Conservation of Momentum I: Inelastic Collisions

• recall: \[ \vec{P}_{\text{system}} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \ldots \]

\[ \vec{F}_{\text{tot}} = \Delta \vec{P}_{\text{system}} \]

if \[ \vec{F}_{\text{tot}} = 0 \] (or equivalently, \[ \vec{F}_{\text{net}} = 0 \]),

then \[ \Delta \vec{P}_{\text{system}} = 0 \]

i.e., \[ \vec{F}_{\text{net}} = 0 \] \(\iff\) \[ \vec{P}_{\text{system}} = \text{constant} \]

i.e., \[ \vec{F}_{\text{net}} = 0 \] \(\iff\) \[ \vec{P}_{f} = \vec{P}_i \]

• Conservation of momentum applied to collisions: if the net external force on a system of objects is zero, then momentum is conserved during any collisions between objects in the system.

• kinds of collisions:

1. inelastic \(\rightarrow\) \(\Delta K < 0\) \(\text{(kinetic energy is lost during collision)}\)

2. elastic \(\rightarrow\) \(\Delta K = 0\) \(\text{(kinetic energy conserved during collision)}\)

3. super-elastic \(\rightarrow\) \(\Delta K > 0\) \(\text{(kinetic energy gained during collision, say from some sort of explosion)}\)
Ex. Completely (or perfectly) inelastic collision

consider two objects that collide and stick. Is energy conserved in such a process?

\[
\text{before: } \quad \begin{array}{c}
\text{A} \\
\downarrow \\
\text{v}_{A_i}
\end{array} \quad \begin{array}{c}
\text{V}_{B_i} \\
\downarrow \\
\text{B}
\end{array} \\
\text{after: } \quad \begin{array}{c}
\text{A} \\
\downarrow \\
\text{v}_{f}
\end{array} \quad \begin{array}{c}
\text{B} \\
\downarrow \\
\text{v}_{f}
\end{array}
\]

System = A + B.

Assume momentum is conserved (no friction or other external forces that ruin momentum conservation).

then \( P_{ix}^{(\text{system})} = m_A v_{A_{ix}} + m_B v_{B_{ix}} \) <- components

\( = m_A v_{A_{i}} - m_B v_{B_{i}} \) <- magnitudes (speeds)

\( P_{fx}^{(\text{system})} = (m_A + m_B) v_{f_{x}} \)

\( = (m_A + m_B) v_{f} \)

conservation of momentum in x-direction:

\( P_{fx}^{(\text{system})} = P_{ix}^{(\text{system})} \Rightarrow (m_A + m_B) v_{f} = m_A v_{A_{i}} - m_B v_{B_{i}} \)

\( \Rightarrow v_{f} = \frac{m_A v_{A_{i}} - m_B v_{B_{i}}}{(m_A + m_B)} \)
\[
V_{Af} = V_{Bf} = V_f = \frac{m_A V_{Ai} - m_B V_{Bi}}{m_A + m_B}
\]

Is kinetic energy conserved? (is it possible?)

\[
K_i^{(\text{system})} = K_i^{(A)} + K_i^{(B)}
\]

\[
= \frac{1}{2} m_A V_{Ai}^2 + \frac{1}{2} m_B V_{Bi}^2
\]

\[
K_f^{(\text{system})} = K_f^{(A)} + K_f^{(B)}
\]

\[
= \frac{1}{2} m_A V_{Af}^2 + \frac{1}{2} m_B V_{Bf}^2
\]

\[
= \frac{1}{2} (m_A + m_B) V_f^2
\]

\[
= \frac{1}{2} (m_A + m_B) \left( \frac{m_A V_{Ai} - m_B V_{Bi}}{m_A + m_B} \right)^2
\]

\[
= \frac{1}{2} \frac{m_A^2 V_{Ai}^2 + m_B^2 V_{Bi}^2 - 2 m_A m_B V_{Ai} V_{Bi}}{(m_A + m_B)^2}
\]

\[
= \frac{1}{2} \left( \frac{m_A}{m_A + m_B} \right) m_A V_{Ai}^2 + \frac{1}{2} \left( \frac{m_B}{m_A + m_B} \right) m_B V_{Bi}^2 - \left( \frac{m_A m_B}{m_A + m_B} \right) V_{Ai} V_{Bi} \quad \text{positive number}
\]

\[
\Rightarrow K_f^{(\text{system})} < K_i^{(\text{system})}, \quad \Delta K^{(\text{system})} < 0.
\]
the previous example shows that when objects stick together after a collision, the collision is necessarily inelastic (i.e., it is not possible for energy to be conserved in such a collision)

H OWEVER the inverse is NOT true!

\[
\text{sticking together } \implies \text{ inelastic}
\]

\[
\begin{array}{c}
\text{inelastic} \\
\not\implies
\end{array}
\begin{array}{c}
\text{sticking together}
\end{array}
\]

- sticking together vs. bouncing: (here "bounce" just means "not sticking together")

\[
\begin{array}{c}
\text{inelastic} \\
\leftrightarrow
\end{array}
\begin{array}{c}
\text{bounce}
\end{array}
\begin{array}{c}
\text{stick together}
\end{array}
\]

\[
\text{elastic} \implies \text{ bounce}
\]

\[
\text{super-elastic} \implies \text{ bounce}
\]

i.e., sticking together implies inelastic, but inelastic does not imply sticking together.

Likewise, elastic implies bounce, and super-elastic implies bounce, but bounce does not imply anything (since it is possible to bounce while being inelastic, elastic or super-elastic.)
Ex: Four puck collisions

Consider a collision between pucks on an air hockey table (frictionless horizontal surface). Puck 1 has mass 5.00 kg and initially moves at 3.00 m/s to the right. Puck 2 has mass 3.00 kg and initially moves to the left at 1.00 m/s.

Consider four different scenarios for what happens after the collision:

(a) After the collision, puck 1 moves to the right at 1.50 m/s.
(b) After the collision, puck 1 moves to the right at 0.90 m/s.
(c) After the collision, puck 1 remains stationary (at rest).
(d) After the collision, puck 1 moves to the left at 1.50 m/s.

One may show that:

<table>
<thead>
<tr>
<th>Case</th>
<th>( V_{ix}^{(1)} )</th>
<th>( V_{fx}^{(1)} )</th>
<th>( V_{ix}^{(2)} )</th>
<th>( V_{fx}^{(2)} )</th>
<th>( \Delta K^{(\text{system})} )</th>
<th>Type of Collision</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>+ 3.00 m/s</td>
<td>+ 1.50 m/s</td>
<td>-1.00 m/s</td>
<td>+ 1.5 m/s</td>
<td>-15.0 J</td>
<td>inelastic</td>
</tr>
<tr>
<td>(b)</td>
<td></td>
<td>+ 0.90 m/s</td>
<td></td>
<td>+ 2.5 m/s</td>
<td>-12.6 J</td>
<td>inelastic</td>
</tr>
<tr>
<td>(c)</td>
<td></td>
<td>0 m/s</td>
<td></td>
<td>+ 4.00 m/s</td>
<td>0 J</td>
<td>elastic</td>
</tr>
<tr>
<td>(d)</td>
<td></td>
<td>-1.5 m/s</td>
<td></td>
<td>+ 6.50 m/s</td>
<td>+45.0 J</td>
<td>super-elastic</td>
</tr>
</tbody>
</table>

Note: both (a) and (b) are inelastic collisions even though in one case the pucks move together after the collision (case (a)), and in the other case the pucks "bounce", i.e., do not move together after the collision (case (b)).
Ex: Puck-brick collision in two dimensions

A hockey puck of mass 0.160 kg slides on a frictionless horizontal surface towards a brick with initial speed of 15.0 m/s. It collides with a brick of mass 0.50 kg initially at rest. After the collision, the puck moves with speed 8.0 m/s at 10.0° relative to its original trajectory, as shown below.

(a) Find the final velocity (magnitude and direction) of the brick.
(b) Is this collision elastic, inelastic, or superelastic?

Before (top-view):

\[ V_{pi} \]

\[ V_{bi} = 0 \]

After (top-view):

\[ V_{pf} \]

\[ V_{bf} \]

(c) take the system to include the puck and brick.

Then \( F_{net} = 0 \), and so \( P_{f}^{(system)} = P_{i}^{(system)} \).

\[ P_{ix}^{(system)} = P_{ix}^{(puck)} + P_{ix}^{(brick)} = mV_{pi,x} + MV_{bi,x} = mV_{pi} \]

\[ P_{iy}^{(system)} = P_{iy}^{(puck)} + P_{iy}^{(brick)} = 0 \]
\[ P_{fx}^{\text{(system)}} = P_{fx}^{\text{(puck)}} + P_{fx}^{\text{(brick)}} = m_v pf_x + M v_{bf,x} \]
\[ = m v_{pf} \cos \theta + M v_{bf,x} \]

\[ P_{fy}^{\text{(system)}} = P_{fy}^{\text{(puck)}} + P_{fy}^{\text{(brick)}} = m v_{pf,y} + M v_{bf,y} \]
\[ = m v_{pf} \sin \theta + M v_{bf,y} \]

Conservation of momentum in the \( x \)-direction:

\[ P_{fx}^{\text{(system)}} = P_{ix}^{\text{(system)}} \implies m v_{pf} \cos \theta + M v_{bf,x} = m v_{pi} \]

\[ \implies v_{bf,x} = \frac{m}{M} (v_{pi} - v_{pf} \cos \theta) \]
\[ = \left( \frac{0.160 \text{ kg}}{0.500 \text{ kg}} \right) (15.0 \text{ m/s} - (8.00 \text{ m/s}) \cos (10.0^\circ)) \]
\[ = 2.2789 \text{ m/s} \]

Conservation of momentum in the \( y \)-direction:

\[ P_{fy}^{\text{(system)}} = P_{iy}^{\text{(system)}} \implies m v_{pf} \sin \theta + M v_{bf,y} = 0 \]

\[ \implies v_{bf,y} = -\frac{M}{M} v_{pf} \sin \theta \]
\[ = -\left( \frac{0.160 \text{ kg}}{0.500 \text{ kg}} \right) (8.00 \text{ m/s}) \sin (10.0^\circ) \]
\[ = -0.44454 \text{ m/s} \]
magnitude of $\vec{v}_{bf}$:

$$\vec{v}_{bf} = \sqrt{(v_{bf,x})^2 + (v_{bf,y})^2}$$

$$= \sqrt{(2.2789 \text{ m/s})^2 + (-0.44454 \text{ m/s})^2}$$

$$= 2.3219 \text{ m/s}$$

direction of $\vec{v}_{bf}$:

$$\tan \phi = \left( \frac{|v_{bf,y}|}{|v_{bf,x}|} \right)$$

$$\Rightarrow \phi = \tan^{-1} \left( \frac{|v_{bf,y}|}{|v_{bf,x}|} \right) = \tan^{-1} \left( \frac{0.44454}{2.2789} \right) = 11.038^\circ$$

$\vec{v}_{bf}$ has magnitude of 2.32 m/s, and is directed at $11.0^\circ$ below the +x axis.

(b) $K_i^{(\text{system})} = K_i^{(puck)} + K_i^{(brick)} = \frac{1}{2} M v_{pi}^2 + \frac{1}{2} M x_{i,0}^2$.

$$= \frac{1}{2} (0.160 \text{ kg})(15.0 \text{ m/s})^2$$

$$= 18.0 \text{ J}$$

$K_f^{(\text{system})} = K_f^{(puck)} + K_f^{(brick)} = \frac{1}{2} M v_{pf}^2 + \frac{1}{2} M v_{bf}^2$.

$$= \frac{1}{2} (0.160 \text{ kg})(8.00 \text{ m/s})^2 + \frac{1}{2} (0.500 \text{ kg})(2.3219 \text{ m/s})^2$$

$$= 6.4678 \text{ J}$$
\[ \Delta K^{(\text{system})} = K_f^{(\text{system})} - K_i^{(\text{system})} \]

\[ = 6.47 \text{ J} - 18.0 \text{ J} \]

\[ = -11.53 \text{ J} \]

\[ \therefore \Delta K < 0 \]

\[ \therefore \text{collision is inelastic} \]

(Note: in this case the collision inelastic even though the objects do not move together after the collision, i.e., do not stick together.)

*Ex. A rail car of mass 500.0 kg is coasting along a horizontal frictionless track at speed of 3.00 m/s. A box of mass 100.0 kg is dropped from a height of 60.0 cm into the car.

(a) Find the final velocity of the car + box system after the collision.

(b) If the collision between the box and car lasts for 12.0 ms, find the average force that the car exerts on the box during the collision.

Before: \[ \begin{array}{c}
\text{car} \\
\downarrow \\
\vec{v}_i
\end{array} \]

After: \[ \begin{array}{c}
\text{car + box} \\
\downarrow \\
\vec{v}_f
\end{array} \]

Answers:
(a) \( \vec{v}_f = 2.50 \text{ m/s} \)

(b) \( F_{\text{av}}^{(\text{car on box})} = 35,400 \text{ N} , 53.9^\circ \text{ rel. to } +x \text{ axis} \).