinear problem with a small step

$$J_\varepsilon(u) = G(x(T)) + \varepsilon \sum_{k=0}^{N-1} F(x(k\varepsilon), u(k\varepsilon), k\varepsilon) \to \min,$$

$$x(t+\varepsilon) = f(x(t), u(t), t),\ t = k\varepsilon,\ x(0) = x^0,\ k = \overline{0, N-1},\ T = N\varepsilon,$$

the asymptotic solution of the boundary-layer type is constructed by the direct scheme in [39], [28].

Note, that the second term in $J_\varepsilon(u)$ is an integral sum, therefore the limit problem has in this case the following form:

$$\overline{J}_0(\overline{x}_0, \overline{u}_0) = G(x_0^T) + \int_0^T F(\overline{x}_0(t), \overline{u}_0(t), t)\, dt \to \min,$$

$$\overline{x}_0(t) = f(\overline{x}_0(t), \overline{u}_0(t), t),\ x_0^T = arg \min G(x).$$

In [40], an asymptotic solution of a discrete linear-quadratic problem with a small step ε under a linear inequality-type terminal constraint is constructed by the direct scheme and then this asymptotics is used to find an ε-suboptimal admissible control of order $2(n+1)$, where n is the order of the asymptotics.

Discrete linear-quadratic problems with a cheap control have been studied in [41].

According to [42], if the proper motions in discrete time-varying linear systems with a small step are stable, complete controllability conditions are the same as for stationary systems.

9 Descriptor systems

An implicit discrete systems may appear in singularly perturbed control problems if a small parameter is neglected. Such systems have extensive applications in chemical engineering, telecommunications, and economical systems.

They form a topic for intensive research in the last three decades and are also referred to as descriptor or singular difference systems. The most part of publications is devoted to linear systems. A brief survey of the development of the theory of singular difference systems and singular stochastic difference systems (the solvability of initial value problems and multi-point boundary value problems, the stability and robust stability of solutions) is given in [1]. The reviews of publications devoted to control problems for discrete descriptor systems are presented, for instance, in [85] and [70]. See also the bibliographic index [38].

Control optimality conditions are obtained for discrete optimal control problems by descriptor systems both for program and feedback controls. We present here the results for two problems.

9.1. The papers [75], [76] deal with the problem of minimizing the quadratic functional

$$J(u) = \frac{1}{2}\langle x(N), Vx(N)\rangle + \frac{1}{2}\sum_{k=0}^{N-1}\langle \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}, \begin{pmatrix} W(k) & S(k) \\ S(k)^* & R(k) \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}\rangle \tag{18}$$

on trajectories of a descriptor system of the form

$$\begin{gathered} A(k+1)B(k+1)x(k+1) = C(k)x(k) + D(k)u(k),\ k = \overline{0, N-1}, \\ A(0)B(0)x(0) = z^0. \end{gathered} \tag{19}$$

Here $x(k) \in X$, $u(k) \in U$, $B(k) \in L(X, Y)$, $A(k) \in L(Y, Z)$, $C(k) \in L(X, Z)$, $D(k) \in L(U, Z)$, $V, W(k) \in L(X)$, $S(k) \in L(U, X)$, $R(k) \in L(U)$; X, Y, Z, U are real Hilbert spaces, z^0 is a given element from Z, $V = V^* \geq 0$, $W(k) = W(k)^*$, $R(k) = R(k)^*$, $\begin{pmatrix} W(k) & S(k) \\ S(k)^* & R(k) \end{pmatrix} \geq 0$, $\langle \cdot, \cdot \rangle$ means an inner product in appropriate spaces, the exponent star with the notation of an operator denotes the adjoint operator.

The control $u(k), k = \overline{0, N-1}$, is admissible, if there is a solution of problem (19).

If the control $u_*(k)$, $k = \overline{0, N-1}$, is a solution of the system

$$\begin{gathered} A(k+1)B(k+1)x(k+1) = C(k)x(k) + D(k)u(k), A(0)B(0)x(0) = z^0, \\ B(k)^*A(k)^*\psi(k) = -W(k)x(k) - S(k)u(k) + C(k)^*\psi(k+1), \\ B(N)^*A(N)^*\psi(N) = -Vx(N), \\ 0 = -S(k)^*x(k) + D(k)^*\psi(k+1) - R(k)u(k), k = \overline{0, N-1}, \end{gathered} \tag{20}$$

then $u_*(k)$ is an optimal control for problem (18), (19).

Under some conditions (see [76]) the implicit system (20), following from the control optimality condition, provides for the pair $(B(k)x(k), A(k)^*\psi(k))$ an explicit non-negative standard Hamiltonian system

$$\begin{pmatrix} B(k+1)x(k+1) \\ -A(k)^*\psi(k) \end{pmatrix} = E(k) \begin{pmatrix} B(k)x(k) \\ A(k+1)^*\psi(k+1) \end{pmatrix},\ k = \overline{0, N-1},$$

where $E(k) = \begin{pmatrix} E_1(k) & E_2(k) \\ E_3(k) & -E_1(k)^* \end{pmatrix}$, the operators $E_2(k)$, $E_3(k)$ are symmetric and nonnegative.

From the last statement, it follows that two-point boundary value problem (20) has a unique solution, hence the optimal control problem (18), (19) is solvable.

In contrast to the works [6], [101], [98], devoted to linear-quadratic discrete optimal control problems for standard descriptor systems ($B(k) \equiv I$ in (19)) with constant coefficients in a finite-dimensional case, the regularity of the pencil of the operators from the state equation is not required here. It is also not assumed that the system is causal.

Linear-quadratic discrete optimal control problems for descriptor systems with unbounded operator coefficients in Hilbert spaces have been studied for constant coefficients in [11] and for variable coefficients in [12]. The existence and uniqueness of an optimal control are established under certain restriction on the resolvent growth at infinity.

The sufficient control optimality conditions in the maximum principle form are given in [71] for a discrete descriptor system with values of controls in some set.

Necessary optimality conditions for linear-quadratic discrete control problem, where the state equation is a descriptor system of the order more than one, are given in [99].

9.2. For discrete linear-quadratic optimal control problem (18), (19) with $B(k) \equiv I$ we give the result from [72], [73] concerning an optimal feedback control.

If symmetric operators $K(k)$ are the solution of the problem

$$\begin{gathered} A(k)^* K(k) A(k) = W(k) + C(k)^* K(k+1) C(k) - \\ -(S(k) + C(k)^* K(k+1) D(k)) L(k) (S(k) + C(k)^* K(k+1) D(k))^*, \\ k = \overline{0, N-1},\ A(N)^* K(N) A(N) = V \end{gathered} \tag{21}$$

such that the operators

$$L(k)^{-1} = R(k) + D(k)^* K(k+1) D(k) \tag{22}$$

are positive definite and $x_*(k)$ is a solution of problem (19) with the control $u_*(k)$ given by the formula

$$u_*(k) = -L(k)(S(k)^* + D(k)^* K(k+1) C(k)) x_*(k)$$

then $u_*(k)$, $k = \overline{0, N-1}$, is an optimal control for the problem (18), (19) and the minimal value for the performance index is

$$J(u_*) = \frac{1}{2} \langle z^0, K(0) z^0 \rangle.$$

Similar results are obtained in [73] for the problem with fixed points, periodic problem and for the regulation problem with constant coefficients on an infinite interval.

In contrast to the papers [63], [6], [101], the regularity of the pencil of the operators from the state equation is not required here. In these three papers, an optimal control is determined with the help of the part of the state variable.

In general case, the implicit discrete operator Riccati equation (21) has no symmetric nonnegative definite solution though it may have a solution ensuring the positive definiteness of the operators $L(k)$ of form (22) (see the corresponding example in [73]). In [98], a nonnegative definite solution of equation (21) has been used.

The form of the relations defining the control in the feedback form is identical both for a singular state equation and for a nonsingular state e-quation. This is very convenient while studying singularly perturbed control problems.

Note here paper [132] which deals with H_∞-control for discrete-time singular delay systems of the form

$$Ax(k+1) = Cx(k) + C_d x(k-d) + D_1 u(k) + D_2 w(k),\ z(k) = Gx(k),$$

where $w(k)$ is a disturbance input, $d > 0$ is an integer representing a constant time delay. H_∞-control for discrete descriptor systems is also considered in [152], where a state feedback controller is designed satisfying a prescribed H_∞-norm-bound condition.

Stabilization of discrete descriptor systems is studied, for instance, in [153] and [151].

10 Discretization of continuous-time singularly perturbed problems

In applied sciences, investigations of the dynamical behavior of nonlinear evolutionary equations such as ordinary, functional, or partial differential equations are mainly computional. For that purpose problems are discretized. Discretization of different equations is presented, for instance, in [117]. In [107], p. 39-40, some aspects of singular perturbations in difference equations are examined along with those in differential equations. Difference schemes for singularly perturbed differential equations with turning points or interior layers are presented in [123].

In [63] and [60], a discrete approximation is presented to singularly perturbed linear, time-varying, continuous-time control systems. Discrete-time modeling of such type systems is also shortly discussed in [103] (see the references therein). Discretization schemes for nonlinear singularly perturbed continuous-time control systems, which are linear with respect to a control, can be found in, e.g., [5] and [121].

The authors of the paper [7] present a discretization scheme of two-time scale nonlinear continuous-time systems of the form

$$\frac{dx}{dt} = f(t, x, z, u, \varepsilon),\ x(0) = \eta(\varepsilon),$$

$$\varepsilon \frac{dz}{dt} = g(t, x, z, u, \varepsilon),\ z(0) = \zeta(\varepsilon),$$

$$y = h(t, x, z, \varepsilon).$$

This discretization scheme was adopted from Euler's methodology. It uses two periods, slow and fast, which are determined with respect to slow and fast variables rate, respectively. This scheme enables us to define and then to implement a multi-rate digital control for the considered type of systems. To this end, two reduced order observers are constructed for the slow and fast subsystems.

Note here the paper [26] dealing with the digital control of a particular class of nonlinear singularly perturbed systems which are linear with respect to a fast variable and control. Three distinct multi-rate composite control strategies are discussed. The method used in this paper is a nonlinear extension of the composite technique used for the linear case in [91].

State-feedback H_∞-control problem for linear singularly perturbed systems with norm-bounded uncertainties

$$E_\varepsilon \frac{dx(t)}{dt} = (A + H\Delta F_0)x(t) + (B_1 + H\Delta F_1)w(t) + (B_2 + H\Delta F_2)u(t),$$

$$z(t) = Cx(t) + Du(t),$$

where $u(t)$ is the control input, $w(t)$ is the exogenous disturbance signal, Δ is an uncertain time-varying matrix, is studied in [37]. The fast variables are sampled with fast rates, while for the slow variables both cases of slow and fast sampling are considered. The closed-loop system is represented as a continuous one with time-varying input delay.

11 Other control problems

The stability of hybrid control systems singularly perturbed by fast but continuous actuators is analyzed in [122]. The paper [35] deals with a class of stochastic optimal control problems involving two different time scales. The fast mode of the system is represented by deterministic state equations whereas the slow mode of the system corresponds to a jump disturbance process. The authors show how an approximate optimal control law can be constructed from the solution of the limit control problem.

Based on the stability analysis of both continuous-time and discrete-time fuzzy singularly perturbed systems, stabilizing feedback controller gains are designed in [93]. The reduced-control law, which is only dependent on the slow variables, is also discussed.

The authors of the paper [20] investigate the problem of the state feedback robust stabilization for discrete-time fuzzy singularly perturbed systems with parameter uncertainty, where the i-th rule is formulated as follows:

$$x(k+1) = E_\varepsilon(A_i + \Delta A_i)x(k) + E_\varepsilon B_i u(k),$$

where the matrix ΔA_i represents a time-varying uncertain matrix.

In [127] and [154], the delta-operator approach is applied for obtaining a singularly perturbed unified model which can be used in both continuous-time domain and discrete-time domain. A method of designing a state feedback gain achieving a specified insensitivity of the closed-loop trajectory is proposed in [127]. The condition to find the robust controller is derived for a fuzzy singularly perturbed unified model [154].

The H_∞ - optimal control and risk-sensitive control of linear singularly perturbed discrete-time systems is described in [105].

Two methods for designing state feedback H_∞-controllers for standard discrete-time singularly perturbed systems with polytopic uncertainties in terms of solutions of a set of linear matrix inequalities are considered in [31]. Moreover, a method of evaluating the upper bound of singular perturbation parameter is given (see also [32], [92] for more results regarding to H_∞-control problems of discrete singularly perturbed systems).

Paper [128] deals with an unified approach to H_∞-optimal control of singularly perturbed systems. The key contribution of the paper is to present continuous-time and discrete-time singularly perturbed cases simultaneously under general imperfect state measurements using infinite-horizon formulations. Note that an unified approach is also applied for linear-quadratic regulator design in [126].

The design of a mixed H_2/H_∞-linear state variable feedback suboptimal controller for a discrete-time singularly perturbed system

$$x(k+1) = Ax(k) + B_1u(k) + B_2w(k),$$

$$z(k) = Cx(k) + Du(k),$$

where $x(k) = (x_1(k)', x_2(k)')'$, $A = \begin{pmatrix} I + \varepsilon A_{11} & \varepsilon A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $B_1 = (\varepsilon B'_{11}\ B'_{21})'$, $B_2 = (\varepsilon B'_{12}\ B'_{22})'$, $C = (C_1\ C_2)$, u is the control input, w is the disturbance input, is described in [25] using reduced order slow and fast subsystems. It is shown that the designed controller based on reduced order models and the corresponding performance index both are $O(\varepsilon)$ close to those synthesized using the full order system.

The authors of [108] study the output feedback sliding mode control problem for a sampled data linear system with disturbances. By taking into account the disturbance compensation, a deadbeat high gain output feedback control strategy with additional dynamics is able to attenuate the disturbances. It is shown that the closed loop system exhibits both singularly perturbed and weakly coupled characteristics.

We note here the paper [18], where a sub-optimal controller is designed for nonstandard discrete-time singularly perturbed systems through LMI (Linear Matrix Inequality) methodology. It is different from the general fast-slow decomposition method.

In [101], a discrete minimax problem is considered in the presence of regular and singular perturbations in the dynamics.

The paper [3] is devoted to near-optimal controls of large-scale discrete-time nonlinear dynamic systems driven by Markov chains. The underlying problem is to minimize an expected cost function.

Numerically-asymptotic method of solving discrete optimal control problem with fast and slow variables of the form

$$
\begin{gathered}
J_\varepsilon(u) = \Phi(x(N)) \to \min\nolimits_{u \in U}, \\
x(k+1) = x(k) + \varepsilon(X(k, x(k), y(k)) + A(x(k))\phi(k, u(k)),\ x(0) = x^0, \\
y(k+1) = Y(k, x(k), y(k)),\ y(0) = y^0, \\
k = \overline{0, N-1},
\end{gathered}
$$

is described and justified in [10]. An asymptotically optimal control in the original problem is found by the optimal control in the averaging problem.

Discrete singularly perturbed linear-quadratic problems are recursively solved in [44] and [47]. In particular, a recursive algorithm is developed for the discrete singularly perturbed output feedback stochastic control problem. Nonlinear algebraic matrix equations are decomposed to ones corresponding to slow and fast modes, so that only low-order systems are involved in algebraic computations. Moreover, such a decomposition removes the ill-conditioning of the higher order system. The proposed algorithm gives the accuracy of $O(\varepsilon^n)$, where ε is a small perturbation parameter and n is the number of iterations.

The controllability and observability of linear discrete-time systems satisfying the two-time-scale property are considered in [134]. A method is proposed for designing two state feedback gains which are used for separate placement of slow and fast eigenvalues. This method is based on the separability of the slow and fast controls.

12 Applications

The monographs [66], [102] and the overviews [104], [103], [29], [155] contain the survey of publications devoted to singularly perturbed continuous problems arising in fluid dynamics, electrical circuits and machines, electrical power systems, aerospace systems, chemical reactions, diffusion and reactor control systems, biology and biochemistry, and other applications. Numerous references are given.

We present here some applications of the discrete singularly perturbed control problems theory.

The efficiency of the recursive method for solving discrete singularly perturbed linear-quadratic control problems, studied in [44] and [47], is demonstrated in these monographs with examples of the discrete model of an aircraft F-8 and a fifth order discrete model for a steam power system. Some results concerning the investigation of the discretized model of an F-8 aircraft are also presented in [46], [8], and [18]. The stabilization bounds for a fifth-order discrete model of a steam power system obtained by three design procedures are given in [86]. The effectiveness of the proposed in [87] controller is illustrated in this paper by the same discrete model for a steam power system.

The problem of the form (14), (15) for the discrete linearized model of an F-15 aircraft has been solved in [43] using the pure-slow and pure-fast algebraic Riccati equations. There is the ideal proximity with the exact solution.

The L-1011 aircraft model is used by the authors of [108] for sampled-data output feedback sliding-mode control design.

Time-scale analysis and synthesis control methodology for continuous-time model predictive control is presented in [157]. In this method, low-order slow and fast subsystems are derived from a higher-order plant with a two-time-scale character which is a wind energy conversion system with permanent magnet synchronous generators. Then slow and fast subcontrollers are designed separately and a composite model predictive control is obtained. The discrete-time model predictive controller is designed using the discretized wind energy conversion system with a sample interval. The performance of the discrete-time model predictive controller is compared with that of the continuous-time model predictive controllers.

The authors of the paper [122] demonstrate the effect of the fast actuator dynamics on hybrid feedback algorithms that, in the absence of actuator dynamics, globally asymptotically stabilize the inverted position of a pendulum on a cart and the position and orientation of a mobile robot, respectively.

The problem of fault diagnosis of discrete singularly perturbed systems is studied in [133] by designing residuals based on reduced slow subsystems. Two examples of applications are given to demonstrate the efficiency of the proposed method. The first of them deals with a real laboratory two tanks system with single input and single output. In the second one, a discrete-time singularly perturbed system with two outputs and one input is considered.

A reduced-order model for control design applications of a seventh-order two-time-scale discrete-time model of synchronous machines is developed in [22]. A tracking control law based on sliding-mode principles and singular perturbation techniques, which allows to track a reference rotor angle for a synchronous generator, is proposed in [121].

Obtained in [20] results, concerning the state feedback robust stabilization for discrete-time fuzzy singularly perturbed systems with parameter uncertainty, are applied in this paper for a tunnel diode circuit. It is shown that the proposed method is effective in improving the upper bound of a small parameter.

Many control schemes for dc-dc converters begin with the assertion that induced currents are "fast" states and capacitor voltage are "slow" states. This assertion must be true for power factor correction converters to allow independent control of current and voltage. In [62], separation criteria were derived for boost, buck, and buck-boost controllers in both continuous and discrete-time formulations.

The zero-order approximation for a asymptotic solution of a discrete linear-quadratic problem of DC motor control with low inductance is constructed in [57].

Using the boundary-layer function method, the second order asymptotic solution of a time-invariant discrete linear-quadratic optimal control problem for the third order power system model is constructed in [64]. The comparison of optimal solutions of a discrete linear-quadratic optimal control problem and suboptimal asymptotic solutions of the orders up two is given in [65] for a sampled fifth order power system model with two small parameters.

The algorithm of design of observers and stabilizing feedback controllers for singularly perturbed discrete systems, proposed in [59], is applied in this paper to a ninth-order boiler model.

The algorithm for constructing an uniform asymptotic approximation of the zero-order to an optimal control leading to balanced growth trajectories for a discrete model with a small step combining properties of the dynamical models by Leontief and by Neumann is proposed in [24].

The paper [36] deals with a two-factor production model where one factor is characterized by a continuous state variable. It corresponds to the fast mode of the system in the form of a controlled diffusion process. The second production factor is characterized by a discrete variable. It corresponds to the slow mode of the system in the form of a controlled jump process. The authors define the limit-control problem approximated numerically and implement a numerical technique, using approximating Markov chains and observe the claimed convergence for the numerical solutions.

13 Conclusion

The present paper is the first review of publications for the last years entirely devoted to study discrete control systems with the help of perturbations theory approaches.

Note that reviews [103] and [155] cover mainly the works in English and almost do not reflect the works, ideas and approaches of the Russian-speaking authors. While prof. Vasil'eva A.B. (Moscow State University) and her followers in the former USSR and Russia were the first to study the discrete dynamic systems with the help of singular perturbations theory in 60-70s of the last century.

This overview is constructed in such a way that in each section several papers are selected that most clearly demonstrate the subject of this section.

The authors apologize for any omission of references that should have been included in this survey. They very hope that this paper will be useful for readers dealing with discrete systems containing parameters.

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References

[1] P. K. Anh, N. H. Du, and L. C. Loi, Singular difference equations: an overview, *Vietnam J. of Mathematics*, **35**, (2007), no. 4, 339-372.

[2] B. D. O. Anderson, R. R. Bitmead, C. R. Johnson, Jr., P. V. Kokotovic, R. L. Kosut, I. M. Y. Mareels, L. Praly, and B. D. Riedle, Stability of Adaptive Systems: Passivity and Averaging Analysis, the MIT Press Cambridge, Massachusetts, London, England, 1986.

[3] G. Badowski, G. Yin, and Q. Zhang, Near-optimal controls of discrete-time dynamic systems driven by singularly-perturbed markov chains, *J. of Optimization Theory and Applications*, **116**, (2003), iss. 1, 131-166.

[4] N. Kh. Bagirova, A. B. Vasil'eva, and M. I. Imanaliev, On an asymptotic solution of an optimal control problem, *Differentsial'nye Uravneniya (Differential Equations)*, **3**, (1967), no. 11, 1895-1902 (in Russian).

[5] J. P. Barbot, S. Monaco, D. Normand-Cyrot, and N. Pantalos, Discretization schemes for nonlinear singularly perturbed systems, *Proc. of the 30th Conference on Decision and Control*, Brighton, England, December 11-13, 1991, pp. 443-448.

[6] D. J. Bender and A. J. Laub, The linear-quadratic optimal regulator for descriptor systems: discrete-time case, *Automatica*, **23**, (1987) 71-85.

[7] M. Bidani and M. Djemaï, A multirate digital control via a discrete-time observer for non-linear singularly perturbed continuous-time systems, *Int. J. Control*, **75**, (2002), no. 8, 591-613.

[8] M. Bidani, N. E. Radhy, and B. Bensassi, Optimal control of discrete-time singularly perturbed systems, *Int. J. Control*, **75**, (2002), no. 13, 955-966.

[9] G. L. Blankenship, Singularly perturbed difference equations in optimal control problems, *IEEE Trans. Automat. Control*, **26**, (1981) 911-917.

[10] I. A. Boitsova, Numerically-asymptotic method of solving discrete optimal control problems with fast and slow variables, *Vestnik BGU. Seriya 1, Fizika, Matematika, Informatika (Bulletin of BSU. Series 1, Physics, Mathematics, Informatics)*, (2011), no. 1, 105-110 (in Russian).

[11] M. F. Bondarenko and L. A. Vlasenko, A linear quadratic regulator problem for descriptor lumped and distributed systems with discrete time, *Problemy Upravleniya i Informatiki (Problems of Control and Informatics)*, (2010), no. 1, 76-85 (in Russian): Engl. transl.: *J. of Automation and Information Sciences*, **42**, (2010), iss. 1, 32-41.

[12] M. F. Bondarenko, L. A. Vlasenko, and A. G. Rutkas, Discrete optimal control of descriptor systems with variable parameters, *Problemy Upravleniya i Informatiki (Problems of Control and Informatics)*, (2011), no. 3, 5-12 (in Russian); English transl.: *J. of Automation and Information Sciences*, **43**, (2011), iss. 5, 1-9.

[13] R. Bouyekhf, A. El Hami, and A. El Moudni, Optimal control of a particular class of singularly perturbed nonlinear discrete-time systems, *IEEE Trans. Automat. Control*, **46**, (2001), no. 7, 1097-1101.

[14] R. Bouyekhf and A. El Moudni, On analysis of discrete singularly perturbed nonlinear systems: Application to the study of stability properties, *J. Franklin Inst.*, **334**, (1997), no. 2, 199-212.

[15] R. Bouyekhf and A. El Moudni, Stabilization and regulation of class of non-linear singularly perturbed discrete-time systems, *J. Franklin Inst.*, **335**, (1998), no. 5, 963-982.

[16] R. Bouyekhf, A. El Moudni, A. El Hami, N. Zerhouni, and M. Ferney, Reduced order modelling of two-time-scale discrete non-linear systems, *J. Franklin Inst.*, **333**, (1996), no. 4, 499-512.

[17] V. F. Butuzov and A. B. Vasil'eva, Differential and difference equation systems with a small parameter for the case in which the unperturbed (degenerate) system is in the spectrum, *Differentsial'nye Uravneniya*, **6**, (1970), no. 4, 650-664 (in Russian); English transl.: *Differential Equations*, **6**, (1971) 499-510.

[18] C. Cai and Y. Zou, Sub-optimal linear quadratic control for non-standard discrete-time singularly perturbed systems, *8th International Conference on Control, Automation, Robotics and Vision*, Kunming, China, 6-9th December 2004, **3**, (2004), pp. 1649-1653.

[19] S.-J. Chen and J.-L. Lin, Maximal stability bounds of discrete-time singularly perturbed systems, *Control and Cybernetics*, **33**, (2004), no. 1, 95-108.

[20] J. Chen, F. Sun, Y. Yin, and C. Hu, State feedback robust stabilization for discrete-time fuzzy singularly perturbed systems with parameter uncertainty, *IET Control Theory Appl.*, **5**, (2011), iss. 10, 1195-1202.

[21] J.-S. Chiou, Stability bound of discrete multiple time-delay singularly perturbed systems, *Int. J. Systems Sci.*, **37**, (2006), no. 14, 1069-1076.

[22] G. K. Choudhary, N. P. Singh, and Y. P. Singh, Two-time-scale discrete-time modelling and decomposition of synchronous machines, *Int. J. Systems Sci.*, **26**, (1995), iss. 5, 1029-1039.

[23] C. Comstock and G. C. Hsiao, Singular perturbation for difference equation, *Rocky Mountain J. Math.*, **6**, (1976) 561-567.

[24] Y. E. Danik and M. G. Dmitriev, Magistral trajectories in economics and singular perturbations, *Trudy ISA RAN (Proc. ISA RAS)*, **65**, (2015), no. 1, 61-68 (in Russian).

[25] K. B. Datta and A. RaiChaudhuri, H_2/H_∞ control of discrete singularly perturbed systems: the state feedback case, *Automatica*, **38**, (2002) 1791-1797.

[26] M. Djemai, J. P. Barbot, and H. K. Khalil, Digital multirate control for a class of non-linear singularly perturbed systems, *Int. J. Control*, **72**, (1999), no. 10, 851-865.

[27] M. G. Dmitriev, A boundary layer in optimal control problems, *Izv. Akad. Nauk SSSR, Tekhn. Kibernet.*, (1983), no. 4, 63-69 (in Russian); English transl.: *Engineering Cybernetics*, **21**, (1984), no. 3, 134-140.

[28] M. G. Dmitriev, S. V. Belokopytov, and M. A. Gaipov, An asymptotic solution of a nonlinear discrete optimal control problem with a small step without control constraints (Proof) II, *Izv. Akad. Nauk TSSR, FTKh i GN*, (1990), no. 2, 10-18 (in Russian).

[29] M. G. Dmitriev and G. A. Kurina, Singular perturbations in control problems, *Avtomatika i Telemekhanika*, (2006), no. 1, 3-51 (in Russian); English transl.: *Automation and Remote Control*, **67**, (2006), no. 1, 1-43.

[30] M. G. Dmitriev and G. A. Kurina, On some singularly perturbed optimal control problems, *Problemy Ustoichivosti i Upravleniya/ Sbornik nauchnyh statei, posvyashchennyi 80-letiyu akademika Vladimira Mefod'evicha Matrosova (Problems of Stability and Control / Collection of Papers Dedicated to the 80th Anniversary of Academician V.M. Matrosov)*, Fizmatlit, Moscow (2013), 148-160 (in Russian).

[31] J. Dong and G.-H. Yang, Robust H_∞ control for standard discrete-time singularly perturbed systems, *IET Control Theory Applications*, **1**, (2007), no. 4, 1141-1148.

[32] J. Dong and G.-H. Yang, H_∞ control for fast sampling discrete-time singularly perturbed systems, *Automatica*, **44**, (2008) 1385-1393.

[33] S. A. Dubovik and A. A. Kabanov, A measure of stability against singular perturbations and robust properties of linear systems, *Problemy Upravleniya i Informatiki (Problems of Control and Informatics)*, (2010), no. 3, 17-28 (in Russian); English transl.: *J. of Automation and Information Sciences*, **42**, (2010), iss. 6, 55-66.

[34] S. V. Emelyanov, S. K. Korovin, and I. G. Mamedov, Discrete singularly perturbed dynamical systems, *Variable Structure Control Systems: Discrete and Digital*, CRC Press, Boca Raton, FL, (1995) 177-232.

[35] J. A. Filar, V. Gaitsgory, and A. B. Haurie, Control of singularly perturbed hybrid stochastic systems, *IEEE Trans. Automat. Control*, **46**, (2001), no. 2, 179-190.

[36] J. A. Filar and A. Haurie, A two-factor stochastic production model with two time scales, *Automatica*, **37**, (2001), iss. 10, 1505-1513.

[37] E. Fridman, Robust sampled-data H_∞ control of linear singularly perturbed systems, *IEEE Trans. Automat. Control*, **51**, (2006), no. 3, 470-475.

[38] R. Gabasov, F. M. Kirillova, and I. K. Asmykovich, Deskriptornye Sistemy Upravleniya. Bibliograf. Ukazatel' (Descriptor Control Systems: A Bibliographic Index), AN BSSR, Institut Matematiki, Minsk, 1988 (in Russian).

[39] M. A. Gaipov, Asymptotics of the solution of a nonlinear discrete optimal control problem with small step without constraints on the control (formalism) I, *Izv. Akad. Nauk TSSR, FTKh i GN*, (1990), no. 1, 9-16 (in Russian).

[40] M. A. Gaipov, An asymptotic solution of a discrete singularly perturbed optimal control problem under phase constraints, *Informatika i Sistemnyi Analiz (Informatics and System Analysis)*, Ashkhabad, 1990, pp. 43-53 (in Russian).

[41] M. A. Gaipov, Asymptotics of Solutions of Some Discrete Optimal Control Problems with Singularly Perturbed Constraints, PhD Thesis, Irkutsk State University, Russia, 1992 (in Russian).

[42] M. A. Gaipov and M. G. Dmitriev, On controllability of a linear discrete nonstationary system with a small step, *Izv. Ross. Akad. Nauk, Teor. Sist. Upravlen.*, (1998), no. 2, 44-46 (in Russian); English transl.: *J. of Computer and Systems Sciences International*, **37**, (1998), no. 2, 204-206.

[43] Z. Gajic and M. Lim, Optimal Control of Singularly Perturbed Linear Systems and Applications. High-Accuracy Techniques, Marcel Dekker, New York, 2000.

[44] Z. Gajic, D. Petkovski, and X. Shen, Singularly Perturbed and Weakly Coupled Linear Control Systems: A Recursive Approach, Lect. Notes Control Inform. Sci., **140**, Springer, 1990.

[45] Z. Gajic and X. Shen, Parallel reduced-order controllers for stochastic linear singularly perturbed discrete systems, *IEEE Trans. Automat. Control*, **36**, (1991), no. 1, 87-90.

[46] Z. Gajic and X.-M. Shen, Study of the discrete singularly perturbed linear-quadratic contol problem by a bilinear transformation, *Automatica*, **27**, (1991), iss. 6, 1025-1028.

[47] Z. Gajić and X. Shen, Parallel Algorithms for Optimal Control of Large Scale Linear Systems, Springer-Verlag, London Limited, 1993.

[48] V. Ya. Glizer, A diference optimal control problem with a small step, *Differentsial'nye Uravneniya (Differential Equations)*, **21**, (1985), no. 8, 1440-1442 (in Russian).

[49] V. Ya. Glizer, An asymptotic solution of a difference optimal control problem with a small step and moving trajectory right end, *Differentsial'nye Uravneniya (Differential Equations)*, **24**, (1988), no. 8, 1457-1459 (in Russian).

[50] V. Ya. Glizer and M. G. Dmitriev, Asymptotics of solution of some discrete optimal control problems with a small step, *Differentsial'nye Uravneniya*, **15**, (1979), no. 9, 1681-1691 (in Russian); English transl.: *Differential Equations*, **15**, (1979), no. 9, 1200-1207.

[51] A. Haurie and F. Moresino, Singularly perturbed piecewise deterministic games, *SIAM J. on Control and Optimization*, **47**, (2008), no. 1, 73-91.

[52] F. C. Hoppensteadt and W. L. Miranker, Multitime methods for systems of difference equations, *Stud. Appl. Math.*, **56**, (1976/77), no. 3, 273-289.

[53] F.-H. Hsiao, J.-D. Hwang, and S.-T. Pan, Stabilization of discrete singularly perturbed systems under composite observer-based control, *J. of Dyn. Syst., Meas. and Contr.*, **123**, (2001), no. 1, 132-139.

[54] F.-H. Hsiao, J.-D. Hwang, and S.-T. Pan, D-stability problem of discrete singularly perturbed systems, *Int. J. Systems Sci.*, **34**, (2003), iss. 3, 227-236.

[55] F. H. Hsiao, S. T. Pan, and J. D. Hwang, D-stabilization control for discrete-time two-time-scale systems, *IEEE Control Systems Magazine*, **20**, (2000) 56-66.

[56] F.-H. Hsiao, S.-T. Pan, and C.-C. Teng, An efficient algorithm for finding the D-stability bound of discrete singularly perturbed systems with multiple time delays, *Int. J. of Control*, **72**, (1999), iss. 1, 1-17.

[57] A. A. Kabanov, Synthesis of terminal control by discrete singularly perturbed systems, *Avtomatizatsiya protsessov i upravlenie (Automation of processes and control)*, (2009), iss. 95, 141-145 (in Russian).

[58] H. Kando, State estimation of stochastic singularly perturbed discrete-time systems, *Optimal Control Applications and Methods*, **18**, (1997), iss. 1, 15-28.

[59] H. Kando and T. Iwazumi, Design of observers and stabilising feedback controllers for singularly perturbed discrete systems, *IEE Proceedings*, **132**, (1985), no. 1, 1-10.

[60] H. Kando and T. Iwazumi, State feedback controller design of two-time-scale discrete systems via singular perturbation approach, *Int. J. Systems Sci.*, **18**, (1987), no. 8, 1537-1554.

[61] B.-S. Kim, Y.-J. Kim, and M.-T. Lim, LQG control for nonstandard singularly perturbed discrete time systems, *Proc. of the 41st IEEE Conference on Design and Control*, Las Vegas, Nevada USA, December 10-13, 2002, pp. 3736-3741.

[62] J. W. Kimball and P. T. Krein, Singular perturbation theory for DC-DC converters and applications to PFC converters, *IEEE Trans. Power Electronics*, **23**, (2008), no. 6, 2970-2981.

[63] M. Kimura, On the matrix Riccati equation for a singularly perturbed linear discrete control system, *Int. J. Control*, **38**, (1983), no. 5, 959-975.

[64] G. Kishore Babu and M. S. Krishnarayalu, Suboptimal control of singularly perturbed discrete control system, *Int. Electrical Engineering J.*, **5**, (2014), no. 5, 1413-1419.

[65] G. Kishore Babu and M. S. Krishnarayalu, Suboptimal control of singularly perturbed two parameter discrete control system, *Int. Electrical Engineering J.*, **5**, (2014), no. 11, 1594-1604.

[66] P. V. Kokotović, H. K. Khalil, and J. O'Reilly, Singular Perturbation Methods in Control. Analysis and Design, Academic Press, London, 1986.

[67] S. K. Korovin, I. G. Mamedov, and I. P. Mamedova, Uniform stability relatively to a small parameter and stabilization of discrete singularly perturbed dynamic systems, *Izv. Akad. Nauk SSSR, Tekhn. Kibern.*, (1989), no. 1, 21-29 (in Russian).

[68] M. S. Krishnarayalu, Singular perturbation methods for a class of initial and boundary value problems in multiparameter classical digital control systems, *ANZIAM J.*, **46**, (2004), no. 1, 67-77.

[69] M. Krishnarayalu and D. S. Naidu, Singular perturbation method for boundary value problems in two-parameter discrete control systems, *Int. J. Systems Sci.*, **19**, (1988), no. 10, 2131-2143.

[70] G. A. Kurina, Singular perturbations of control problems with equation of state not solved for the derivative (A survey), *Izv. RAN. Tekhn. Kibern.*, (1992), no. 4, 20-48 (in Russian); English transl.: *J. Comput. and Syst. Sci. Int.*, **31**, (1993), no. 6, 17-45.

[71] G. A. Kurina, On sufficient control optimality conditions for discrete descriptor systems, *Avtomatika i Telemekhanika*, (1993), no. 8, 52-55 (in Russian); English transl.: *Automation and Remote Control*, **54**, (1993), no. 8, 1223-1226.

[72] G. Kurina, Linear-quadratic optimal control problems for discrete descriptor systems, *System Structure and Control, 1st IFAC/IEEE Symposium on System Structure and Control*, Prague, Czech Republic, August 27-31, **1**, 2001, pp. 261-266.

[73] G. A. Kurina, Feedback control for discrete descriptor systems, *Syst. Sci.*, **28**, (2002), no. 2, 29-40.

[74] G. A. Kurina, Asymptotics of solution of optimal control problems for discrete weakly controllable systems, *Prikladnaya Matematika i Mekhanika*, **66**, (2002), no. 2, 214-227 (in Russian); English transl.: *J. Appl. Maths. Mechs.*, **66**, (2002), no. 2, 201-213.

[75] G. A. Kurina, Solvability of linear-quadratic discrete optimal control problems for descriptor system in Hilbert space, *Proc. 10th IEEE Mediterranean Conference on Control and Automation*, Instituto Superior Tećnico, July 9-12, Lisboa, Portugal, 2002.

[76] G. A. Kurina, Linear-quadratic discrete optimal control problems for descriptor systems in Hilbert space, *J. Dyn. Control Syst.*, **10**, (2004), no. 3, 365-375.

[77] G. A. Kurina, M. G. Dmitriev, and D. S. Naidu, Discrete singularly perturbed optimal control problems, *13th Viennese Workshop Optimal Control and Dynamic Games. May 13th-May 16th, 2015*, Vienna University of Technology, 2015, p. 126.

[78] G. A. Kurina and E. Yu. Dolgopolova, Singulyarnye Vozmushcheniya v Zadachakh Upravleniya. Bibliograf. Ukazatel' (1982-2002) (Singular Perturbations in Control Problems: A Bibliographic Index for 1982-2002), VGLTA, Voronezh, 2004 (in Russian).

[79] G. Kurina and N. Nekrasova, Asymptotic solution of singularly perturbed nonlinear discrete periodic optimal control problem, *Proc. of the 44th IEEE Conference on Decision and Control, and the European Control Conference, CDC-ECC'05, 2005*, Seville, Spain, December 12-15, 2005, pp. 3531-3536.

[80] G. A. Kurina and N. V. Nekrasova, Asymptotics of solution for a class of singularly perturbed discrete optimal control problems, *Vestnik Voronezhskogo Gosudarstvennogo Universiteta, Seriya: Fizika. Matematika* (Bulletin of Voronezh State University, Series: Physics and Mathematics), (2006), no. 2, 222-229 (in Russian).

[81] G. Kurina and N. Nekrasova, Asymptotic solution of control problems for discrete weakly controllable systems, *Preprints of the 3rd IFAC Symposium on System, Structure and Control SSSC07*, Foz do Iguassu, Brazil, October 17-19th, 2007, pp. 1/6-6/6.

[82] G. Kurina and N. Nekrasova, Asymptotics of solution for singularly perturbed nonlinear discrete periodic optimal control problems, *Systems Science*, **34**, (2008), no. 1, 23-37.

[83] G. A. Kurina and N. V. Nekrasova, Asymptotic solution of a discrete optimal control problem for one class of weakly controllable systems, *Izvestiya RAN. Teoriya i Systemy Upravleniya*, (2009), no. 1, 34-44 (in Russian); English transl.: *J. of Computer and Systems Sciences International*, **48**, (2009), iss. 1, 30-40.

[84] C. H. Lee, T. H. S. Li, and F. C. Kung, D-stability analysis for discrete systems with a time delay, *Systems and Control Letters*, **19**, (1992) 213-219.

[85] F. L. Lewis, A survey of linear singular systems, *Circuits, Systems and Signal Processing*, **5**, (1986), no. 1, 3-36.

[86] T-H. S. Li and J.-H. Li, Optimal static output feedback stabilization of singularly perturbed discrete-time systems, *IMA J. Math. Control Info.*, **11**, (1994), iss. 3, 213-230.

[87] J.-H. Li and T-H. S. Li, Discrete two-time-scale dynamic output feedback control, *Proc. of the Amerlcan Control Conference*, Seattle, Washington, June 1995, pp. 2895-2899.

[88] M. T. Lim, Z. Gajic, and X. Shen, New methods for optimal control and filtering of singularly perturbed linear discrete stochastic systems, *Proc. of the American Control Conference*, Seattle, Washington, USA, June 21-23, **1**, 1995, pp. 534-538.

[89] M.-T. Lim, C.-G. Kang, and B.-S. Kim, Optimal control of linear nonstandard singularly perturbed discrete systems, *Dyn. of Continuous, Discret. and Impuls. Syst., Series B: Applications and Algorithms*, **9**, (2002) 163-174.

[90] B. Litkouhi and H. Khalil, Infinite-time regulators for singularly perturbed difference equations, *Int. J. Control*, **39**, (1984), no. 3, 587-598.

[91] B. Litkouhi and H. K. Khalil, Multirate and composite control of two-time-scale discrete-time systems, *IEEE Trans. Automat. Control*, **30**, (1985), no. 7, 645-651.

[92] D. Liu, L. Liu, and Y. Yang, H_∞ control of discrete-time singularly perturbed systems via static output feedback, *Abstract and Applied Analysis*, (2013), Article ID 528695, 1-9, http://dx.doi.org/10.1155/2013/528695.

[93] H. P. Liu, F.C. Sun, and Z. Q. Sun, Stability analysis and synthesis of fuzzy singularly perturbed systems, *IEEE Trans. Fuzzy Systems*, **13**, (2005) 273-284.

[94] C. Loescharataramdee and M. E. Sawan, Stability robustness bounds of discrete two-time-scale systems, *J. of the Franklin Institute*, **336**, (1999) 973-981.

[95] M. S. Mahmoud, Stabilization of discrete systems with multiple-time scales, *IEEE Trans. Automat. Control*, **31**, (1986), no. 2, 159 -162.

[96] I. Malloci, J. Daafouz, and C. Iung, Stability and stabilization of two time scale switched systems in discrete time. *IEEE Trans. Automat. Control*, **55**, (2010), no. 6, 1434-1438.

[97] G. P. Mantas and N. J. Krikelis, Linear quadratic optimal control for discrete descriptor systems, *J. Optim. Theory Appl.*, **61**, (1989) 221-241.

[98] V. L. Mehrmann,The Autonomous Linear Quadratic Problem. Theory and Numerical Solution, Lect. Notes Control Inform. Sci., **163**, Springer, 1991.

[99] V. Mehrmann and L. Scholz, Self-conjugate differential and difference operators arising in the optimal control of descriptor systems, *Operators and Matrices*, **8**, (2014), no. 3, 659-682.

[100] K. Mizukami and H. Xu, A reduced-order method to design incentive strategy for the discrete system with slow and fast modes, *American Control Conference*, Boston, USA, June 26-28, 1991, pp. 1252-1257.

[101] S. K. Myshkov and V. V. Karelin, Minimax control in the singularly perturbed linear-quadratic stabilization problem, *"Stability and Control Processes" in Memory of V.I. Zubov (SCP), 2015 Int. Conference*, St. Petersburg, Russia, October 5-9, 2015, pp. 328-331.

[102] D. S. Naidu, Singular Perturbation Methodology in Control Systems, IEE Control Eng. ser., **34**, Peter Peregrinus Ltd., London, United Kingdom, 1988.

[103] D. S. Naidu, Singular perturbations and time scales in control theory and applications: An overview, *Dyn. of Continuous, Discret. and Impuls. Syst., Series B: Applications and Algorithms*, **9**, (2002) 233-278.

[104] D. S. Naidu and A. J. Calise, Singular perturbations and time scales in guidance and control of aerospace systems: a survey, *J. Guidance, Control and Dynamics*, **24**, (2001), no. 6, 1057-1078.

[105] D. S. Naidu, C. D. Charalambous, K. L. Moore, and M. A. Abdelrahman, H_∞-optimal control of singularly perturbed discrete-time systems and risk-sensitive control, *Proc. of the 33rd IEEE Conference on Decision and Control*, Lake Buena Vista, Florida, USA, December 14-16, **2**, 1994, pp. 1706-1711.

[106] D. S. Naidu, D. B. Price, and J. L. Hibey, Singular perturbations and time-scales (SP-TS) in discrete control systems - an overview, *Proc. of the 26th IEEE Conference on Decision and Control*, Los Angeles, CA, USA, December 9-11, 1987, pp. 2096-2103.

[107] D. S. Naidu and A. K. Rao, Singular Perturbation Analysis of Discrete Control Systems, Lect. Notes Math., **1154**, Springer-Verlag, Berlin, 1985.

[108] T. Nguyen, W. C. Su, and Z. Gajic, Singular perturbation analysis of discrete-time output feedback sliding mode control with disturbance attennuation, *2009 American Control Conference*, St. Louis, MO, USA, June 10-12, 2009, pp. 757-762.

[109] S.-T. Pan, The order reduction and robust D-stability analysis of discrete uncertain time-delay systems by time-scale separation, *Control and Cybernetics*, **32**, (2003), no. 4, 743-760.

[110] S. J. Pan and J. S.-H. Tsai, The application of genetic algorithm on control of discrete multiple time-delay singularly perturbed systems, *Eighth Int. Conference on Intelligent Systems Design and Applications*, Kaohsiung, Taiwan, November 26-28, 2008, pp. 531-536.

[111] I. S. Panchenko and B. G. Razumeiko, On asymptotic behaviour of solutions of difference boundary value problem where the step is a small parameter, *Differentsial'nye Uravneniya (Differential Equations)*, **13**, (1977), no. 4, 718-723 (in Russian).

[112] K. S. Park and J. T. Lim, Stability analysis of nonstandard nonlinear singularly perturbed discrete systems, *IEEE Trans. Circuits and Systems II: Express Briefs*, **58**, (2011), no. 5, 309-313.

[113] K. S. Park and J. T. Lim, Robust stability of non-standard nonlinear singularly perturbed discrete systems with uncertainties, *Int. J. Systems Sci.*, **45**, (2014), iss. 3, 616-624.

[114] K. S. Park and J. T. Lim, Exponential stability of singularly perturbed discrete systems with time-delay, *Int. J. Innovative Computing, Information and Control*, **9**, (2013), no. 2, 865-874.

[115] R. G. Philips, Reduced order modelling and control of two-time scale discrete systems, *Int J. Control*, **31**, (1980), no. 4, 765-780.

[116] C. Pötzsche, Slow and fast variables in non-autonomous difference equations, *J. Difference Equations and Applications*, **9**, (2003), no. 5, 473-487.

[117] C. Pötzsche, Geometric Theory of Discrete Nonautonomous Dynamical Systems, *Lect. Notes Math.*, **2002**, Springer-Verlag, Berlin, Heidelberg, 2010.

[118] A. Rao and D. Naidu, Singular perturbation method for Kalman filter in discrete systems, *IEE Proc., Part D*, **131**, (1984) 39-46.

[119] K. Rehnamai, H. Zarabadipour, and M. E. Sawan, Stakelberg strategies for discrete-time systems with slow and fast modes, *Fifth Int. Conf. on Systems Engineering*, Wichita State Univ., USA, September 9-11, 1987, pp. 139-150.

[120] V. R. Saksena, J. O$'$Reilly, and P. V. Kokotovic, Singular perturbations and time-scale methods in control theory: survey 1976 - 1983, *Automatica*, **20**, (1984), no. 3, 273-293.

[121] A. E. Sanchez-Orta, J. De León-Morales, and E. Lopes-Toledo, Discrete-time nonlinear control scheme for small synchronous generator, *Electrical Power and Energy Systems*, **24**, (2002) 751-764.

[122] R. G. Sanfelice and A. R. Teel, On singular perturbations due to fast actuators in hybrid control systems, *Automatica*, **47**, (2011), iss. 4, 692-701.

[123] K. K. Sharma, P. Rai, and K. C. Patidar, A review on singularly perturbed differential equations with turning points and interior layers, *Appl. Math. Comput.*, **219**, (2013) 10575-10609.

[124] X. Shen, V.-G. Gourishankar, Q. Xia, and M. Rao, Composite control of discrete singularly perturbed systems with stochastic jump parameters, *J. Franklin Inst.*, **331**, (1994), no. 2, 217-227.

[125] P. Shi, Limit Hamilton-Jacobi-Isaacs equations for singularly perturbed zero-sum dynamic (discrete time) games, *SIAM J. on Control and Optimization*, **41**, (2002), iss. 3, 826-850.

[126] K.-H. Shim and M. E. Sawan, Linear-quadratic regulator design for singularly perturbed systems by unified approach using δ operators, *Int. J. Systems Sci.*, **32**, (2001), iss. 9, 1119-1125.

[127] K.-H. Shim and M. E. Sawan, Singularly perturbed unified time systems with low sensitivity to model reduction using delta operators, *Int. J. Systems Sci.*, **37**, (2006), iss. 4, 243-251.

[128] H. Singh, R. H. Brown, and D. S. Naidu, Unified approach to H_∞-optimal control of singularly perturbed systems: imperfect state measurements, *Proc. of the American Control Conference*, San Diego, CA, USA, June 02-04, 1999, **4**, pp. 2909-2913.

[129] H. Singh and Magdi S. Mahmoud, Stackelberg strategies for singularly perturbed linear quadratic problems in discrete-time systems, *J. of Systems Analysis and Modelling Simulation*, **39**, (2001), iss. 4, 479-492.

[130] T. J. Su and W. J. Shyr, Robust D-stability for linear uncertain discrete time-delay systems, *IEEE Trans. Automat. Control*, **39**, (1994) 425-428.

[131] F. Sun, Y. Hu, and H. Liu, Stability analysis and robust controller design for uncertain discrete-time singularly perturbed systems, *Dyn. of Continuous, Discret. and Impuls. Systems, Series B: Applications and Algorithms*, **12**, (2005), no. 5-6, 849-865.

[132] X. Sun, Q. Zhang, C. Yang, Z. Su, and Y. Shao, H_∞ control for discrete-time singular delay systems, *2010 Chinese Control and Decision Conference*, Xuzhou, Chaina, 26-28 May, 2010, pp. 1075-1080.

[133] A. Tellili, N. Abdelkrim, B. Jaouadi, and M. N. Abdelkrim, Diagnosis of discrete-time singularly perturbed systems based on slow subsystem, *Acta Mechanica et Automatica*, **8**, (2014), no. 4, 175-180.

[134] M. T. Tran and M. E. Sawan, Controllability, observability and eigenvalue placement for discrete-time systems with slow and fast modes, *Proc. of the American Control Conference*, San Francisco, CA, USA, June 22-24, 1983, pp. 439-440.

[135] M. T. Tran and M. E. Sawan, Nash Strategies for discrete-time systems with slow and fast modes, *Int. J. Systems Sci.*, **14**, (1983), no. 12, 1355-1371.

[136] H. Trinh and M. Aldeen, Robust stability of singularly perturbed discrete-delay systems, *IEEE Trans. Automat. Control*, **40**, (1995), no. 9, 1620-1623.

[137] G. A. Tsyganov, Asymptotics of Solution of Difference Systems of Conditionally Stable Type. Ph. D. Thesis, University of People's Friendship of Patrice Lumumba, Moscow, Russia, 1973 (in Russian).

[138] A. B. Vasil'eva, On correspondence between some properties of solutions of linear difference systems and systems of ordinary linear differential equations, *Trudy Seminara po Teorii Differentsial'nyh Uravnenii s Otklonyayushchimsya Argumentom (Proc. of Seminar on Theory of Differential Equations with Deviating Argument)*, **5**, (1967), 21-44 (in Russian).

[139] A. B. Vasil'eva, On difference systems near to differential difference one, *Trudy Seminara po Teorii Differentsial'nyh Uravnenii s Otklonyayushchimsya Argumentom (Proc. of Seminar on Theory of Differential Equations with Deviating Argument)*, **6**, (1968), 37-43 (in Russian).

[140] A. B. Vasil'eva and M. G. Dmitriev, Singular perturbations in optimal control problems, Itogi Nauki i Tekhniki, Mat. Analiz, **20**, (1982), 3-77 (in Russian); English transl.: *J. Sov. Math.*, **34**, (1986), no. 3, 1579-1629.

[141] A. B. Vasil'eva and M. V. Faminskaya, Boundary value problem for singularly perturbed differential and difference systems when an unperturbed system is on the spectrum. *Differenstial'nye Uravneniya (Differential Equations)*, **13**, (1977), no. 4, 738-742 (in Russian).

[142] A. B. Vasil'eva and E. Yuldashev, Linear difference systems with a small delay, *Differenstial'nye Uravneniya (Differential Equations)*, **6**, (1970), no. 12, 2267-2269 (in Russian).

[143] N. V. Voropaeva, Decomposition of discrete systems wth slow and fast variables, *Izvestiya RAEN, Seriya MMMIU (Bulletin of RANS, MMMIU Series)*, **2**, (1998), no. 4, 142-149 (in Russian).

[144] N. V. Voropaeva, Decomposition of problems of optimal control and estimation for discrete systems with fast and slow variables, *Avtomatika i Telemekhanika*, (2008), no. 6, 15-25 (in Russian); English transl.: *Automation and Remote Control*, **69**, (2008), no. 6, 920-928.

[145] N. V. Voropaeva, Decomposition of discrete nonlinear systems with fast and slow variables, *Vestnik SamGU. Estestvennonauchnaya seriya (Bulletin of SSU. The Natural Science Series)*, (2010), no. 4 (78), 28-41 (in Russian).

[146] N. V. Voropaeva, Decomposition of control problems for multi-tempo systems with discrete-time, *XII Vserossiiskoe Soveshchanie po problemam upravleniya VSPU-2014 (XII Russian Meeting on Control Problems)*, Moscow, Russia, June 16-19, 2014, pp. 842-848 (in Russian).

[147] N. V. Voropaeva and V. A. Sobolev, Decomposition of Multi-tempo Systems, SMS, Samara, 2000 (in Russian).

[148] N. V. Voropaeva, and V. A. Sobolev, Decomposition of multi-tempo discrete control systems, *Mehatronika, Avtomatizatsiya, Upravlenie (Mechatronics, Automation, Control)*, (2004), no. 8, 2-6 (in Russian).

[149] N. V. Voropaeva and V. A. Sobolev, Decomposition of motions in discrete multitempo control systems, *Optimizatsiya, Upravlenie, Intellect (Optimization, Control, Intelligence)*, (2005), no. 2 (10), 58-64 (in Russian).

[150] N. V. Voropaeva and V. A. Sobolev, Decomposition of a linear-quadratic optimal control problem with fast and slow variables, *Avtomatika i Telemekhanika*, (2006), no. 8, 3-11 (in Russian); English transl.: *Automation and Remote Contol*, **67**, (2006), no. 8, 1185-1193.

[151] X.-H. Xia and W.-Q. Wang, Decentralized stabilization of discrete singular largescale systems with non-causality, *J. of Nanjing Univ. Sci. and Technol. Natur. Sci.*, (2006), no.1, 65-69 (in Chinese).

[152] S. Xu and C. Yang, H_∞ state feedback control for discrete singular systems, *IEEE Trans. Automat. Control*, **45**, (2000), no. 7, 1405-1409.

[153] B. Yao, Q.-L. Zhang, and Y.-P. Chen, Robust quadratic stability and stabilization with integrity for uncertain discrete singular systems, *Facta Universitas. Ser. Mech. Eng.*, **2**, (2004), no. 1, 25 - 34.

[154] Y. Yuan, F. Sun, H. Liu, and B. Xu, Robust control for fuzzy singular perturbed unified model, *American Control Conference*, Montréal, Canada, June 27-29, 2012, pp. 5628-5633.

[155] Y. Zhang, D. S. Naidu, C. Cai, and Y. Zou, Singular perturbation and time scales in control theories and applications: An overview 2002-2012, *Int. J. Information and Systems Sciences*, **9**, (2014), no. 1, 1-36.

[156] Y. Zhang, D. S. Naidu, C. Cai, and Y. Zou, Composite control of a class of nonlinear singularly perturbed discrete time systems via D-SDRE, *Int. J. Systems Sci.*, **47**, (2016), iss. 11, 2632-2641.

[157] Y. Zhang, H. Nguyen, D. S. Naidu, Y. Zou, and C. X. Cai, Time scale analysis and synthesis for model predictive control, *WSEAS Trans. Systems and Control*, **9**, (2014) 130-139.

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