2. Cross Product (or vector product)

- The cross product is a way of multiplying two vectors together which yields a third vector (perpendicular to the plane spanned by the original two vectors).

- We denote the cross product as: \( \vec{A} \times \vec{B} = \vec{C} \)

- Magnitude in the geometric representation: as with the dot product, decompose one vector into a component parallel to the other vector and a component perpendicular.

\[
\vec{A} = \vec{A}_{\parallel} + \vec{A}_{\perp}
\]

this time, however, we multiply the magnitudes of \( \vec{B} \) and \( \vec{A}_{\perp} \) (rather than \( \vec{A}_{\parallel} \), as in the dot product). This gives the magnitude of the cross product between \( \vec{A} \) and \( \vec{B} \):

\[
|\vec{A} \times \vec{B}| = |\vec{A}_{\perp}| |\vec{B}| = (A \sin \phi) B = AB \sin \phi
\]

(\( \because \) from above: \( \vec{A}_{\perp} \Rightarrow \vec{A}_1 = A \sin \phi \))
Alternatively: \[ \vec{B} = \vec{B}_\parallel + \vec{B}_\perp \]

\( \vec{B}_\parallel \) parallel to \( \vec{A} \)
\( \vec{B}_\perp \) perpendicular to \( \vec{A} \)

\[ \vec{B}_\parallel \]
\[ \vec{B}_\perp \]
\[ \phi \]
\[ \vec{B} \]

\[ \Rightarrow \]

\[ \begin{align*}
B_\parallel & = B \sin \phi \\
\uparrow & \\
B_\perp & = B \cos \phi
\end{align*} \]

\[ |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}_\perp| = A (B \sin \phi) = AB \sin \phi \]

Either way, same result:

\[ |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \phi \]

(\text{magnitude of cross-product in geometric rep.})

** as with dot product, \( \phi \) is the angle between \( \vec{A} \) and \( \vec{B} \) when they are lined up tail to tail

(IMPORTANT!!!)
direction in the geometric representation: right hand rule! (our first of many!)

direction of $\vec{A} \times \vec{B}$ is defined to be perpendicular to the plane spanned by $\vec{A}$ and $\vec{B}$ (any two vectors span, or define, a plane)

plane in which both $\vec{A}$ and $\vec{B}$ lie

so there are two possibilities. Which to choose? By convention the cross product is defined to be the direction perpendicular to the plane that is determined by the following rule, called the "right hand rule":

1. take your right hand and point it in the direction of the first vector (in the case of $\vec{A} \times \vec{B}$, that means $\vec{A}$)

2. by closing your hand, imagine rotating the first vector into the second vector (here: $\vec{A}$ rotated into the direction of $\vec{B}$) [note: you must do this through the smaller angle, less than 180°]

3. your thumb points in the direction of the cross product ($\vec{A} \times \vec{B}$)
Ex. \( \vec{A}, \vec{B}, \) and \( \vec{C} \) all lie in the plane of the paper.

Determine the direction of the following cross products:

\[
\begin{align*}
\text{(a) } \vec{A} \times \vec{B} & \\
\text{(b) } \vec{B} \times \vec{C} & \\
\text{(c) } \vec{C} \times \vec{A} &
\end{align*}
\]

\( \vec{A} \times \vec{B} \)

with right hand, rotate \( \vec{A} \) into \( \vec{B} \), then thumb points out of the page

\( \therefore \vec{A} \times \vec{B} \) is out of the page, \( \bigcirc \)

\( \vec{B} \times \vec{C} \)

is into the page, \( \bigotimes \)

\( \vec{C} \times \vec{A} \)

is out of the page, \( \bigcirc \)

Note: \( \vec{A} \times \vec{B} = - \vec{B} \times \vec{A} \), i.e., order matters when using cross product!

\[\text{notation:}\]

\( \bigcirc \) = vector coming out of the page (arrow coming towards you)

\( \bigotimes \) = vector going into the page (arrow tail going away from you)
algebraic representation:

recall our unit vectors:

\[ \hat{\ell} \times \hat{\ell} = |\hat{\ell}| |\hat{\ell}| \sin \phi_{\ell\ell} = (1) \cdot (1) \sin 90^\circ = 0 \]

likewise \( \hat{\ell} \times \hat{j} = 0 \) and \( \hat{k} \times \hat{k} = 0 \)

However all other combinations are non-zero:

\[ |\hat{\ell} \times \hat{j}| = |\hat{\ell}| |\hat{j}| \sin \phi_{\ell j} = (1) \cdot (1) \sin 90^\circ = 1 \]

direction by RHR is in \( z \)-direction.

\( \therefore \hat{\ell} \times \hat{j} \) gives a vector with magnitude 1 in \( z \)-direction.

But that is precisely what \( \hat{k} \) is!

\[ \Rightarrow \hat{\ell} \times \hat{j} = \hat{k} \]

likewise \( \hat{j} \times \hat{k} = \hat{\ell} \) and \( \hat{k} \times \hat{\ell} = \hat{j} \)
so all together:

\[
\begin{align*}
\hat{i} \times \hat{j} &= \hat{k} \\
\hat{j} \times \hat{k} &= \hat{i} \\
\hat{k} \times \hat{i} &= \hat{j}
\end{align*}
\]

\[
\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0
\]

all other combinations vanish

pneumonic:
(tool for remembering)

\[
\begin{array}{ccc}
\hat{k} & \rightarrow \\
\hat{i} & + \\
\hat{j} & \rightarrow
\end{array}
\]

going with the flow, + (e.g., $\hat{i} \times \hat{j} = \hat{k}$)

going against the flow, - (e.g., $\hat{i} \times \hat{k} = -\hat{j}$)

(alphabetical order flows clockwise)

now we can use these results to find $\vec{A} \times \vec{B}$ for any two vectors in terms of their components:

if $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

and $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$

then $\vec{A} \times \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$

\[
= A_x B_x (\hat{i} \times \hat{i}) + A_x B_y (\hat{i} \times \hat{j}) + A_x B_z (\hat{i} \times \hat{k}) + A_y B_x (\hat{j} \times \hat{i}) + A_y B_y (\hat{j} \times \hat{j}) + A_y B_z (\hat{j} \times \hat{k}) + A_z B_x (\hat{k} \times \hat{i}) + A_z B_y (\hat{k} \times \hat{j}) + A_z B_z (\hat{k} \times \hat{k})
\]

\[
= A_x B_y \hat{k} + A_x B_z (-\hat{j}) + A_y B_x (-\hat{k}) + A_y B_z \hat{i} + A_z B_x \hat{j} + A_z B_y (-\hat{i})
\]
\[ \vec{A} \times \vec{B} = \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_z B_x - A_x B_z) + \hat{k}(A_x B_y - A_y B_x) \]

Methods to compute:

1. memorize the above formula and apply each time (notice the cyclic pattern: xyz, then yzx, then zxy)

2. foil out by hand each time (essentially reproducing the above formula for yourself each time).

3. using determinants

**Determinant Method**

I will not spend much/any time in lecture discussing the determinant method, since it is often overkill in this class (unnecessary), students often make sign mistakes when using it, and students often don't remember (or have never learned) anything about determinants. However, for the record, here it is:

Recall: if \[ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \], then \[ \det M = 1M1 = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \]

Example:

\[ \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (3)(2) = 4 - 6 = -2 \]

For a 3x3 matrix, calculating the determinant is more involved.

If \[ M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \], then...
\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{vmatrix}
\]

\[
= \begin{vmatrix}
  a & b \\
  d & e \\
  g & h \\
\end{vmatrix} \\
\text{re-copy first two columns here}
\]

\[
= \text{sum (multiply diagonals sloping from top to bottom)} \\
- \text{sum (multiply diagonals sloping from bottom to top)}
\]

\[
= (aei + bfg + cdh) - (gec + hfa + idb)
\]

\[
= aei + bfg + cdh - gec - hfa - idb
\]
alternatively, "expand by minors"

\[
det M = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}
\]

\[
= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}
\]

\[
\begin{vmatrix} e & f \\ h & i \end{vmatrix}
- b \begin{vmatrix} d & f \\ g & i \end{vmatrix}
+ c \begin{vmatrix} d & e \\ g & h \end{vmatrix}
\]

choose any column or row. Here I will take the top row.

\[
\begin{array}{cccc}
+ & - & + \\
- & + & - \\
+ & - & + \\
\end{array}
\]

here, I am expanding along this first row, so the signs are +, -, +.

obtained by crossing out the row and column containing \(a\) and then taking the determinant of the remaining elements.

\[
\begin{vmatrix} e & f \\ h & i \end{vmatrix}
\]

obtained by crossing out row/cell containing element \(b\).

\[
\begin{vmatrix} d & f \\ g & i \end{vmatrix}
\]

obtained by crossing out row/cell containing \(c\).

\[
\begin{vmatrix} d & e \\ g & h \end{vmatrix}
\]

\[
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}
\]

take determinant of these.
\[ \begin{align*}
&= a(ei - hf) - b(di - gf) + c(dh - ge) \\
&= ae i - ah f - bdi + bg f + cdh - cge \\
&= ae i + bg f + cdh - gec - hfa - idb
\end{align*} \]

So how do we apply all of this to calculating cross product???

**Trick!** to compute \( \mathbf{A} \times \mathbf{B} \), put unit vectors in first row, components of first vector in second row, and components of second vector in third row.

then

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
A_x & A_y & A_z \\
B_x & B_y & B_z
\end{vmatrix}
\]

\[
\begin{align*}
&= \hat{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + \hat{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \\
&= \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_x B_z - A_z B_x) + \hat{k}(A_x B_y - A_y B_x)
\end{align*}
\]

**Note:**  **SAME** AS SIMPLY FOILING OUT \((A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})\)
NOTE: in this course, simply foiling out is the best method (easiest, quickest, and least prone to errors)

Ex. Consider \( \vec{A} = 5\hat{i} + 3\hat{j} \), \( \vec{B} = -3\hat{j} + 2\hat{k} \).

Find \( \vec{A} \times \vec{B} \).

Using method 2 (Foil)
\[
\vec{A} \times \vec{B} = (5\hat{i} + 3\hat{j}) \times (-3\hat{j} + 2\hat{k})
\]
\[
= 5(-3)\hat{i} \times \hat{j} + 5(2)\hat{i} \times \hat{k} \\
+ 3(-3)\hat{j} \times \hat{j} + 3(2)\hat{j} \times \hat{k}
\]
\[
= -15\hat{k} + 10(-\hat{j}) + 6\hat{i}
\]
\[
= 6\hat{i} - 10\hat{j} - 15\hat{k}
\]

Using method 1 (memorize formula)
\[
\vec{A} \times \vec{B} = \hat{i}(A_yB_z - A_zB_y) + \hat{j}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)
\]
\[
= \hat{i}(3(2) - 0(-3)) + \hat{j}(0(2) - 5(2)) + \hat{k}(5(-3) - 3(0))
\]
\[
= \hat{i}(6 - 0) + \hat{j}(0 - 10) + \hat{k}(-15 - 0)
\]
\[
= 6\hat{i} - 10\hat{j} - 15\hat{k}
\]

Hard part is getting the formula right!
Using method 3 (determinant)

\[
\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 3 & 0 \\ 0 & -3 & 2 \end{vmatrix}
\]

\[
= \hat{i} \begin{vmatrix} 30 \\ -32 \end{vmatrix} - \hat{j} \begin{vmatrix} 50 \\ 02 \end{vmatrix} + \hat{k} \begin{vmatrix} 53 \\ 03 \end{vmatrix}
\]

\[
= \hat{i} (3(2) - 0) - \hat{j} (10 - 0) + \hat{k} (-15 - 0)
\]

\[
= 6 \hat{i} - 10 \hat{j} - 15 \hat{k}
\]

All three give the same result (as the must). Choose the one that works best for you.

My personal recommendation (based on years of seeing students mess up methods 1 and 3): just foil it out. In this course, we rarely take cross products of two vectors, each having 3 components. So the other methods are a bit overkill, and contain many pitfalls.