Conservation of Energy I: Gravitational Potential Energy

- gravity is a conservative force

- gravitational potential energy defined by:

\[
\Delta U_{\text{grav}} = -W_{\text{grav}}
\]

\[
= - \int F_{\text{grav}} \cdot d\vec{s}
\]

\[
= - \int_{y_1}^{y_2} F_y \, dy
\]

\[
= - \int_{y_1}^{y_2} (-mg) \, dy
\]

\[
= +mg \,(y_2 - y_1)
\]

\[
= U_2 - U_1 \quad \Rightarrow \quad U_{\text{grav}} = mg \, y
\]

(note: this assumes y-axis is directed upwards.)

\[
\vec{F}_{\text{grav}} = F_y \, \hat{j} = -mg \, \hat{j}
\]

\[
d\vec{s} = dx \, \hat{i} + dy \, \hat{j} + dz \, \hat{k}
\]

(note: no assumptions are made here about the path taken between endpoints)

\[
\text{y-coordinate in a coordinate system where y-axis is directed upwards.}
\]

- zero of gravitational potential energy is arbitrary.

i.e., you may choose \( y = 0 \) wherever you like without affecting the physics. Only changes in potential energy have real (measurable) significance.
* recall: **SYSTEMS** have potential energy, not individual objects within a system.

→ gravitational potential energy is only defined for systems in which the masses in the system interact via gravitational forces.

→ gravitational potential energy near the surface of the earth is only defined for systems that include the earth.

**Ex**

a ball in free-fall by itself (i.e., no other objects in "the system" besides the ball) does not have gravitational potential energy.

**Ex**

the ball + earth system does have gravitational potential energy.

**Ex**

Block sliding down an inclined plane without friction

\[ \Delta E = \Delta K + \Delta U = W_{\text{internal}} + W_{\text{external}} \]

(Note: \(W_{\text{normal}} = 0\) because \(N \cdot a = 0\)).

if system = block, then \(\Delta E = \Delta K + \Delta U = W_{\text{internal}} + W_{\text{external}}\)

\(\Delta K = mgh\).
if system = block + earth, then \[ \Delta E = \Delta K + \Delta U = W_{\text{internal non-conservative}} + W_{\text{external}} \]

\[\Rightarrow \Delta K + \Delta U = 0\]

\[\Rightarrow \Delta K = -\Delta U = -(mgx_f^o - mgy_i)\]

\[= -( -mgH)\]

\[= mgH\]

.: same result, but different interpretation depending on what system you consider

Ex: Cart on frictionless track.

\[V_A = 6.00 \text{ m/s}\]

\[m = 1.8 \text{ kg}\]

\[\text{How fast does the cart move when it reaches points B and C?}\]

if system = cart, then no potential energy stored in the system, external work done on the system by gravity.

if system = cart + earth, then system stores gravitational potential energy, gravitational force is considered an internal force, and so does not contribute to work done on the system by external forces.
let's choose system = cart + earth.  (Note: "earth" means the physical ball of mass called the earth, NOT everything on top of the earth, for example not the track that the cart rides along in this case)

Then at A: \( E_A = K_A + U_A \)
\[= \frac{1}{2}mv_A^2 + mg y_A \]

at B: \( E_B = K_B + U_B = \frac{1}{2}mv_B^2 + mg y_B \)

at C: \( E_C = K_C + U_C = \frac{1}{2}mv_c^2 + mg y_c \)

between A and B: \( W_{\text{internal}} \) non-conservative \( = 0 \)
\( W_{\text{external}} = W_{\text{normal}} = 0 \)

between B and C: ditto

\[ \Rightarrow \Delta E_{A \to B} = 0 \Rightarrow E_A = E_B \Rightarrow \frac{1}{2}mv_A^2 + mg y_A \]
\[= \frac{1}{2}mv_B^2 + mg y_B \]

\[ \Rightarrow v_B = \sqrt{v_A^2 + 2g(y_A - y_B)} \]
\[= \sqrt{(6.00 \text{ m/s})^2 + 2(9.8 \text{ m/s}^2)(0.80 \text{ m} - 1.50 \text{ m})} \]
\[= 3.53 \text{ m/s} \]
likewise $\Delta E_{A\rightarrow C} = 0$  
(or $\Delta E_{B\rightarrow C} = 0$)

$\Rightarrow$  $E_A = E_C$

$\Rightarrow$  $\frac{1}{2}mv_A^2 + mg y_A = \frac{1}{2}mv_C^2 + mg y_C$

$\Rightarrow$  $v_C = \sqrt{v_A^2 + 2g(y_A - y_C)}$

$= \sqrt{(6.00 \text{m/s})^2 + 2(9.8 \text{m/s}^2)(0.30 \text{m} - 2.80 \text{m})}$

$= \sqrt{-13.0 \text{m}^2/\text{s}^2}$

$\uparrow$  no real solution
$\Rightarrow$  cart doesn't make it to point C!

**TIP:** thinking in terms of potential energy is often more convenient than thinking in terms of external work.

Therefore, whenever dealing with energy methods in earth's gravitational field, include the earth in your system of interest.
Ex: Cart on track with friction

\[ V_A = 6.00 \text{ m/s} \]
\[ m = 1.8 \text{ kg} \]

\[ \begin{array}{c}
\text{A} \\
\downarrow 0.30 \text{m} \\
\text{B}
\end{array} \]

if \( V_B = 1.40 \text{ m/s} \), what work was done by friction?

System = cart + earth \[ \Rightarrow \begin{cases} 
W_{\text{external non-conservative}} &= 0 \\
W_{\text{external conservative}} &= 0 \\
W_{\text{external non-conservative}} &= W_{\text{friction}} \neq 0
\end{cases} \]

at A: \[ E_A = K_A + U_A = \frac{1}{2} mv_A^2 + mg y_A \]

at B: \[ E_B = K_B + U_B = \frac{1}{2} mv_B^2 + mg y_B \]

between A and B: \[ W_{\text{other}} = W_{\text{external non-conservative}} = W_{\text{friction}} \]

\[ \Rightarrow \Delta E = W_{\text{friction}} \]

\[ W_{\text{friction}} = E_B - E_A = \frac{1}{2} mv_B^2 + mg y_B - \frac{1}{2} mv_A^2 - mg y_A \]

\[ = \frac{1}{2} m (v_B^2 - v_A^2) + mg (y_B - y_A) \]

\[ = \frac{1}{2} (1.8 \text{ kg})(1.40 \text{ m/s})^2 - (6.00 \text{ m/s})^2 + (1.8 \text{ kg})(1.8 \text{ m/s})(1.20 \text{ m}) = -9.5 \text{ J} \]
**Example: Modified Atwood machine without friction**

**FBDs:**

![FBD Diagram]

**Is mechanical energy conserved?**

<table>
<thead>
<tr>
<th>System</th>
<th>$W_{\text{internal}}$ non-conservative</th>
<th>$W_{\text{external}}$</th>
<th>$\Delta E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>block A</td>
<td>0 (no internal forces)</td>
<td>$W_N = 0$</td>
<td>$\Delta E &gt; 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$W_{\text{grav}} = 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$W_T &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>block B</td>
<td>0 (no internal forces)</td>
<td>$W_T &lt; 0$</td>
<td>$\Delta E &gt; 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$W_{\text{grav}} &gt; 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$</td>
<td>W_{\text{grav}}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\therefore W_{\text{external}} &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>block A, block B, string</td>
<td>$W_{TA} &gt; 0$</td>
<td>$W_N = 0$</td>
<td>$\Delta E &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>$W_{TB} &lt; 0$</td>
<td>$W_{\text{grav}} &gt; 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>W_{TA}</td>
<td>=</td>
</tr>
<tr>
<td></td>
<td>$\therefore W_{\text{internal}} = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>block A, block B, earth</td>
<td>0 (no internal non-conservative forces)</td>
<td>$W_N = 0$</td>
<td>$\Delta E = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$W_{TA} &gt; 0$</td>
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</table>
Ex Modified Atwood machine with friction

(Ideal string, ideal pulley)

Assume friction is insufficient to keep system at rest when released from rest.

Find speed of blocks after moving a distance D.

given: $m_A, m_B, D, \mu_k$

Start:

\[ y_A \quad y_B \]

\[ o = y_{Ai} \quad y_{Bi} = 0 \]

(Note: you can use different coordinate systems for different objects)

end:

\[ y_A \quad y_B \]

\[ o = y_{Ai} \quad y_{Bi} = 0 \]

assume system = block A, block B, string, earth

FBD's:

\[ A \]

\[ \vec{N} \text{ (external)} \quad \vec{F}_h \text{ (internal)} \]

\[ \vec{M}_G \text{ (internal)} \]

\[ B \]

\[ \vec{T}_B \text{ (internal)} \]

\[ m_B \vec{g} \text{ (internal)} \]
between start and end:

\[ W_{\text{internal non-conservative}} = W_{TA} + W_{TB} \]

\[ = TA \cdot D - TB \cdot D \]

\[ = TD - TD \]

\[ = 0 \]

\[ \vec{F}_A \cdot \vec{d} = 0 \]

\[ W_{\text{external}} = W_N + W_{fr} \]

\[ = \vec{F}_N \cdot \vec{d} \]

\[ = |\vec{F}_N| |\vec{d}| \cos \phi \]

\[ = (\mu_k m_A g \cos \phi)(D) \cos 180^\circ \]

\[ = - \mu_k m_A g D \cos \theta \]

at start: \[ E_i = K_{Ai} + K_{Bi} + U_{Ai} + U_{Bi} \]

\[ = 0 \]

Zero by choice of coordinates

at end: \[ E_f = K_{Af} + K_{Bf} + U_{Af} + U_{Bf} \]

\[ = \frac{1}{2} m_A v_{Af}^2 + \frac{1}{2} m_B v_{Bf}^2 + m_A g D \sin \theta + m_B g (-D) \]

\[ = \frac{1}{2} (m_A + m_B) v_f^2 + (m_A \sin \theta - m_B) g D \]
ΔE = W_{\text{internal, non-conservative}} + W_{\text{external}}

\Rightarrow \quad \frac{1}{2} (m_A + m_B) v_f^2 + (m_A \sin \theta - m_B) gD = -\mu_k m_A gD \cos \theta

\Rightarrow \quad \frac{1}{2} (m_A + m_B) v_f^2 = \left[ m_B - m_A (\sin \theta + \mu_k \cos \theta) \right] gD

\Rightarrow \quad v_f = \sqrt{2gD \left( \frac{m_B - m_A (\sin \theta + \mu_k \cos \theta)}{m_A + m_B} \right)}