Work

- Work is a scalar quantity related to force acting over a distance, or affecting some change in configuration.

- First pass at a definition: \[ W = F \cdot d \]
  - "force times distance"
  - This is ok if the force is constant, the displacement of an object is along a straight line, and the force is along the direction of displacement.

Ex

\[
\begin{array}{c}
\text{Start:} \\
\text{end:}
\end{array}
\]

\[ m \rightarrow \vec{F} \quad \text{let } F = 2.50 \, \text{N}, \quad d = 3.00 \, \text{m}, \quad m = 5.00 \, \text{kg}. \]

Then work done by pulling force \( \vec{F} \) is:

\[ W_F = F \cdot d = (2.50 \, \text{N})(3.00 \, \text{m}) = 7.50 \, \text{J} \]

SI unit of work is the Joule (J)

1 J = 1 N·m
But what if force is not directed along the displacement?

\[ \text{Ex} \]

\[ \text{start: } \vec{F} \quad \text{end: } \vec{F} \]

\[ d \]

\[ W = \text{?} = Fd \]

In this case, work is the distance times the magnitude of the component of the force in the same direction as the displacement.

\[ \Rightarrow W = (F \cos \theta) \, d \]

\[ = Fd \, \cos \theta \]

But we have already seen a vector way of multiplying two vectors that yields the same result: dot product:

recall: \[ \vec{A} \cdot \vec{B} = |\vec{A}| \, |\vec{B}| \cos \phi_{AB} \]

angle between \( \vec{A} \) and \( \vec{B} \) when they are lined up tail to tail.

or

\[ = A_x B_x + A_y B_y + A_z B_z \]
Applying this to \( \vec{F} \) and \( \vec{d} \):

\[
\vec{F} \cdot \vec{d} = |\vec{F}| |\vec{d}| \cos \phi
\]

angle between \( \vec{F} \) and \( \vec{d} \) when they are lined up tail to tail

\[
\vec{d} = \Delta \vec{r} = \vec{r}_a - \vec{r}_i
\]

force vector
displacement vector
different notation for the same thing
final position vector
initial position vector

(in the example above, \( \phi = 0 \))

So we have an improved definition of work involving the dot product:

- second pass at a definition: \( W = \vec{F} \cdot \vec{d} \)
  
  = "force dot displacement"

  note: "times" \( \neq \) "dot"

- this definition is ok if the force is constant and the displacement of an object is along a straight line, even if the angle between them is non-zero
  (but remains constant throughout the motion)
Ex: Find work done by each force, as well as total work.

\[ \text{let } m = 5.00 \text{ kg} \]
\[ \theta = 12.0^\circ \]
\[ d = 3.0 \text{ m} \]
\[ \mu_s = 0.6 \]
\[ \mu_k = 0.3 \]
\[ V_0 = 7.50 \text{ m/s} \]

(a) Work done by gravity?
(b) Work done by normal force?
(c) Work done by friction?
(d) Total work?
(e) BONUS. Is the block slowing down, speeding up, or moving at constant speed?

(a) First let's draw the free-body diagram of the block.

(chosen because acceleration is either up the incline, down the incline, or zero. We don't yet know which, but this is a good choice regardless.)

To calculate work done by gravity we can proceed in multiple ways.

way #1: find magnitude of component of gravitational force in the direction of displacement and multiply
Way #2 find magnitude of component of displacement in the direction of gravitational force and multiply

\[ W_{\text{grav}} = mg \cdot (d \cdot \sin \theta) = m gd \sin \theta \]

Way #3 use dot product as in geometric description

\[ W_{\text{grav}} = \vec{F}_{\text{grav}} \cdot \vec{d} \]
\[ = mg \cdot \vec{d} \]
\[ = |m| \cdot |d| \cdot \cos \phi \]
\[ = m gd \cos (90^\circ - \theta) \]
\[ = m gd \sin \theta \]

\[ \text{trig:} \]
\[ \cos(a + b) = \cos a \cdot \cos b + \sin a \cdot \sin b \]
Way #4 use dot product as in the algebraic description:

\[ \vec{d} = d \hat{i} \]

\[ \vec{mg} = (mg \sin \theta) \hat{i} - (mg \cos \theta) \hat{j} \]

\[
W_{\text{grav}} = \vec{F}_{\text{grav}} \cdot \vec{d} \\
= \vec{mg} \cdot \vec{d} \\
= [(mg \sin \theta) \hat{i} - (mg \cos \theta) \hat{j}] \cdot [d \hat{i}] \\
= (mgd \sin \theta) \hat{i} \cdot \hat{i} - (mgd \cos \theta) \hat{j} \cdot \hat{i} \\
= mgd \sin \theta
\]

These are four different ways of calculating the same dot product. All yield the same result (as they must):

\[ W_{\text{grav}} = mgd \sin \theta \]
(b) \( W_{\text{normal}} = \vec{N} \cdot \vec{d} = 0 \), since \( \vec{N} \) is perpendicular to \( \vec{d} \)

recall: (dot product of perpendicular vectors vanishes)

geometrically: \( \vec{N} \cdot \vec{d} = \vec{N} d \cos \phi = \vec{N} d \cos 90^\circ = 0 \)

algebraically: \( \vec{N} \cdot \vec{d} = (N\hat{j}) \cdot (d \hat{e}) = \vec{N} d (\hat{f} \cdot \hat{t})^\circ = 0 \)

(c) \( W_{\text{friction}} = \vec{f}_k \cdot \vec{d} \)

return to FBD:

Here we need to solve Newton's 2nd law equations to determine \( N \) (needed to determine \( f_k \)).

2nd law eqns: \[ \begin{align*}
\Sigma F_x &= mg \sin \theta - f_k = ma_x \quad \text{①} \\
\Sigma F_y &= N - mg \cos \theta = ma_y \quad \text{②}
\end{align*} \]

② \( \Rightarrow N = mg \cos \theta \) \( \Rightarrow f_k = \mu_k N = \mu_k mg \cos \theta \)

\[ \Rightarrow W_{\text{friction}} = f_k \cdot d = |f_k| |d| \cos \phi = (\mu_k mg \cos \theta)(d) \cos 180^\circ \]

\[ = -\mu_k mg d \cos \theta \] (since \( f_k \) is antiparallel to \( d \))
(d) total work:

\[ W_{\text{total}} = W_{\text{grav}} + W_{\text{normal}} + W_{\text{friction}} \]

\[ = mgd \sin \theta - \mu_k mgd \cos \theta \]

Numeric:

\[ W_{\text{grav}} = mgd \sin \theta = (5.00 \text{ kg})(9.8 \text{ m/s}^2)(3.00 \text{ m}) \sin(12.0^\circ) = 30.6 \text{ J} \]

\[ W_{\text{normal}} = 0 \]

\[ W_{\text{friction}} = - \mu_k mgd \cos \theta = -(0.3)(5.00 \text{ kg})(9.8 \text{ m/s}^2)(3.00 \text{ m}) \cos(12.0^\circ) \]

\[ = -43.1 \text{ J} \]

\[ W_{\text{tot}} = W_{\text{grav}} + W_{\text{normal}} + W_{\text{friction}} \]

\[ = 30.6 \text{ J} + 0 - 43.1 \text{ J} \]

\[ = -12.5 \text{ J} \]

(e) speeding up or slowing down?

From Newton's 2nd law equation (1) (see above) \( \Rightarrow \ a_x = g(\sin \theta - \mu_k \cos \theta) \)

\[ = (9.8 \text{ m/s}^2) \cdot (\sin 12^\circ - (0.3) \cos 12^\circ) \]

\[ = -0.838 \text{ m/s}^2 \]

\[ \Rightarrow a \text{ is directed up the ramp.} \text{ Since } \vec{v} \text{ is directed down the ramp, this means the block slows down.} \]
• the above ("second pass") definition of work is still not adequate in general.

if force is not constant in magnitude, or not constant in direction, or if the path is not a straight line, then the above definitions breakdown

• third (and final) pass at a definition of work:

\[ W = \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} \]

\[(\text{infinitesimal displacement along path}) \]

(work done by force \( \vec{F} \) on object as it moves from \( r_1 \) to \( r_2 \).)

in general \( \vec{F}, d\vec{r}, \) and \( \phi \) all change along the path