Newton's Second Law for Rotational Dynamics

R. Stated without proof:

\[ \tau_{\text{net}} = \sum \tau_i = I \alpha \]

for a fixed axis along the z-direction (which is the convention we've been sticking to) this becomes a component equation:

\[ \tau_{\text{net},z} = \sum \tau_z = I \alpha_z \]

Note that the signs of \( \tau_z \) and \( \alpha_z \) depend on a choice of coordinates (corresponding to choosing a sign for clockwise vs. counter-clockwise rotation).

Also I is the moment of inertia of the system in question about a rotation axis that passes through the point around which the torque is being calculated.

2nd law might be better written as: \( \sum \tau^{(0)} = I^{(0)} \alpha \) to indicate that both \( \tau \) and \( I \) depend on which point (0) they are calculated around.
How to use Newton's Second law for rotational problems?

1. Draw an extended free body diagram for each relevant body.
2. Identify the point of interest for calculating torque around. (and axis of rotation)
3. Choose coordinates for each diagram.
4. Label any relevant angles on each diagram.
5. Write out Newton's second law equation for torques about your chosen axis. (double-check signs!)
6. Identify any constraints or auxiliary relationships between forces or accelerations in the problem.
7. Check that (# Eq'ns) = (# unknowns).
8. Solve for unknowns.
Derivation of Newton's Second Law

for Rotational Motion (Advanced: may be omitted or skipped without loss of continuity.)

Above we simply stated, without proof, that Newton’s second law for rotation takes the form

\[ \vec{\tau}_{\text{net}} = I \vec{\alpha} \]

We simply took Newton’s second law for linear motion

\[ \vec{F}_{\text{net}} = m \vec{\alpha} \]

and transformed the linear variables into rotational variables:

\[ \vec{F}_{\text{net}} \rightarrow \vec{\tau}_{\text{net}} \]

\[ m \rightarrow I \]

\[ \vec{\alpha} \rightarrow \vec{\alpha} \]

This gives the correct result, but it is not a rigorous derivation. In any case, one should always double-check that such fast and loose reasoning actually does give the correct answer.
Start by considering a single point mass rotating about a fixed axis with a fixed orbital radius.

The first thing to notice is that $\vec{ω} \times \vec{r}$ is in the same direction as $\vec{r}$. Furthermore we saw before that $|\vec{ω}| = |\vec{ω}| |\vec{r}|$. So in fact both magnitude and direction of $\vec{ω} \times \vec{r}$ are identical to the magnitude and direction of $\vec{r}$.

Therefore

$$\vec{a} = \vec{ω} \times \vec{r}$$

Taking a time derivative yields:

$$\vec{a} = \frac{d}{dt} (\vec{ω} \times \vec{r})$$

$$= \frac{dω}{dt} \times \vec{r} + \vec{ω} \times \frac{d\vec{r}}{dt}$$

$$= \vec{a}_r \times \vec{r} + \vec{ω} \times \vec{v}$$

**tangential acceleration**  
**radial acceleration**

$$a_r = \omega \times \vec{v} = \omega^2 r$$
To derive Newton's second law for rotational motion, we take Newton's second law for linear motion (right now just for a single point particle)

\[ F_{\text{net}} = m\ddot{a} \]

and cross it into \( \vec{r} \) into it:

\[ \vec{r} \times F_{\text{net}} = \vec{r} \times (m\ddot{a}) \]

It should be clear that \( \vec{r} \times F_{\text{net}} = \vec{\tau}_{\text{net}} \), so all that remains is to calculate the right-hand side:

\[ \vec{r} \times (m\ddot{a}) = \begin{pmatrix} \vec{r} \times (m\ddot{a}) \end{pmatrix} \]

above

\[ = m \left[ \vec{r} \times (\vec{\omega} \times \vec{r} + \vec{\omega} \times \ddot{\vec{r}}) \right] \]

\[ = m \left( \vec{r} \times \vec{\omega} \times \vec{r} \right) + \frac{m\vec{r} \times \ddot{\vec{r}}}{6} \text{ triple-product property} \]

\[ = m \left[ \vec{\omega} (\vec{r} \cdot \vec{r}) - \vec{r} (\vec{\omega} \cdot \vec{r}) \right] \]

\[ = m r^2 \ddot{\vec{r}} \]

\[ = I \ddot{\vec{r}} \]

\[ - \text{QED} \]

in general: \( \vec{A} \times \vec{B} \times \vec{C} = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B}) \)
(proof can be found in any vector-calculus textbook)
For a system of particles, the derivation is essentially the same. The crucial assumption, however, is that all of the particles are rotating about the same fixed axis and that they all maintain a fixed orbital radius about this axis. In other words, it is crucial that the system of particles form a rigid body.

Assuming that, we proceed as before:

\[
\vec{R}_{cm} \times \vec{F}_{net}^{sys} = \vec{R}_{cm} \times \sum_{i} \vec{F}_{net,i} = \sum_{i} \vec{R}_{cm} \times \vec{F}_{net,i} = \sum_{i} \left( \vec{r}_i - \vec{r}_{cm} \right) \times \vec{F}_{net,i} = \sum_{i} \vec{r}_i \times \vec{F}_{net,i} - \sum_{i} \vec{r}_{cm} \times \vec{F}_{net,i} = \sum_{i} \vec{F}_{net,i}
\]

\[
\vec{R}_{cm} \times M\vec{a}_{cm} = \vec{R}_{cm} \times \sum_{i} m_i \vec{a}_i = \sum_{i} \vec{R}_{cm} \times m_i \vec{a}_i = \sum_{i} \left( \vec{r}_i - \vec{r}_{cm} \right) \times m_i \vec{a}_i = \sum_{i} m_i \left( \vec{r}_i \times \vec{a}_i \right) - \sum_{i} \vec{r}_{cm} \times \vec{F}_{net,i} = \sum_{i} m_i \left[ \vec{r}_i \times \left( \vec{a}_i \times \vec{r}_i + \vec{\omega}_i \times \vec{v}_i \right) \right] - \sum_{i} \vec{r}_i \times \vec{F}_{net,i}
\]
\[
\begin{align*}
\sum \vec{m}_i \left( \vec{r}_i \times \vec{\alpha} \times \vec{r}_i \right) + \sum \vec{m}_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) - \sum \vec{h}_i \times \vec{F}_{net,i} \\
= \sum \vec{m}_i \left( \vec{\alpha} (\vec{r}_i \times \vec{r}_i) - \vec{r}_i (\vec{r}_i \cdot \vec{\alpha}) \right) - \sum \vec{h}_i \times \vec{F}_{net,i} \\
= \sum \vec{m}_i \vec{r}_i^2 \vec{\alpha} - \sum \vec{h}_i \times \vec{F}_{net,i} \\
= \vec{\alpha} \sum \vec{m}_i \vec{r}_i^2 - \sum \vec{h}_i \times \vec{F}_{net,i} \\
= I_{sys} \vec{\alpha} - \sum \vec{h}_i \times \vec{F}_{net,i}
\end{align*}
\]

\[\therefore \ \vec{r}_{cm} \times \left( \vec{F}_{net} \right) = M \vec{\alpha}_{cm} \]

\[\Rightarrow \ \vec{\omega}_{net} - \sum \vec{h}_i \times \vec{F}_{net,i} = I_{sys} \vec{\alpha} - \sum \vec{h}_i \times \vec{F}_{net,i} \]

\[\Rightarrow \ \vec{\omega}_{net} = I \vec{\alpha} \]

- Q.E.D.