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Decision Support

A sensitivity analysis algorithm for hierarchical decision models

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Abstract

In this paper, a comprehensive algorithm is developed to analyze the sensitivity of hierarchical decision models (HDM), including the analytic hierarchy process and its variants, to single and multiple changes in the local contribution matrices at any level of the decision hierarchy. The algorithm is applicable to all HDM that use an additive function to derive the overall contribution vector. It is independent of pairwise comparison scales, judgment quantification techniques and group opinion combining methods. The *allowable range/region of perturbations*, *contribution tolerance*, *operating point sensitivity coefficient*, *total sensitivity coefficient* and *the most critical decision element* at a certain level are identified in the HDM SA algorithm. An example is given to demonstrate the application of the algorithm and show that HDM SA can reveal information more significant and useful than simply knowing the rank order of the decision alternatives.

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1. Introduction

As the world has become more complex, decision problems have followed suit and must contend with increasingly complex relationships and interactions among the decision elements. To assist decision makers and analysts, different methods have been developed to decompose problems into hierarchical levels and formulate hierarchical decision models (HDM). In addition to the well-known analytic hierarchy process (AHP) developed by Saaty (1980), several other models based on the same basic concept of dealing with multiple decision levels but using different pairwise comparison scales and judgment quantification techniques were developed concurrent with or shortly after the introduction of AHP (i.e., Kocaoglu, 1976, 1983; Chu et al., 1979; Johnson et al., 1980; Hihn, 1980; Belton and Gear, 1983, 1985; Jensen, 1984; Ra, 1988; Lootsma, 1999).

In HDM, the local contributions of decision elements at one level to decision elements on the next higher level, derived from different judgment quantification methods, are supplied as intermediate input to the hierarchical model. Decisions obtained by evaluating the final ranking of alternatives are based on the local contributions. However, values of the local contributions are seldom known at a 100% confidence level and

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are subject to variations as the environment changes. Besides, the various pairwise comparison scales and judgment quantification techniques employed in HDM usually yield different local contribution values, and thus different results for the same problem (Triantaphyllou, 2000), and various group-opinion combining methods (i.e., Ferrell, 1985; Barzilai and Lootsma, 1997; Saaty, 2000; Hastie and Kameda, 2005) may change the current decision. Therefore, the solution of a problem is not complete with the mere determination of a rank order of decision alternatives. In order to develop an overall strategy to meet the various contingencies, one needs to conduct a sensitivity analysis (SA) for the HDM results.

SA has been regarded as a fundamental concept in the effective use and implementation of quantitative decision models (Dantzig, 1963; Evans, 1984). It has several important roles and serves different purposes in the decision-making process (e.g., Dantzig, 1963; Howard, 1968; Alexander, 1989; Kelton et al., 1998; Harrell et al., 2000; Saltelli, 2004; Reilly, 2000). SA can even provide information more significant and useful than simply knowing the model solution (Phillips et al., 1976). Conducting a SA for HDM can: (i) help visualize the impact of changes at the policy and strategy levels on decisions at the operational level; (ii) test the robustness of the recommended decision (Ho, 2004); (iii) identify the critical elements of the decision (Armacost and Hosseini, 1994; Triantaphyllou and Sanchez, 1997); (iv) generate scenarios of possible rankings of decision alternatives under different conditions (Winebrake and Creswick, 2003); (v) help judgment providers (the experts) reach consensus (Yeh et al., 2001); and (vi) offer answers to “what if” questions.

There is considerable literature on the development of SA for various operations research and management science models (Triantaphyllou and Sanchez, 1997). However, literature on the SA for HDM is limited. Most literature in the field of HDM has been focused on the applications side (Golden et al., 1989; Forman and Gass, 2001). Theoretical studies have been geared toward analyzing and solving the ranking irregularity problems (Belton and Gear, 1983, 1985; Triantaphyllou, 2000) and comparing and evaluating different HDM (Triantaphyllou, 2000). Some studies in this group relevant to SA tried to identify the ranges in which the values in the AHP pairwise comparison matrix can vary without causing the rank reversal problem (Arbel and Vargas, 1990; Moreno-Jimenez and Vargas, 1993; Sugihara and Tanaka, 2001; Farkas et al., 2004).

In much of the literature where HDM, especially AHP, were applied to help solve problems, a basic SA was conducted by incrementally changing the numerical values of specific problems and graphically showing the corresponding trend of changes in the model result (Partovi, 1994; Borthick and Scheiner, 1998; Yeh et al., 2001; Winebrake and Creswick, 2003; Ho, 2004). Such literature constitutes the first group of sensitivity studies of HDM, namely the numerical incremental analysis, which is an iteration-based and data-dependant process. *Expert Choice* (1990), software based on AHP, belongs to this group since it offers a basic SA function with which users can alter one of the criteria weights and see graphically how the global priorities change. However, the function is very limited: it does not allow users to change values at levels other than the first level of the decision hierarchy, nor does it allow more than one change at a time.

Other researchers employ a simulations approach to study the sensitivity of HDM. They replace values in the local contribution matrix with probability distributions and calculate the expected value of ranks after hundreds of simulation runs (Hauser and Tadikamalla, 1996; Butler et al., 1997). The probabilistic input introduces stochasticity to the output, thus making the model non-deterministic. The algorithm proposed in this paper focuses on deterministic additive HDM but is also applicable to non-deterministic additive HDM: If the probability distributions of the contribution values are known in an interval, our algorithm can determine the probability of rank changes and generate scenarios of rankings while the contribution value varies within the interval. This is significantly more information than that provided by the expected ranks obtained in the simulations approach.

The third group of sensitivity studies of HDM is through mathematical deduction, which is usually used when simple closed-form expressions can describe the relationship between inputs and outputs. Compared to numerical incremental analysis and the simulations approach, mathematical deduction has better performance (rich with information, precisely defined threshold value to any decimal place, 100% accurate once the deduction process has been verified), less computational complexity (fast, does not depend on repetitive iterations or large replications) and equal generality (same assumptions). Since performance, computational complexity and generality are the three characteristics to be compared while evaluating systems methods (Klir, 2001), mathematical deduction is identified to be the approach preferred overall to study the sensitivity of HDM. Major studies that employed mathematical deduction are discussed below.

Armocost and Hosseini (1994) defined the *determinant attribute* in an AHP decision hierarchy as the one that most differentiates the final ranking of the alternatives. Masuda (1990) and Huang (2002) investigated the situation when the entire AHP contribution matrix is perturbed and proposed two different sensitivity coefficients, both as measurements of the likelihood of rank changes: The closer to zero the coefficient is, the less likely the rank changes will occur. Triantaphyllou and Sanchez (1997) studied the threshold of a single change in the first-level contribution vector of a weighted sum model (including AHP) and a weighted product model, and proposed a sensitivity coefficient without normalizing the perturbation threshold (see Appendix A). For multiplicative AHP, Aguaron and Moreno-Jimenez (2000) proposed a local stability index as the reciprocal of a local stability interval in which a judgment, alternative, or matrix can vary without changing the ranking of alternatives' local priorities (ω_i).

To summarize, although researchers frequently understand the importance of SA for HDM, no study has developed a comprehensive algorithm to examine the sensitivity of HDM results in a fast, accurate and precise way. To close the research gap, we propose an HDM SA algorithm to study the model's robustness to changes in every local contribution matrix at different levels. The algorithm is independent from the pairwise comparison scales and judgment quantification techniques and is applicable to all HDM based on an additive relationship. A comprehensive SA for HDM using the multiplicative function proposed by Barzilai and Lootsma (1997), which is less widely used and is invalid in certain situations (Vargas, 1998), will be studied in future research.

The next section begins by introducing the model structure of additive HDM and clarifying the notations used in this paper. Five groups of propositions that constitute the HDM SA algorithm are then presented to define several sensitivity indicators of HDM in different situations. The SA of adding new decision alternatives to an existing model is also addressed. Data from a recent Ph.D. dissertation (Ho, 2004) is employed as a simple example to illustrate the application of the algorithm and show the significant insight gained through HDM SA. Contributions and future work conclude the paper.

2. The HDM SA algorithm

2.1. HDM model structure and notations

Since mathematical deduction in symbolic form is employed to study the sensitivity of HDM results to variations at different levels of the decision hierarchy, instead of the typical three-level model used in most of the literature (e.g., Armocost and Hosseini, 1994; Triantaphyllou, 2000), a "MOGSA (mission-objective-goal-strategy-action) model" (Cleland and Kocaoglu, 1981), which consists of five decision levels, is used to represent the general HDM model structure in this paper. In applications, the levels of the hierarchy can be extended or reduced according to specific needs. The notations used in this paper are as follows (see Fig. 1):

- O_ℓ the ℓ th objective, $\ell = 1, 2, \dots, L$
- G_k the k th goal, $k = 1, 2, \dots, K$

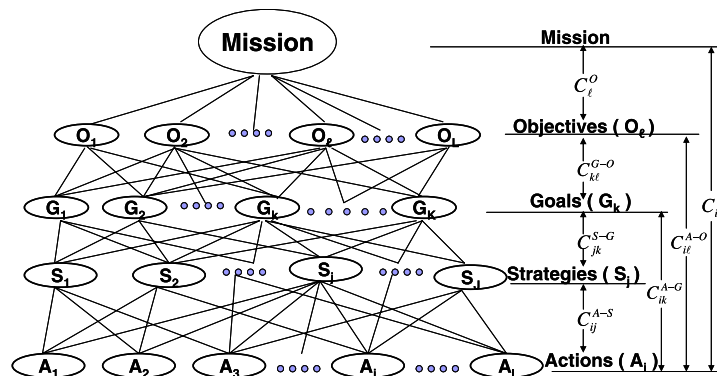


Fig. 1. HDM model structure.

S_j	the j th strategy, $j = 1, 2, \dots, J$
A_i	the i th action, $i = 1, 2, \dots, I$
L	number of objectives
K	number of goals
I	number of actions
J	number of strategies
C_i^A	overall contribution of the i th action A_i to the mission
r	the rank of i . A_r ranks before A_{r+n} , which indicates $C_r^A > C_{r+n}^A$
$C_{i\ell}^{A-O}$	contribution of the i th action A_i to the ℓ th objective O_ℓ
C_{ik}^{A-G}	contribution of the i th action A_i to the k th goal G^k
C_{ij}^{A-S}	contribution of the i th action A_i to the j th strategy S_j
C_j^S	overall contribution of the j th strategy to the mission
C_{jk}^{S-G}	contribution of the j th strategy to the k th goal
$C_{k\ell}^{G-O}$	contribution of the k th goal to the ℓ th objective
C_ℓ^O	contribution of the ℓ th objective to the mission

The terms “criteria weights” (Ra, 1988; Triantaphyllou and Sanchez, 1997; Yeh et al., 2001; Winebrake and Creswick, 2003), “priority” (Saaty, 1980, 2000; Masuda, 1990; Armacost and Hosseini, 1994; Aguaron and Moreno-Jimenez, 2000; Huang, 2002) and “performance values” (Triantaphyllou and Sanchez, 1997) used in the literature are called “contributions” in this study because they are actually measurements of the contribution of a decision element to another element on a higher level.

At the bottom level of the decision hierarchy, actions are the decision alternatives under evaluation. They are ranked according to their overall contribution to the mission, denoted as C_i^A , which is calculated by taking the sum-product of all the local contribution matrices between M and A levels:

$$C_i^A = \sum_{\ell=1}^L C_{i\ell}^{A-O} \times C_\ell^O = \sum_{\ell=1}^L \sum_{k=1}^K C_{ik}^{A-G} \times C_{k\ell}^{G-O} \times C_\ell^O = \sum_{\ell=1}^L \sum_{k=1}^K \sum_{j=1}^J C_{ij}^{A-S} \times C_{jk}^{S-G} \times C_{k\ell}^{G-O} \times C_\ell^O. \quad (1)$$

All the values in the matrices are normalized so that the contributions to each decision element add up to 1:

$$\sum_{\ell=1}^L C_\ell^O = 1, \quad \sum_{k=1}^K C_{k\ell}^{G-O} = 1, \quad \sum_{j=1}^J C_{jk}^{S-G} = 1, \quad \sum_{i=1}^I C_{ij}^{A-S} = 1. \quad (2)$$

(Note that the major difference between additive HDM and multiplicative HDM lies in this aggregation step.)

2.2. Assumptions

All the assumptions that apply to additive HDM are applicable in this study. In addition, it is assumed that when perturbations are induced on any of the contributions, the values of other related contributions will be changed according to their original ratio scale relationships, so the contributions of different decision elements to a higher-level decision element still add up to 1. For example, if M perturbations $P_{k_m^* \ell^*}^{G-O}$ ($m = 1, 2, \dots, M$) are induced on contributions of M goals, $G_{k_m^*}$'s, to a specific objective, O_{ℓ^*} , the new values of $C_{k_m^* \ell^*}^{G-O}$'s will be

$$C_{k_m^* \ell^*}^{G-O}(\text{new}) = C_{k_m^* \ell^*}^{G-O} + P_{k_m^* \ell^*}^{G-O}. \quad (3a)$$

The new values of other $C_{k\ell}^{G-O}$'s will be

$$C_{k\ell}^{G-O}(\text{new}) = C_{k\ell}^{G-O} + P_{k\ell}^{G-O}, \quad \text{with } P_{k\ell}^{G-O} = - \sum_{m=1}^M P_{k_m^* \ell^*}^{G-O} \times \frac{C_{k\ell}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_1^*, \dots, k_M^*}}^K C_{k\ell}^{G-O}}. \quad (3b)$$

(* indicates that perturbation(s) are induced on contribution(s) related to that specific decision element.)

2.3. Tolerance analysis

Tolerance is defined as the allowable range in which a contribution value can vary without changing the rank order of decision alternatives. To determine the tolerance of each contribution, the *allowable range of perturbations* on the contribution is calculated first. The allowable range of perturbations corresponds to the “slack” or “allowable increase and decrease”, as used in the sensitivity analysis of linear programming (Murty, 1976; Phillips et al., 1976).

The logic behind deducting the allowable range of perturbations is: Suppose originally A_r ranks before A_t , indicating ($C_r^A > C_t^A$); the rank order of A_r and A_t will be preserved if the new value of C_r^A is still greater than or equal to the new value of C_t^A . Therefore, the relationships between the perturbation(s) and the contributions can be found by representing the new values of C_r^A and C_t^A with an expression containing the original contributions and the induced perturbation(s). For details of the mathematical deductions, please refer to Appendix B.

As noted in the literature (Triantaphyllou and Sanchez, 1997; Aguaron and Moreno-Jimenez, 2000; Barron and Schmidt, 1988), decision makers may be interested in either the ranking of all decision alternatives or only the top choice in different cases. In this paper three situations are considered to preserve the current rank order of: (i) a pair of decision alternatives, (ii) all decision alternatives, and (iii) the best alternative.

In an effort to offer a comprehensive algorithm, we present three groups of propositions in the following subsections, covering situations when multiple and single perturbations are induced on local contribution matrices from the top to the bottom level of the decision hierarchy. Tolerance of the local contributions at each level is also defined.

2.3.1. First level contribution vector

Theorem 1. Let $P_{\ell_m}^O \left(-C_{\ell_m}^O \leq P_{\ell_m}^O \leq 1 - C_{\ell_m}^O, \sum_{m=1}^M P_{\ell_m}^O \leq 1 - \sum_{m=1}^M C_{\ell_m}^O, m = 1, 2, \dots, M \right)$ denote M perturbations induced on M of the C_{ℓ}^O 's, which are $C_{\ell_m}^O$; the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \geq P_{\ell_1}^O \times \lambda_{\ell_1}^O + P_{\ell_2}^O \times \lambda_{\ell_2}^O + \dots + P_{\ell_m}^O \times \lambda_{\ell_m}^O + \dots + P_{\ell_M}^O \times \lambda_{\ell_M}^O, \tag{4a}$$

where $\lambda = C_r^A - C_{r+n}^A$, (4b)

$$\lambda_{\ell_m}^O = C_{r+n, \ell_m}^{A-O} - C_{r, \ell_m}^{A-O} - \sum_{\substack{\ell=1 \\ \ell \neq \ell_1^*, \dots, \ell_m^*, \dots, \ell_M^*}}^L C_{r+n, \ell}^{A-O} \times \frac{C_{\ell}^O}{\sum_{\substack{\ell=1 \\ \ell \neq \ell_1^*, \dots, \ell_m^*, \dots, \ell_M^*}}^L C_{\ell}^O} + \sum_{\substack{\ell=1 \\ \ell \neq \ell_1^*, \dots, \ell_m^*, \dots, \ell_M^*}}^L C_{r, \ell}^{A-O} \times \frac{C_{\ell}^O}{\sum_{\substack{\ell=1 \\ \ell \neq \ell_1^*, \dots, \ell_m^*, \dots, \ell_M^*}}^L C_{\ell}^O}. \tag{4c}$$

The top choice will remain at the top rank if the above condition is satisfied for all $r = 1$ and $n = 1, 2, \dots, I - 1$. The rank order of all A_i 's will remain unchanged if the above condition is satisfied for all $r = 1, 2, \dots, I - 1$, and $n = 1$.

Theorem 1 defines an M dimensional allowable region for M perturbations induced in the first level contribution vector C_{ℓ}^O . As long as the values of the perturbations fall into this allowable region, current rank orders will remain unchanged. When ($M = 1$), which means only one C_{ℓ}^O value is perturbed, the threshold of the perturbation can be determined by Corollary 1.1.

Corollary 1.1. Let $P_{\ell^*}^O \left(-C_{\ell^*}^O \leq P_{\ell^*}^O \leq 1 - C_{\ell^*}^O \right)$ denote the perturbation induced on one of the C_{ℓ}^O 's, which is $C_{\ell^*}^O$; the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \geq P_{\ell^*}^O \lambda^O, \tag{5a}$$

where $\lambda = C_r^A - C_{r+n}^A$, (5b)

$$\lambda^O = C_{r+n, \ell^*}^{A-O} - C_{r, \ell^*}^{A-O} - \sum_{\ell=1, \ell \neq \ell^*}^L C_{r+n, \ell}^{A-O} \times \frac{C_{\ell}^O}{\sum_{\ell=1, \ell \neq \ell^*}^L C_{\ell}^O} + \sum_{\ell=1, \ell \neq \ell^*}^L C_{r, \ell}^{A-O} \times \frac{C_{\ell}^O}{\sum_{\ell=1, \ell \neq \ell^*}^L C_{\ell}^O}. \tag{5c}$$

The top choice will remain at the top rank if the above condition is satisfied for all $r = 1$ and $n = 1, 2, \dots, I - 1$. The rank order of all A_i 's will remain unchanged if the above condition is satisfied for all $r = 1, 2, \dots, I - 1$, and $n = 1$.

Thresholds of the single perturbation $P_{\ell^*}^O$, denoted as $\varepsilon_{\ell^-}^O$ (negative) and $\varepsilon_{\ell^+}^O$ (positive), to preserve current ranking of interested A_i 's can be calculated from (5a)–(5c). Combining the feasibility constraint ($-C_{\ell^*}^O \leq P_{\ell^*}^O \leq 1 - C_{\ell^*}^O$), which protects any $C_{\ell^*}^O$ value from going below zero or above one, the allowable range of perturbations on $C_{\ell^*}^O$, denoted as $[\delta_{\ell^-}^O, \delta_{\ell^+}^O]$, can be derived as $[\text{Max}\{-C_{\ell^*}^O, \varepsilon_{\ell^-}^O\}, \text{Min}\{1 - C_{\ell^*}^O, \varepsilon_{\ell^+}^O\}]$. Then, the tolerance of the corresponding contribution $C_{\ell^*}^O$ is $[\delta_{\ell^-}^O + C_{\ell^*}^O, \delta_{\ell^+}^O + C_{\ell^*}^O]$. As long as the value of $C_{\ell^*}^O$ is within this tolerance range, the final ranking of A_i 's under consideration will remain unchanged. To derive the allowable range of perturbations or the tolerance of a $C_{\ell^*}^O$, I inequalities need to be satisfied in both cases: to either preserve the top-ranked alternative only or to preserve the rank order for all A_i 's. I is the number of decision alternatives.

2.3.2. Middle levels of the decision hierarchy

Theorem 2 and its corollaries are applicable to perturbation(s) induced in middle-level contribution matrices, such as C_{kl}^{G-O} and C_{jk}^{S-G} in the MOGSA model. Notations used in this group of propositions are from the C_{kl}^{G-O} matrix.

Theorem 2. Let $P_{k_m^* \ell_x^*}^{G-O}$ ($-C_{k_m^* \ell_x^*}^{G-O} \leq P_{k_m^* \ell_x^*}^{G-O} \leq 1 - C_{k_m^* \ell_x^*}^{G-O}, \sum_{m=1}^M P_{k_m^* \ell_x^*}^{G-O} \leq 1 - \sum_{m=1}^M C_{k_m^* \ell_x^*}^{G-O}, m = 1, 2, \dots, M$) denote M perturbations induced on M of the C_{kl}^{G-O} 's (contributions of M goals $G_{k_m^*}$ to the x th changing objective $O_{\ell_x^*}$), $P_{k_t^* \ell_\beta^*}^{G-O}$ ($-C_{k_t^* \ell_\beta^*}^{G-O} \leq P_{k_t^* \ell_\beta^*}^{G-O} \leq 1 - C_{k_t^* \ell_\beta^*}^{G-O}, \sum_{t=1}^T P_{k_t^* \ell_\beta^*}^{G-O} \leq 1 - \sum_{t=1}^T C_{k_t^* \ell_\beta^*}^{G-O}, t = 1, 2, \dots, T$) denote T perturbations induced on T of the C_{kl}^{G-O} 's (contributions of T goals $G_{k_t^*}$ to the β th changing objective $O_{\ell_\beta^*}$), $P_{k_q^* \ell_\gamma^*}^{G-O}$ ($-C_{k_q^* \ell_\gamma^*}^{G-O} \leq P_{k_q^* \ell_\gamma^*}^{G-O} \leq 1 - C_{k_q^* \ell_\gamma^*}^{G-O}, \sum_{q=1}^Q P_{k_q^* \ell_\gamma^*}^{G-O} \leq 1 - \sum_{q=1}^Q C_{k_q^* \ell_\gamma^*}^{G-O}, q = 1, 2, \dots, Q$) denote Q perturbations induced on Q of the C_{kl}^{G-O} 's (contributions of Q goals $G_{k_q^*}$ to the γ th changing objective $O_{\ell_\gamma^*}$); the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \geq \sum_{m=1}^M (P_{k_m^* \ell_x^*}^{G-O} \times \lambda_{k_m^* \ell_x^*}^{G-O}) + \sum_{t=1}^T (P_{k_t^* \ell_\beta^*}^{G-O} \times \lambda_{k_t^* \ell_\beta^*}^{G-O}) + \sum_{q=1}^Q (P_{k_q^* \ell_\gamma^*}^{G-O} \times \lambda_{k_q^* \ell_\gamma^*}^{G-O}), \tag{6a}$$

$$\text{where } \lambda = C_r^A - C_{r+n}^A, \tag{6b}$$

$$\lambda_{k_m^* \ell_x^*}^{G-O} = C_{\ell_x^*}^O \times \left[C_{r+n, k_m^*}^{A-G} - C_{rk_m^*}^{A-G} + \sum_{\substack{k=1 \\ k \neq k_m^*}}^K (C_{rk}^{A-G} - C_{r+n, k}^{A-G}) \times \frac{C_{kl_x^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{kl_x^*}^{G-O}} \right], \tag{6c}$$

$$\lambda_{k_t^* \ell_\beta^*}^{G-O} = C_{\ell_\beta^*}^O \times \left[C_{r+n, k_t^*}^{A-G} - C_{rk_t^*}^{A-G} + \sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{\ell_\beta^*}^O (C_{rk}^{A-G} - C_{r+n, k}^{A-G}) \times \frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{k\ell_\beta^*}^{G-O}} \right], \tag{6d}$$

$$\lambda_{k_q^* \ell_\gamma^*}^{G-O} = C_{\ell_\gamma^*}^O \times \left[C_{r+n, k_q^*}^{A-G} - C_{rk_q^*}^{A-G} + \sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{\ell_\gamma^*}^O (C_{rk}^{A-G} - C_{r+n, k}^{A-G}) \times \frac{C_{k\ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{k\ell_\gamma^*}^{G-O}} \right]. \tag{6e}$$

The top choice will remain at the top rank if the above condition is satisfied for all $r = 1$ and $n = 1, 2, \dots, I - 1$. The rank order of all A_i 's will remain unchanged if the above condition is satisfied for all $r = 1, 2, \dots, I - 1$, and $n = 1$.

Theorem 2 deals with a general situation when different numbers (M, T, Q) of the local contributions to three objectives ($O_{\ell_x^*}, O_{\ell_\beta^*}$ and $O_{\ell_\gamma^*}$) are perturbed (see Fig. 2). It defines a $(M + T + Q)$ dimensional allowable region for the $(M + T + Q)$ perturbations induced in the local contribution matrix C_{kl}^{G-O} . When contributions to more than three objectives need to be changed, (6a) can be extended by adding more $\sum_{x=1}^X (P_{k_x^* \ell_\theta^*}^{G-O} \times \lambda_{k_x^* \ell_\theta^*}^{G-O})$'s following the same pattern, using x to represent the number of perturbations induced for each C_{kl}^{G-O} and θ to differentiate the new $O_{\ell_\theta^*}$ to which the x contributions will be perturbed. When there is only one C_{kl}^{G-O} being changed, the threshold of such change can be determined by Corollary 2.1.

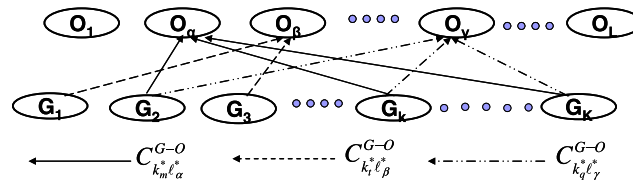


Fig. 2. Contributions of multiple goals to multiple objectives.

Corollary 2.1. Let $P_{k^* \ell^*}^{G-O} (-C_{k^* \ell^*}^{G-O} < P_{k^* \ell^*}^{G-O} < 1 - C_{k^* \ell^*}^{G-O})$ denote a perturbation induced on one of the C_{kl}^{G-O} 's, which is $C_{k^* \ell^*}^{G-O}$ (contribution of a specific goal G_{k^*} to a specific objective O_{ℓ^*}); the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \geq P_{k^* \ell^*}^{G-O} \times \lambda_{k_1 \ell}^{G-O}, \tag{7a}$$

where $\lambda = C_r^A - C_{r+n}^A$, (7b)

$$\lambda_{k_1 \ell}^{G-O} = C_{\ell^*}^O \times \left[C_{r+n, k^*}^{A-G} - C_{rk^*}^{A-G} + \left(\sum_{\substack{k=1 \\ k \neq k^*}}^K C_{rk}^{A-G} - \sum_{\substack{k=1 \\ k \neq k^*}}^K C_{r+n, k}^{A-G} \right) \times \frac{C_{k \ell^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k^*}}^K C_{k \ell^*}^{G-O}} \right]. \tag{7c}$$

The top choice will remain at the top rank if the above condition is satisfied for all $r = 1$ and $n = 1, 2, \dots, I - 1$. The rank order of all A_i 's will remain unchanged if the above condition is satisfied for all $r = 1, 2, \dots, I - 1$, and $n = 1$.

The thresholds of $P_{k^* \ell^*}^{G-O}$ in both directions, denoted as $\varepsilon_{k \ell -}^{G-O}$ and $\varepsilon_{k \ell +}^{G-O}$, can be derived from (6a)–(6c). Then, the allowable range of perturbations on $C_{k^* \ell^*}^{G-O}$ is $[\delta_{k \ell -}^{G-O}, \delta_{k \ell +}^{G-O}]$, where $(\delta_{k \ell -}^{G-O} = \text{Max}\{-C_{k^* \ell^*}^{G-O}, \varepsilon_{k \ell -}^{G-O}\})$ and $(\delta_{k \ell +}^{G-O} = \text{Min}\{1 - C_{k^* \ell^*}^{G-O}, \varepsilon_{k \ell +}^{G-O}\})$. The tolerance of contribution $C_{k^* \ell^*}^{G-O}$ is $[\delta_{k \ell -}^{G-O} + C_{k^* \ell^*}^{G-O}, \delta_{k \ell +}^{G-O} + C_{k^* \ell^*}^{G-O}]$.

Corollary 2.2. Let $P_{k_m^* \ell^*}^{G-O} (-C_{k_m^* \ell^*}^{G-O} < P_{k_m^* \ell^*}^{G-O} < 1 - C_{k_m^* \ell^*}^{G-O}, \sum_{k=1, k \neq k_m^*}^K C_{k \ell^*}^{G-O} - 1 < \sum_{m=1}^M P_{k_m^* \ell^*}^{G-O} < \sum_{k=1, k \neq k_m^*}^K C_{k \ell^*}^{G-O}, m = 1, 2, \dots, M)$ denote M perturbations induced on M of the C_{kl}^{G-O} 's, which are $C_{k_m^* \ell^*}^{G-O}$ (contributions of specific goals G_{k^*} 's to a specific objective O_{ℓ^*} , see Fig. 3); the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \geq P_{k_1^* \ell^*}^{G-O} \times \lambda_{k_1 \ell}^{G-O} + P_{k_2^* \ell^*}^{G-O} \times \lambda_{k_2 \ell}^{G-O} + \dots + P_{k_m^* \ell^*}^{G-O} \times \lambda_{k_m \ell}^{G-O} + \dots + P_{k_M^* \ell^*}^{G-O} \times \lambda_{k_M \ell}^{G-O}, \tag{8a}$$

where $\lambda = C_r^A - C_{r+n}^A$, (8b)

$$\lambda_{k_m \ell}^{G-O} = C_{\ell^*}^O \times \left[C_{r+n, k_m^*}^{A-G} - C_{rk_m^*}^{A-G} + \sum_{\substack{k=1 \\ k \neq k_1^* \dots k_M^*}}^K (C_{rk}^{A-G} - C_{r+n, k}^{A-G}) \times \frac{C_{k \ell^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_1^* \dots k_M^*}}^K C_{k \ell^*}^{G-O}} \right]. \tag{8c}$$

The top choice will remain at the top rank if the above condition is satisfied for all $r = 1$ and $n = 1, 2, \dots, I - 1$. The rank order of all A_i 's will remain unchanged if the above condition is satisfied for all $r = 1, 2, \dots, I - 1$, and $n = 1$.

2.3.3. Bottom level of the decision hierarchy

Theorem 3 and its corollaries deal with perturbations induced in matrix C_{ij}^{A-S} , which is the bottom level of the decision hierarchy. Since the decision alternatives' level is involved in the analysis, situations are

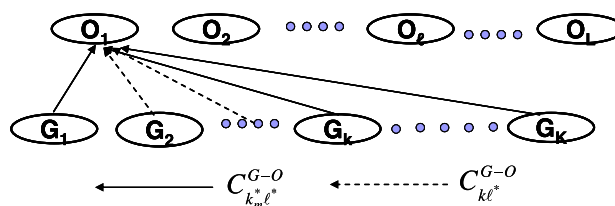


Fig. 3. Contributions of multiple goals G_{k^*} to a specific objective O_{ℓ^*} .

differentiated when the perturbation(s) are induced on the decision alternatives being compared or not. Consequently, more complex propositions are developed to address various situations.

Theorem 3. Let $P_{i_m^* j_\alpha^*}^{A-S} \left(-C_{i_m^* j_\alpha^*}^{A-S} \leq P_{i_m^* j_\alpha^*}^{A-S} \leq 1 - C_{i_m^* j_\alpha^*}^{A-S}, \sum_{m=1}^M P_{i_m^* j_\alpha^*}^{A-S} \leq 1 - \sum_{m=1}^M C_{i_m^* j_\alpha^*}^{A-S}, m = 1, 2, \dots, M \right)$ denote M perturbations induced in M of the $C_{ij_\alpha}^{A-S}$'s (contributions of M actions $A_{i_m^*}$ to the α th changing strategy $S_{j_\alpha^*}$), $P_{i_t^* j_\beta^*}^{A-S} \left(-C_{i_t^* j_\beta^*}^{A-S} \leq P_{i_t^* j_\beta^*}^{A-S} \leq 1 - C_{i_t^* j_\beta^*}^{A-S}, \sum_{t=1}^T P_{i_t^* j_\beta^*}^{A-S} \leq 1 - \sum_{t=1}^T C_{i_t^* j_\beta^*}^{A-S}, t = 1, 2, \dots, T \right)$ denote T perturbations induced in T of the $C_{ij_\beta}^{A-S}$'s (contributions of T actions $A_{i_t^*}$ to the β th changing strategy $S_{j_\beta^*}$), $P_{i_q^* j_\gamma^*}^{A-S} \left(-C_{i_q^* j_\gamma^*}^{A-S} \leq P_{i_q^* j_\gamma^*}^{A-S} \leq 1 - C_{i_q^* j_\gamma^*}^{A-S}, \sum_{q=1}^Q P_{i_q^* j_\gamma^*}^{A-S} \leq 1 - \sum_{q=1}^Q C_{i_q^* j_\gamma^*}^{A-S}, q = 1, 2, \dots, Q \right)$ denote Q perturbations induced in Q of the $C_{k\ell_\gamma}^{G-O}$'s (contributions of Q actions $A_{i_q^*}$ to the γ th changing strategy $S_{j_\gamma^*}$); the original ranking of A_r and A_{r+n} will not reverse if:

$$C_{r^*}^A - C_{r+n^*}^A \geq -C_{j_\alpha^*}^S \times P_{r^* j_\alpha^*}^{A-S} - C_{j_\alpha^*}^S \times \sum_{m=1}^M P_{i_m^* j_\alpha^*}^{A-S} \times \frac{C_{r+n, j_\alpha^*}^{A-S}}{\sum_{i=1, i \neq i_m^*}^I C_{ij_\alpha^*}^{A-S}} - C_{j_\beta^*}^S \times \sum_{t=1}^T P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{r+n, j_\beta^*}^{A-S}}{\sum_{i=1, i \neq i_t^*}^I C_{ij_\beta^*}^{A-S}} - C_{j_\gamma^*}^S \times \sum_{q=1}^Q P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{r+n, j_\gamma^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^I C_{ij_\gamma^*}^{A-S}}$$

(when some perturbations are induced on C_{rj}^{A-S} 's but not on $C_{r+n, j}^{A-S}$'s), (9a)

$$C_r^A - C_{r+n^*}^A \geq C_{j_\alpha^*}^S \times P_{r+n^* j_\alpha^*}^{A-S} + C_{j_\alpha^*}^S \times \sum_{m=1}^M P_{i_m^* j_\alpha^*}^{A-S} \times \frac{C_{r, j_\alpha^*}^{A-S}}{\sum_{i=1, i \neq i_m^*}^I C_{ij_\alpha^*}^{A-S}} + C_{j_\beta^*}^S \times \sum_{t=1}^T P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{r, j_\beta^*}^{A-S}}{\sum_{i=1, i \neq i_t^*}^I C_{ij_\beta^*}^{A-S}} + C_{j_\gamma^*}^S \times \sum_{q=1}^Q P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{r, j_\gamma^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^I C_{ij_\gamma^*}^{A-S}}$$

(when some perturbations are induced on $C_{r+n, j}^{A-S}$'s but not on C_{rj}^{A-S} 's), (9b)

$$C_r^A - C_{r+n}^A \geq C_{j_\alpha^*}^S \times \sum_{m=1}^M P_{i_m^* j_\alpha^*}^{A-S} \times \frac{C_{r, j_\alpha^*}^{A-S} - C_{r+n, j_\alpha^*}^{A-S}}{\sum_{i=1, i \neq i_m^*}^I C_{ij_\alpha^*}^{A-S}} + C_{j_\beta^*}^S \times \sum_{t=1}^T P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{r, j_\beta^*}^{A-S} - C_{r+n, j_\beta^*}^{A-S}}{\sum_{i=1, i \neq i_t^*}^I C_{ij_\beta^*}^{A-S}} + C_{j_\gamma^*}^S \times \sum_{q=1}^Q P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{r, j_\gamma^*}^{A-S} - C_{r+n, j_\gamma^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^I C_{ij_\gamma^*}^{A-S}}$$

(when some perturbations are induced on both C_{rj}^{A-S} 's and $C_{r+n, j}^{A-S}$'s), (9c)

$$C_{r^*}^A - C_{r+n^*}^A \geq C_{j_\alpha^*}^S \times \left(P_{r+n^* j_\alpha^*}^{A-S} - P_{r^* j_\alpha^*}^{A-S} \right) \quad \text{(when no perturbation is induced on } C_{rj}^{A-S} \text{ nor } C_{r+n, j}^{A-S} \text{).}$$
 (9d)

The top choice will remain at the top rank if all the above conditions, (9a)–(9d), are satisfied for all $r = 1$ and $n = 1, 2, \dots, I - 1$. The original ranking for all A_i 's will remain unchanged if all the above conditions, (9a)–(9d), are satisfied for all $r = 1, 2, \dots, I - 1$, and $n = 1$.

Theorem 3 deals with a general situation when different numbers (M, T, Q) of the local contributions to three strategies ($S_{j_\alpha^*}, S_{j_\beta^*}$ and $S_{j_\gamma^*}$) are perturbed (see Fig. 4). When contributions to more than three strategies

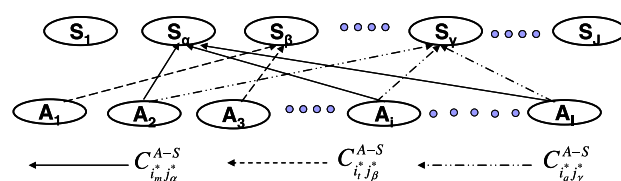


Fig. 4. Contributions of multiple actions to multiple strategies.

need to be changed, (9a)–(9c) can be extended by adding more $\left(C_{\theta}^{A-S} \times \sum_{x=1}^X P_{i_x, \theta}^{A-S} \times \frac{C_{ij_{\theta}}^{A-S}}{\sum_{i=1, i \neq i_x}^I C_{ij_{\theta}}^{A-S}} \right)$ following the same pattern, using x to represent the number of perturbations induced for each $C_{ij_{\theta}}^{A-S}$ and θ to differentiate the new $S_{j_{\theta}}^*$ to which the x contributions will be perturbed. When only one C_{ij}^{A-S} value is perturbed, the threshold of such a perturbation can be determined based on Corollary 3.1.

Corollary 3.1. Let $P_{i^* j^*}^{A-S} \left(-C_{i^* j^*}^{A-S} \leq P_{i^* j^*}^{A-S} \leq 1 - C_{i^* j^*}^{A-S} \right)$ denote the perturbation induced on one of the C_{ij}^{A-S} 's, which is $C_{i^* j^*}^{A-S}$ (contribution of a specific action A_{i^*} to a specific strategy S_{j^*}); the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \geq P_{i^* j^*}^{A-S} \lambda_{ij}^{A-S}, \tag{10a}$$

$$\text{where } \lambda = C_r^A - C_{r+n}^A, \tag{10b}$$

$$\lambda_{ij}^{A-S} = \frac{C_{rj^*}^S \left(C_{rj^*}^{A-S} - C_{r+n, j^*}^{A-S} \right)}{\sum_{i=1, i \neq i^*}^I C_{ij^*}^{A-S}} \quad \left(\text{if } P_{i^* j^*}^{A-S} \text{ is induced on neither } C_{rj}^{A-S} \text{ nor } C_{r+n, j}^{A-S} \right), \tag{10c}$$

$$\text{or } \lambda_{ij}^{A-S} = C_{j^*}^S \times \left(1 + \frac{C_{rj^*}^{A-S}}{\sum_{i=1, i \neq r+n}^I C_{ij^*}^{A-S}} \right) \quad \left(\text{if } P_{i^* j^*}^{A-S} \text{ is induced on } C_{r+n, j}^{A-S} \right), \tag{10d}$$

$$\text{or } \lambda_{ij}^{A-S} = -C_{j^*}^S \times \left(1 + \frac{C_{r+n, j^*}^{A-S}}{\sum_{i=1, i \neq r}^I C_{ij^*}^{A-S}} \right) \quad \left(\text{if } P_{i^* j^*}^{A-S} \text{ is induced on } C_{rj}^{A-S} \right). \tag{10e}$$

The top choice will remain at the top rank if all the above conditions are satisfied for all $r = 1$ and $n = 1, 2, \dots, I - 1$. The original ranking for all A_i 's will remain unchanged if all the above conditions, (10a)–(10e), are satisfied for all $r = 1, 2, \dots, I - 1$, and $n = 1$.

The thresholds of $P_{i^* j^*}^{A-S}$ in both directions, denoted as ϵ_{ij-}^{A-S} and ϵ_{ij+}^{A-S} , can be derived from (10a)–(10e). The allowable range of perturbations on C_{ij}^{A-S} is $[\delta_{ij-}^{A-S}, \delta_{ij+}^{A-S}]$, where $\left(\delta_{ij-}^{A-S} = \text{Max}\{-C_{ij}^{A-S}, \epsilon_{ij-}^{A-S}\} \right)$ and $\left(\delta_{ij+}^{A-S} = \text{Min}\{1 - C_{ij}^{A-S}, \epsilon_{ij+}^{A-S}\} \right)$. The tolerance of contribution $C_{i^* j^*}^{A-S}$ is $[\delta_{ij-}^{A-S} + C_{i^* j^*}^{A-S}, \delta_{ij+}^{A-S} + C_{i^* j^*}^{A-S}]$.

Corollary 3.2. Let $P_{i_m^* j^*}^{A-S} \left(-C_{i_m^* j^*}^{A-S} < P_{i_m^* j^*}^{A-S} < 1 - C_{i_m^* j^*}^{A-S}, \sum_{i=1, i \neq i_m^*}^I C_{ij^*}^{A-S} - 1 < \sum_{m=1}^M P_{i_m^* j^*}^{A-S} < \sum_{i=1, i \neq i_m^*}^I C_{ij^*}^{A-S}, m = 1, 2, \dots, M \right)$ denote M perturbations induced on M of the C_{ij}^{A-S} 's, which are $C_{i_m^* j^*}^{A-S}$ (contributions of M specific actions A_{i^*} 's to a specific strategy S_{j^*} , see Fig. 5); the original ranking of A_r and A_{r+n} will not reverse if:

$$C_r^A - C_{r+n}^A \geq \sum_{m=1}^M P_{i_m^* j^*}^{A-S} \times C_{j^*}^S \times \frac{C_{rj^*}^{A-S} - C_{r+n, j^*}^{A-S}}{\sum_{i=1, i \neq i^*}^I C_{ij^*}^{A-S}} \quad \left(\text{if } P_{i_m^* j^*}^{A-S} \text{ s are induced on neither } C_{rj}^{A-S} \text{ nor } C_{r+n, j}^{A-S} \right) \tag{11a}$$

$$\text{or } C_r^A - C_{(r+n)^*}^A \geq \left(P_{(r+n)^* j^*}^{A-S} - P_{r^* j^*}^{A-S} \right) \times C_{j^*}^S \quad \left(\text{if } P_{i_m^* j^*}^{A-S} \text{ s are induced on both } C_{rj}^{A-S} \text{ and } C_{r+n, j}^{A-S} \right) \tag{11b}$$

$$\text{or } C_r^A - C_{(r+n)^*}^A \geq \sum_{m=1, i \neq r+n}^M P_{i_m^* j^*}^{A-S} \times C_{j^*}^S \times \frac{C_{rj^*}^{A-S}}{\sum_{i=1, i \neq i^*}^I C_{ij^*}^{A-S}} + P_{(r+n)^* j^*}^{A-S} \times C_{j^*}^S \times \left(1 + \frac{C_{rj^*}^{A-S}}{\sum_{i=1, i \neq i^*}^I C_{ij^*}^{A-S}} \right) \tag{11c}$$

(if one of the $P_{i_m^* j^*}^{A-S}$'s, which is $P_{(r+n)^* j^*}^{A-S}$ in this case, is induced on $C_{r+n, j}^{A-S}$)

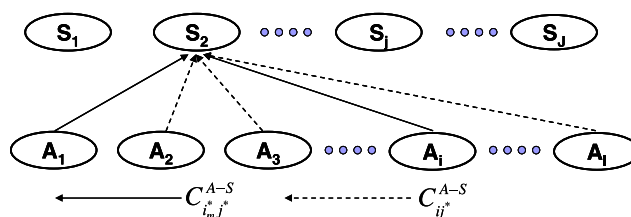


Fig. 5. Contributions of multiple actions A_{i^*} to a specific strategy S_{j^*} .

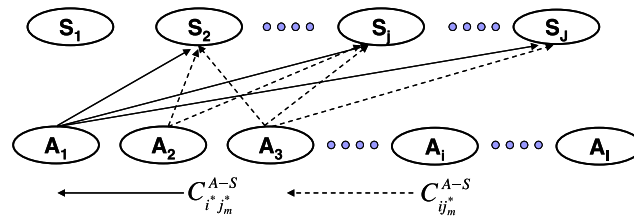


Fig. 6. Contributions of a specific action A_{i^*} to specific strategies S_{j^*} .

$$\text{or } C_{r^*}^A - C_{r+n}^A \geq - \sum_{m=1, i \neq r}^M P_{i^* j_m^*}^{A-S} \times C_{j^*}^S \times \frac{C_{r+n, j^*}^{A-S}}{\sum_{i=1, i \neq i^*}^I C_{ij_m^*}^{A-S}} - P_{r^* j_m^*}^{A-S} \times C_{j^*}^S \times \left(1 + \frac{C_{r+n, j^*}^{A-S}}{\sum_{i=1, i \neq i^*}^I C_{ij_m^*}^{A-S}} \right) \quad (11d)$$

(if one of the $P_{i^* j_m^*}^{A-S}$'s, which is $P_{r^* j_m^*}^{A-S}$ in this case, is induced on C_{rj}^{A-S}).

The top choice will remain at the top rank if (11a)–(11d), are satisfied for all $r = 1$ and $n = 1, 2, \dots, I - 1$. The original ranking for all A_i 's will remain unchanged if all the above conditions, (11a)–(11d), are satisfied for all $r = 1, 2, \dots, I - 1$, and $n = 1$.

Corollary 3.3. Let $P_{i^* j_m^*}^{A-S} \left(-C_{i^* j_m^*}^{A-S} < P_{i^* j_m^*}^{A-S} < 1 - C_{i^* j_m^*}^{A-S}, m = 1, 2, \dots, M \right)$ denote M perturbations induced on M of the C_{ij}^{A-S} 's (contributions of a specific action A_{i^*} to M specific strategies S_{j^*} 's, see Fig. 6); the original ranking of A_r and A_{r+n} will not reverse if:

$$C_r^A - C_{r+n}^A \geq \sum_{m=1}^M C_{j_m^*}^S \times P_{i^* j_m^*}^{A-S} \times \frac{C_{rj_m^*}^{A-S} - C_{r+n, j_m^*}^{A-S}}{\sum_{i=1}^I C_{ij_m^*}^{A-S}} \quad (12a)$$

(when perturbations are induced on neither C_{rj}^{A-S} 's nor $C_{r+n, j}^{A-S}$'s)

$$\text{or } C_r^A - C_{r+n}^A \geq \sum_{m=1}^M C_{j_m^*}^S \times P_{r+n, j_m^*}^{A-S} \times \left(1 + \frac{C_{rj_m^*}^{A-S}}{\sum_{i=1, i \neq t}^I C_{ij_m^*}^{A-S}} \right) \quad (12b)$$

(when perturbations are induced on $C_{r+n, j}^{A-S}$'s)

$$\text{or } C_{r^*}^A - C_{r+n}^A \geq - \sum_{m=1}^M C_{j_m^*}^S \times P_{r^* j_m^*}^{A-S} \times \left(1 + \frac{C_{r+n, j_m^*}^{A-S}}{\sum_{i=1, i \neq r}^I C_{ij_m^*}^{A-S}} \right) \quad (12c)$$

(when perturbations are induced on C_{rj}^{A-S} 's).

The top choice will remain at the top rank if (12a)–(12c) are satisfied for all $r = 1$ and $t = r + 1, r + 2, \dots, r + I - 1$. The original ranking for all A_i 's will remain unchanged if all the above conditions, (12a)–(12c), are satisfied for all $r = 1, 2, \dots, I - 1$ and $t = r + 1$.

2.3.4. Summary

The above three groups of propositions define the allowable region of perturbations and tolerance of contributions at any level of an additive decision hierarchy. Table 1 summarizes the level(s) of the contribution vector/matrix and the number of induced perturbations that each proposition deals with. The number of inequalities that have to be satisfied in each situation is also specified.

When the perturbation number equals two, a two-dimensional allowable region for the two perturbations is defined by the inequalities. When it increases to three, the allowable region for the three perturbations is a three-dimensional polyhedron, as shown in Fig. 7, with its hyperplanes defined by the inequalities. The origin, where the values of the three perturbations are all zero, represents no changes.

2.4. Sensitivity coefficients

Different sensitivity coefficients (SC) for HDM have been proposed in the literature (Masuda, 1990; Triantaphyllou and Sanchez, 1997; Huang, 2002). Masuda (1990) defined the SC as the standard deviation of the

Table 1
Summary of Propositions 1 to 3.3

Theorems (T) and Corollaries (C)	Level (s) in HDM	Number of perturbations	Number of inequalities*	
			Condition 1	Conditions 2 and 3
T 1	Top	M	$2 + M$	$I + M$
C 1.1	Top	1	2	I
T 2 (Fig. 2)	Middles	$M + T + Q$	$M + T + Q + 4$	$I + M + T + Q + 2$
C 2.1	Middles	1	2	I
C 2.2 (Fig. 3)	Middles	M	$2 + M$	$I + M$
T 3 (Fig. 4)	Bottom	$M + T + Q$	$M + T + Q + 4$	$I + M + T + Q + 2$
C 3.1	Bottom	1	2	I
C 3.2 (Fig. 5)	Bottom	M	$2 + M$	$I + M$
C 3.3 (Fig. 6)	Bottom	M	$1 + M$	$I + M - 1$

* Condition 1: Rank order of a pair of decision alternative is of concern.
 Condition 2: Rank order of all the decision alternatives is of concern.
 Condition 3: Rank order of the top choice is of concern.

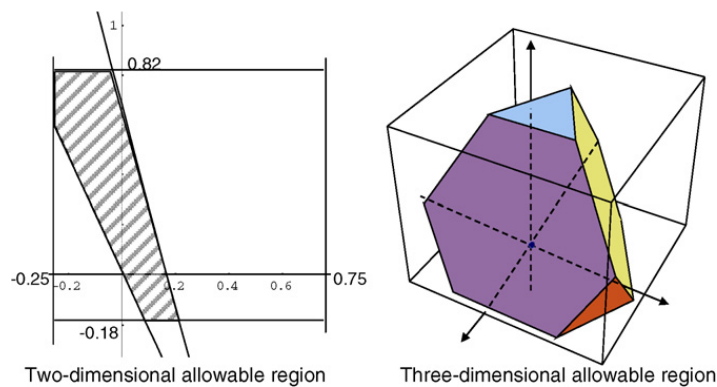


Fig. 7. The allowable region for perturbations.

“extreme vector” of an AHP model. Huang (2002) showed that Masuda’s definition was invalid in certain situations and defined another SC based on Masuda’s work, also as a measurement of the likelihood of range changes. The SC proposed by Triantaphyllou and Sanchez (1997) is the reciprocal of the smallest percentage by which the contribution must change to reverse the alternatives’ ranking. Similar to the sensitivity coefficient concept, a local stability index is defined by Aguaron and Moreno-Jimenez (2000) as the reciprocal of the local stability interval in multiplicative AHP.

In this paper, to give as complete information as possible, two sensitivity coefficients are proposed: the *operating point sensitivity coefficient* (OPSC) and the *total sensitivity coefficient* (TSC). The OPSC is defined as the shortest distance from the current contribution value to the edges of its tolerance. It is dependent on the contribution’s current value (the operating point) and directions of the change (increasing or decreasing). TSC specifies that the shorter the tolerances of a decision element’s contributions are, the more sensitive the final decision is to variations of that decision element. Evans (1984) noted that if the current parametric value is located near the center of P^* (allowable region), then the decision is robust. The OPSC defined in this paper indicates the robustness of the current decision, while the TSC reveals more about how flexible the input values can be without changing the decision. They give different but equally important information and thus should be used together.

Theorem 4.1. *If the allowable range of perturbations on C_ℓ^O is $[\delta_{\ell-}^O, \delta_{\ell+}^O]$ to preserve the final ranking of A_i ’s, the OPSC and TSC of O_ℓ are*

$$OPSC(O_\ell) = \text{Min}\{|\delta_{\ell-}^O|, |\delta_{\ell+}^O|\}, \tag{13a}$$

$$TSC(O_\ell) = |\delta_{\ell+}^O - \delta_{\ell-}^O|. \tag{13b}$$

Theorem 4.2. *If the allowable range of perturbations on C_{kl}^{G-O} is $[\delta_{kl-}^{G-O}, \delta_{kl+}^{G-O}]$ to preserve the final ranking of A_i 's, the OPSC and TSC of G_k are*

$$\text{OPSC}(G_k) = \text{Min}_{1 \leq \ell \leq L} \{|\delta_{k\ell-}^{G-O}|, |\delta_{k\ell+}^{G-O}|\}, \quad (14a)$$

$$\text{TSC}(G_k) = \text{Min}_{1 \leq \ell \leq L} \{|\delta_{k\ell+}^{G-O} - \delta_{k\ell-}^{G-O}|\}. \quad (14b)$$

Theorem 4.3. *If the allowable range of perturbations on C_{ij}^{A-S} is $[\delta_{ij-}^{A-S}, \delta_{ij+}^{A-S}]$ to preserve the final ranking of A_i 's, the OPSC and TSC of A_i are:*

$$\text{OPSC}(A_i) = \text{Min}_{1 \leq j \leq J} \{|\delta_{ij-}^{A-S}|, |\delta_{ij+}^{A-S}|\}, \quad (15a)$$

$$\text{TSC}(A_i) = \text{Min}_{1 \leq j \leq J} \{|\delta_{ij+}^{A-S} - \delta_{ij-}^{A-S}|\}. \quad (15b)$$

The smaller the sensitivity coefficients of a decision element are, the more sensitive the decision is to variations of that element. If the TSC of a decision element is one, meaning the tolerance is from zero to one, the decision is not sensitive at all to changes that occur to the contributions of this element. In addition, the TSC of a contribution is also the probability of varying that contribution value between zero and one without changing the current rankings of A_i 's.

The above theorems are based on “one-way SA” in which the influence of an input to the decision is analyzed while keeping other inputs at their base values (Clemen, 1996; Reilly, 2000). Extending the analysis to multiple simultaneous changes, we can study the sensitivity of a certain decision level in the hierarchy. Recall that in the *tolerance analysis* section, an M -dimensional allowable region is defined for M perturbations induced on any local contribution vectors to preserve the final ranking of A_i 's. Based on the same logic, the shortest distance from the origin to all hyperplanes of the M -dimensional polyhedron and the polyhedron's volume determine the robustness of the current model regarding changes to the M contribution values. As to what TSC reveals in the one-dimensional analysis, since the volume of the M perturbations' feasible region is one, the volume of the M -dimensional polyhedron is also the probability of keeping A_i 's rank orders unchanged when the M contributions vary from zero to one.

2.5. Critical decision elements

In several previous studies, researchers tried to identify the most influential variables with respect to the rank ordering of the alternatives (Howard, 1968) or “determinant attribute” that strongly contributes to the choice among alternatives (Armacost and Hosseini, 1994). In this paper, the most critical decision element is defined as the one whose influence on the final decision is most sensitive to perturbations, as defined by Triantaphyllou and Sanchez (1997). Extending their definition to multiple levels of the decision hierarchy, we get:

Theorem 5. *The most critical decision element at a given level of the decision hierarchy for current ranking of A_i 's is the decision element corresponding to the smallest TSC and OPSC at that level.*

In situations when the smallest TSC and OPSC do not occur on the same decision element, there can be two different decision elements, and each one can be considered the most critical in different situations. Additional analysis can also be carried out to determine which one is more critical.

2.6. Adding new decision alternatives

There are situations where new decision elements need to be added after a hierarchical decision model has been built. Adding new decision elements to the middle levels of the decision hierarchy will change all the contribution matrices. In this case, it is suggested that a new decision hierarchy be constructed and the overall contribution vector be recalculated. However, introducing new decision alternatives only changes the bottom level of the decision hierarchy; and SA can be applied to that special case.

With the assistance of Corollary 3.3, the impact of adding a new decision alternative can be studied by assuming that the current contributions of the new decision alternatives are zero and the new contributions are $P_{i^*j_m}^{A-S}$, where $(i = I + 1)$ and $(m = 1, 2, \dots, J)$. The currently top-ranked decision alternative will remain unchanged as long as inequality (12b) is satisfied for $(r = 1)$ and $(n = I)$. The current ranking of all decision alternatives will remain unchanged if (12b) is satisfied for $(r = 1, 2, \dots, I)$ and $(r + n = I + 1)$, with the new decision alternative ranked last. Based on the same logic, adding multiple new decision alternatives can be analyzed using Theorem 3. The entire decision hierarchy does not have to be re-calculated.

3. An example

All the propositions in the tolerance analysis section are verified using data from a recent Ph.D. dissertation by Ho (2004). The verification shows that whenever the perturbations induced to the local contribution matrices go beyond their allowable region, the ranking of the interested decision alternatives will be changed. Due to limited space, the detailed verification process will not be shown in this paper. The purpose of the example here is to demonstrate the use of HDM SA and show insights that are not available or intuitively recognizable without conducting an HDM SA.

Ho's model evaluated five emerging technologies in Taiwan's semiconductor foundry industry by using a hierarchical decision model containing four levels: overall competitive success, competitive goals, technology strategies, and technology alternatives. Applying Corollary 1.1, Theorems 4.1 and 5, the sensitivity of the competitive-goals level is studied in a "one-way" SA (Clemen, 1996). Local contributions of competitive goals to the overall competitive success (C^O), aggregated contributions of technology alternatives to the competitive goals ($C_{i\ell}^{A-O}$), and overall contributions of technology alternatives to the competitive success (C_i^A) are summarized in Tables 2–4.

First, the rank order of all the technology alternatives is considered.

When $r = 1, n = 1$:

Based on (5b), we get $\lambda = C_r^A - C_{r+n}^A = C_{(1)}^A - C_{(2)}^A = C_2^A - C_5^A = 0.235 - 0.2204 = 0.0146$.

(Note that r and $r + n$ are the ranks of the alternatives. In Table 4 we can see that the second and the fifth technology ranked first and second. Therefore, we have $C_2^A - C_5^A$ in the above expression.)

From (5c), we get

$$\begin{aligned} \lambda^O &= C_{51^*}^{A-O} - C_{21^*}^{A-O} - \sum_{\ell=2}^4 C_{51^*}^{A-O} \times \frac{C_{\ell}^O}{\sum_{\ell=2}^4 C_{\ell}^O} + \sum_{\ell=2}^4 C_{21^*}^{A-O} \times \frac{C_{\ell}^O}{\sum_{\ell=2}^4 C_{\ell}^O} \\ &= 0.24 - 0.24 - (0.22 \times 0.25/0.64 + 0.24 \times 0.21/0.64 + 0.24 \times 0.18/0.64) \\ &\quad + (0.2 \times 0.25/0.64 + 0.22 \times 0.22/0.64 + 0.21 \times 0.18/0.64) = 0.0228. \end{aligned}$$

From (5a), we get $P_{1^*}^O \leq \frac{\lambda}{\lambda^O} = \frac{0.0146}{0.0228} = 0.64$.

Table 2

First level contribution vector C_{ℓ}^O

C_{ℓ}^O	Competitive goals O_{ℓ}			
	Cost leadership	Product leadership	Customer leadership	Market leadership
Overall competitive success	0.36	0.25	0.21	0.18

Table 3

Aggregated contribution matrix $C_{i\ell}^{A-O}$

$C_{i\ell}^{A-O}$	Technology alternatives A_i				
	300 mm	90 nm	Hi k	Lo k	Factory integration
Cost leadership	0.19	0.24	0.13	0.19	0.24
Product leadership	0.27	0.22	0.13	0.18	0.20
Customer leadership	0.21	0.24	0.13	0.19	0.22
Market leadership	0.22	0.24	0.13	0.19	0.21

Table 4
Overall contribution vector C_i^A

C_i^A	300 mm	90 nm	Hi k	Lo k	Factory integration
Overall competitive success	0.2196	0.235	0.1321	0.1929	0.2204
Current ranking	(3)	(1)	(5)	(4)	(2)

Repeating the same steps for $n = 1$, and $r = 2, 3, 4$, we get all the inequalities that need to be satisfied, which are $\left\{ \begin{array}{l} P_{1^*}^O \leq 0.64 \text{ (when } r = 1, n = 1) \\ P_{1^*}^O \geq -0.01 \text{ (when } r = 2, n = 1) \\ P_{1^*}^O \leq 0.466 \text{ (when } r = 3, n = 1) \\ P_{1^*}^O \geq -4.23 \text{ (when } r = 4, n = 1) \end{array} \right\}$. Combining them with the feasibility constraint $[-0.36, 0.64]$, the allowable range of $P_{1^*}^O$ is $[-0.01, 0.466]$.

From (13a) and (13b), $OPSC(O_1) = \text{Min}\{0.01, 0.466\} = 0.01$, $TSC(O_1) = |0.466 + 0.01| = 0.476$.

Repeating the same steps for $C_{2^*}^O, C_{3^*}^O$, and $C_{4^*}^O$, the sensitivity of a single change to contributions of competitive goals to overall success is determined, as summarized in Table 5. In this case, the ranks of all technology alternatives are considered.

From Table 5 and Fig. 8, we can see that OPSCs and TSCs give the same information about the criticality order of the decision elements on the competitive-goals level. The smallest OPSC and TSC both occur on O_2 , making “product leadership” the most critical competitive goal to preserve the current ranking of all technology alternatives. Since $TSC(O_2)$ is 0.258, there is a 74.2% chance that the current rank order of the technology alternatives will change when C_2^O , the contribution of “product leadership” to overall success, varies from zero to one.

If we are only concerned with the current top-ranked technology alternative, Corollary 1.1 is applied by taking $r = 1, n = 1, 2, 3, 4$ to calculate the sensitivity indicators for O_ℓ ($\ell = 1, 2, 3, 4$). In this case, both OPSC

Table 5
HDM SA at O level to preserve the ranking of all A_i

	C_1^O	C_2^O	C_3^O	C_4^O
Base values	0.36	0.25	0.21	0.18
Allowable ranges of perturbations	$[-0.01, 0.466]$	$[-0.25, 0.008]$	$[-0.069, 0.79]$	$[-0.18, 0.82]$
Tolerance	$[0.35, 0.8256]$	$[0, 0.258]$	$[0.1413, 1]$	$[0, 1]$
OPSC (O_ℓ)	0.01	0.008	0.069	0.18
TSC (O_ℓ)	0.476	0.258	0.859	1

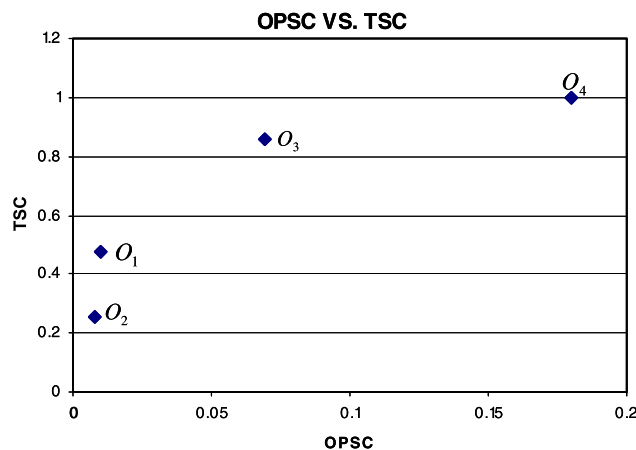


Fig. 8. OPSC and TSC as indicators of the criticality of O_ℓ .

Table 6
HDM SA at O level to preserve the ranking of the top A_i

	C_1^O	C_2^O	C_3^O	C_4^O
Base values	0.36	0.25	0.21	0.18
Allowable ranges of perturbations	[-0.285, 0.64]	[-0.25, 0.177]	[-0.21, 0.79]	[-0.18, 0.82]
Tolerance	[0.075, 1]	[0, 0.427]	[0, 1]	[0, 1]
OPSC (O_ℓ)	0.285	0.177	0.21	0.18
TSC (O_ℓ)	0.925	0.427	1	1

and TSC indicate that O_2 is also the most critical competitive goal for “90 nm linewidth” to remain as the top choice. The HDM SA result is summarized in Table 6.

The HDM SA result shows that the rank of the top technology, “90 nm linewidth”, is not sensitive to changes on C_3^O (contribution of customer leadership to overall success) and C_4^O (contribution of market leadership to overall success). There is zero chance that the technology alternatives’ rank will change when these two values vary from zero to one. However, it is sensitive to value increases on C_2^O (contribution of product leadership to overall success). If the value of C_2^O increases more than 0.177, the inequality defined by (4a) in Definition 1.1 is not satisfied when $r = 1$ and $n = 2$, which indicates that the current rank order of the first- and third-ranked technologies will reverse. Interpreting this from the perspective of how changes at the policy level will affect decisions at the operational level, the sensitivity analysis indicates that if the Taiwan semiconductor foundry industry shifts the emphasis of its competitive goals to product leadership more than 17.7%, then “300 mm wafer” technology should be the top technology to be developed.

In addition, HDM SA also indicates that the current second-ranked technology, “factory integration”, is dominated by “90 nm linewidth” technology, which currently ranks first, regardless of how the contributions of competitive goals change. However, the third-ranked technology, “300 mm wafer”, is sensitive to changes in the competitive goals: it will become the top choice when “product leadership” is emphasized or “cost leadership” is deemphasized. This result may draw the attention of decision makers and cause them to reconsider the resource allocation for these top three technologies, depending on how certain they are of the current contribution value assigned to each competitive goal and how likely the emphasis on the competitive goals will shift.

4. Concluding remarks

In this paper, we propose a comprehensive HDM SA algorithm to analyze the impacts of single and multiple changes to the local contribution vector/matrices at any level of a decision hierarchy. In four groups of propositions, the *allowable range/region of perturbations* and *contribution tolerance* are defined to keep the ranking of interested decision alternatives unchanged; two sensitivity coefficients, *operating point sensitivity coefficient* and *total sensitivity coefficient*, are proposed to evaluate the robustness of a hierarchical model; and *the most critical decision element* at a given level to maintain the current decision is identified. The algorithm is independent of the various pairwise comparison scales, judgment quantification techniques and group opinion combining methods used by different researchers. Even though the tolerance analysis section is based on the additive relationship to aggregate local contribution matrices into an overall contribution vector, the deductive logic can be easily applied to multiplicative HDM.

The tolerance analysis employed mathematical deduction in symbolic form in defining the *allowable range/region of perturbations* and *contribution tolerance*. Compared to other methods employed in the literature, it has better performance (rich with information, precisely defined threshold value to any decimal place, 100% accurate once the deduction process is verified), less computational complexity (fast, does not depend on repetitive iterations or large replications) and equal generality (same assumptions). These propositions are tested and verified by data from a dissertation by Ho (2004) that used an additive hierarchical decision model to evaluate emerging technologies. An example in which three HDM SA propositions are applied to Ho’s model is presented to demonstrate the practical application of the algorithm.

While the HDM SA algorithm deals with changes to the local contribution matrices, which are the intermediate input to HDM and thus make the algorithm independent of the different pairwise comparison scales

and judgment quantification techniques, the disadvantage of such an approach is that it does not reveal the direct impact of judgment changes to the model results. A future extension of this study would be to link the current algorithm to different judgment quantification methods and analyze model's sensitivity to varying judgments. Future work for HDM SA will also address multiple simultaneous changes at different levels of the decision hierarchy. In addition, based on the same deductive logic, SA for multiplicative HDM can be developed to define the allowable range/region of perturbations at any level of the decision hierarchy. Future work also includes applying the whole set of HDM SA algorithms to a large number of problems being addressed by HDM.

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Appendix A. The flaw in Triantaphyllou and Sanchez's SA method

To deduct the threshold value of perturbations on W_1 which will alter the rank order of A_1 and A_2 , Triantaphyllou and Sanchez (1997) defined the perturbation as $\delta_{1,1,2}$, and the new value of W_1 as ($W_1^* = W_1 - \delta_{1,1,2}$). To preserve the property that all weights add up to 1, weights are normalized as follows, with W'_i denoting the normalized value:

$$W'_1 = \frac{W_1^*}{W_1^* + W_2 + \dots + W_n} \tag{A1}$$

$$W'_2 = \frac{W_2}{W_1^* + W_2 + \dots + W_n} \tag{A2}$$

⋮
⋮
⋮

$$W'_n = \frac{W_n}{W_1^* + W_2 + \dots + W_n}. \tag{A3}$$

If we use $\delta_{1,1,2}^*$ to represent the actual threshold instead of the un-normalized threshold $\delta_{1,1,2}$, we have:

$$W_1 - \delta_{1,1,2}^* = W'_1 = \frac{W_1^*}{W_1^* + W_2 + \dots + W_n} = \frac{W_1 - \delta_{1,1,2}}{W_1 - \delta_{1,1,2} + W_2 + \dots + W_n} = \frac{W_1 - \delta_{1,1,2}}{\sum_{i=1}^n W_i - \delta_{1,1,2}}, \tag{A4}$$

$$(W_1 - \delta_{1,1,2}^*) \times \left(\sum_{i=1}^n W_i - \delta_{1,1,2} \right) = W_1 - \delta_{1,1,2}, \tag{A5}$$

$$W_1 \times \left(\sum_{i=1}^n W_i - \delta_{1,1,2} \right) - \delta_{1,1,2}^* \times \left(\sum_{i=1}^n W_i - \delta_{1,1,2} \right) = W_1 - \delta_{1,1,2}, \tag{A6}$$

$$\delta_{1,1,2}^* = \frac{W_1 \times (\sum_{i=1}^n W_i - \delta_{1,1,2}) - W_1 + \delta_{1,1,2}}{\sum_{i=1}^n W_i - \delta_{1,1,2}} = W_1 - \frac{W_1 - \delta_{1,1,2}}{\sum_{i=1}^n W_i - \delta_{1,1,2}}. \tag{A7}$$

The actual threshold, $\delta_{k,i,j}^*$ (shown as $\delta_{1,1,2}^*$ in the above expression), is a value different from $\delta_{k,i,j}$ (shown as $\delta_{1,1,2}$ in the above expression) as was assumed in the Triantaphyllou and Sanchez study; $\delta_{k,i,j}^*$ is a function of $\delta_{k,i,j}$ but not equal to it. For example, if the contribution values, W_i 's, are 0.4, 0.3, 0.2 and 0.1, and the $\delta_{1,1,2}$ defined by Triantaphyllou and Sanchez is 0.1, they conclude that before normalization, W_1 can go down to

0.3 without altering the rank order of the decision alternatives. However, after normalization, W_1 can only go down to 0.33, not to 0.3, and the other contribution values are changed to 0.33, 0.22, and 0.11. The actual threshold of the change on W_1 is $0.07 = (0.4 - \frac{0.4-0.1}{1-0.1})$ instead of 0.1.

Appendix B. Mathematical deduction for Propositions 1.1 to 3.5

B.1. Mathematical deduction for Theorem 1

When M perturbations $P_{\ell_m}^O \left(-C_{\ell_m}^O < P_{\ell_m}^O < 1 - C_{\ell_m}^O, \sum_{\ell=1, \ell \neq \ell_m}^L C_{\ell}^O - 1 < \sum_{m=1}^M P_{\ell_m}^O < \sum_{\ell=1, \ell \neq \ell_m}^L C_{\ell}^O \right)$ are induced on M of the C_{ℓ}^O 's, which are $C_{\ell_m}^O$, the new values of $C_{\ell_m}^O$ are

$$C_{\ell_m}^O(\text{new}) = C_{\ell_m}^O + P_{\ell_m}^O.$$

Based on the assumption, the other C_{ℓ}^O 's will be changed according to their original ratio scales. Therefore, new values of other C_{ℓ}^O 's are

$$C_{\ell}^O(\text{new}) = C_{\ell}^O + P_{\ell}^O, \quad \text{with } P_{\ell}^O = - \sum_{m=1}^M P_{\ell_m}^O \times \frac{C_{\ell}^O}{\sum_{\ell=1, \ell \neq \ell_m}^L C_{\ell}^O}.$$

Therefore, the new values of C_i^A can be represented as

$$\begin{aligned} C_i^A(\text{new}) &= \sum_{m=1}^M (C_{\ell_m}^O + P_{\ell_m}^O) \times C_{i\ell_m}^{A-O} + \sum_{\ell=1, \ell \neq \ell_m}^L (C_{\ell}^O + P_{\ell}^O) \times C_{i\ell}^{A-O} \\ &= \sum_{m=1}^M C_{\ell_m}^O \times C_{i\ell_m}^{A-O} + \sum_{\ell=1, \ell \neq \ell_m}^L C_{\ell}^O \times C_{i\ell}^{A-O} + \sum_{m=1}^M P_{\ell_m}^O \times C_{i\ell_m}^{A-O} - \sum_{\ell=1, \ell \neq \ell_m}^L C_{i\ell}^{A-O} \times \frac{\sum_{m=1}^M P_{\ell_m}^O \times C_{\ell}^O}{\sum_{\ell=1, \ell \neq \ell_m}^L C_{\ell}^O}. \end{aligned}$$

Since $\sum_{m=1}^M C_{\ell_m}^O \times C_{i\ell_m}^{A-O} + \sum_{\ell=1, \ell \neq \ell_1, \dots, \ell_M}^L C_{\ell}^O \times C_{i\ell}^{A-O} = C_i^A$ then

$$C_i^A(\text{new}) = C_i^A + \sum_{m=1}^M P_{\ell_m}^O \times C_{i\ell_m}^{A-O} - \sum_{\ell=1, \ell \neq \ell_m}^L C_{i\ell}^{A-O} \times \frac{\sum_{m=1}^M P_{\ell_m}^O \times C_{\ell}^O}{\sum_{\ell=1, \ell \neq \ell_m}^L C_{\ell}^O}. \tag{B1.1}$$

The ranking of A_r and A_{r+n} will not be reversed if $C_r^A(\text{new}) \geq C_{r+n}^A(\text{new})$. By substituting Eq. (B1.1) in the inequality, we get:

$$\begin{aligned} C_r^A + \sum_{m=1}^M P_{\ell_m}^O \times C_{r\ell_m}^{A-O} - \sum_{\ell=1, \ell \neq \ell_m}^L C_{r\ell}^{A-O} \times \frac{\sum_{m=1}^M P_{\ell_m}^O \times C_{\ell}^O}{\sum_{\ell=1, \ell \neq \ell_m}^L C_{\ell}^O} \\ \geq C_{r+n}^A + \sum_{m=1}^M P_{\ell_m}^O \times C_{r+n, \ell_m}^{A-O} - \sum_{\ell=1, \ell \neq \ell_m}^L C_{r+n, \ell}^{A-O} \times \frac{\sum_{m=1}^M P_{\ell_m}^O \times C_{\ell}^O}{\sum_{\ell=1, \ell \neq \ell_m}^L C_{\ell}^O}, \\ C_r^A - C_{r+n}^A \geq \sum_{m=1}^M P_{\ell_m}^O \times C_{r+n, \ell_m}^{A-O} - \sum_{\ell=1, \ell \neq \ell_m}^L C_{r+n, \ell}^{A-O} \times \frac{\sum_{m=1}^M P_{\ell_m}^O \times C_{\ell}^O}{\sum_{\ell=1, \ell \neq \ell_m}^L C_{\ell}^O} \\ - \sum_{m=1}^M P_{\ell_m}^O \times C_{r\ell_m}^{A-O} + \sum_{\ell=1, \ell \neq \ell_m}^L C_{r\ell}^{A-O} \times \frac{\sum_{m=1}^M P_{\ell_m}^O \times C_{\ell}^O}{\sum_{\ell=1, \ell \neq \ell_m}^L C_{\ell}^O}, \\ C_r^A - C_{r+n}^A \geq \sum_{m=1}^M \left[P_{\ell_m}^O \times \left(C_{r+n, \ell_m}^{A-O} - C_{r\ell_m}^{A-O} - \sum_{\ell=1, \ell \neq \ell_m}^L C_{r+n, \ell}^{A-O} \times \frac{C_{\ell}^O}{\sum_{\ell=1, \ell \neq \ell_m}^L C_{\ell}^O} + \sum_{\ell=1, \ell \neq \ell_m}^L C_{r\ell}^{A-O} \times \frac{C_{\ell}^O}{\sum_{\ell=1, \ell \neq \ell_m}^L C_{\ell}^O} \right) \right]. \end{aligned} \tag{B1.2}$$

The top choice will remain at the top rank if the above condition is satisfied for all $r = 1$ and $n = 1, 2, \dots, I - 1$, which means $C_1^A(\text{new}) \geq C_2^A(\text{new}), C_1^A(\text{new}) \geq C_3^A(\text{new}), \dots, C_1^A(\text{new}) \geq C_I^A(\text{new})$.

The rank order for all A_i 's will remain the same if the above condition is satisfied for all $r = 1, 2, \dots, I - 1$, and $n = 1$, which means $C_1^A(\text{new}) \geq C_2^A(\text{new}) \geq \dots \geq C_r^A(\text{new}) \geq \dots \geq C_I^A(\text{new})$.

B.2. Mathematical deduction for Theorem 2

When M perturbations $P_{k_m^* \ell_x^*}^{G-O}$ ($m = 1, 2, \dots, M$) are induced in M of the $C_{k\ell_x}^{G-O}$'s, denoted as $C_{k_m^* \ell_x^*}^{G-O}$, T perturbations $P_{k_t^* \ell_\beta^*}^{G-O}$ ($t = 1, 2, \dots, T$) are induced in T of the $C_{k\ell_\beta}^{G-O}$'s, denoted as $C_{k_t^* \ell_\beta^*}^{G-O}$, Q perturbations $P_{k_q^* \ell_\gamma^*}^{G-O}$ ($q = 1, 2, \dots, Q$) are induced in Q of the $C_{k\ell_\gamma}^{G-O}$'s, denoted as $C_{k_q^* \ell_\gamma^*}^{G-O}$, based on the assumptions, the new values of $C_{k_m^* \ell_x^*}^{G-O}$'s and other $C_{k\ell_x}^{G-O}$'s will be

$$C_{k_m^* \ell_x^*}^{G-O}(\text{new}) = C_{k_m^* \ell_x^*}^{G-O} + P_{k_m^* \ell_x^*}^{G-O},$$

$$C_{k\ell_x}^{G-O}(\text{new}) = C_{k\ell_x}^{G-O} + P_{k\ell_x}^{G-O}, \quad \text{with } P_{k\ell_x}^{G-O} = - \sum_{m=1}^M P_{k_m^* \ell_x^*}^{G-O} \times \frac{C_{k\ell_x}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{k\ell_x}^{G-O}}.$$

The new values of $C_{k_t^* \ell_\beta^*}^{G-O}$'s and of other $C_{k\ell_\beta}^{G-O}$'s will be

$$C_{k_t^* \ell_\beta^*}^{G-O}(\text{new}) = C_{k_t^* \ell_\beta^*}^{G-O} + P_{k_t^* \ell_\beta^*}^{G-O},$$

$$C_{k\ell_\beta}^{G-O}(\text{new}) = C_{k\ell_\beta}^{G-O} + P_{k\ell_\beta}^{G-O}, \quad \text{with } P_{k\ell_\beta}^{G-O} = - \sum_{t=1}^T P_{k_t^* \ell_\beta^*}^{G-O} \times \frac{C_{k\ell_\beta}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{k\ell_\beta}^{G-O}}.$$

The new values of $C_{k_q^* \ell_\gamma^*}^{G-O}$'s and of other $C_{k\ell_\gamma}^{G-O}$'s will be

$$C_{k_q^* \ell_\gamma^*}^{G-O}(\text{new}) = C_{k_q^* \ell_\gamma^*}^{G-O} + P_{k_q^* \ell_\gamma^*}^{G-O},$$

$$C_{k\ell_\gamma}^{G-O}(\text{new}) = C_{k\ell_\gamma}^{G-O} + P_{k\ell_\gamma}^{G-O}, \quad \text{with } P_{k\ell_\gamma}^{G-O} = - \sum_{q=1}^Q P_{k_q^* \ell_\gamma^*}^{G-O} \times \frac{C_{k\ell_\gamma}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{k\ell_\gamma}^{G-O}}.$$

Therefore, the new values of C_i^A 's can be represented as

$$C_i^A(\text{new}) = \sum_{\ell=1}^L \sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_\ell^O C_{k\ell}^{G-O} C_{ik}^{A-G} + \sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{\ell_x}^O \left(C_{k\ell_x}^{G-O} - \sum_{m=1}^M P_{k_m^* \ell_x^*}^{G-O} \times \frac{C_{k\ell_x}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{k\ell_x}^{G-O}} \right) C_{ik}^{A-G}$$

$$\quad \ell \neq \ell_x^* \quad k \neq k_m^*$$

$$\quad \ell \neq \ell_\beta^* \quad k \neq k_t^*$$

$$\quad \ell \neq \ell_\gamma^* \quad k \neq k_q^*$$

$$+ \sum_{m=1}^M C_{\ell_x}^O \left(C_{k_m^* \ell_x^*}^{G-O} + P_{k_m^* \ell_x^*}^{G-O} \right) C_{ik_m^*}^{A-G} + \sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{\ell_\beta}^O \left(C_{k\ell_\beta}^{G-O} - \sum_{t=1}^T P_{k_t^* \ell_\beta^*}^{G-O} \times \frac{C_{k\ell_\beta}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{k\ell_\beta}^{G-O}} \right) C_{ik}^{A-G}$$

$$+ \sum_{t=1}^T C_{\ell_\beta}^O \left(C_{k_t^* \ell_\beta^*}^{G-O} + P_{k_t^* \ell_\beta^*}^{G-O} \right) C_{ik_t^*}^{A-G} + \sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{\ell_\gamma}^O \left(C_{k\ell_\gamma}^{G-O} - \sum_{q=1}^Q P_{k_q^* \ell_\gamma^*}^{G-O} \times \frac{C_{k\ell_\gamma}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{k\ell_\gamma}^{G-O}} \right) C_{ik}^{A-G}$$

$$+ \sum_{q=1}^Q C_{\ell_\gamma}^O \left(C_{k_q^* \ell_\gamma^*}^{G-O} + P_{k_q^* \ell_\gamma^*}^{G-O} \right) C_{ik_q^*}^{A-G},$$

$$\begin{aligned}
 C_i^A(\text{new}) &= \sum_{\ell=1}^L \sum_{k=1}^K C_{\ell}^O C_{k\ell}^{G-O} C_{ik}^{A-G} - \sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{\ell_z^*}^O \left(\sum_{m=1}^M P_{k_m^* \ell_z^*}^{G-O} \times \frac{C_{k\ell_z^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{k\ell_z^*}^{G-O}} \right) C_{ik}^{A-G} + \sum_{m=1}^M C_{\ell_z^*}^O P_{k_m^* \ell_z^*}^{G-O} C_{ik_m^*}^{A-G} \\
 &\quad - \sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{\ell_\beta^*}^O \left(\sum_{t=1}^T P_{k_t^* \ell_\beta^*}^{G-O} \times \frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{k\ell_\beta^*}^{G-O}} \right) C_{ik}^{A-G} + \sum_{t=1}^T C_{\ell_\beta^*}^O P_{k_t^* \ell_\beta^*}^{G-O} C_{ik_t^*}^{A-G} \\
 &\quad - \sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{\ell_\gamma^*}^O \left(\sum_{q=1}^Q P_{k_q^* \ell_\gamma^*}^{G-O} \times \frac{C_{k\ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{k\ell_\gamma^*}^{G-O}} \right) C_{ik}^{A-G} + \sum_{q=1}^Q C_{\ell_\gamma^*}^O P_{k_q^* \ell_\gamma^*}^{G-O} C_{ik_q^*}^{A-G}.
 \end{aligned}$$

Since $C_i^A = \sum_{\ell=1}^L \sum_{k=1}^K C_{\ell}^O C_{k\ell}^{G-O} C_{ik}^{A-G}$, then

$$\begin{aligned}
 C_i^A(\text{new}) &= C_i^A - \sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{\ell_z^*}^O \left(\sum_{m=1}^M P_{k_m^* \ell_z^*}^{G-O} \times \frac{C_{k\ell_z^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{k\ell_z^*}^{G-O}} \right) C_{ik}^{A-G} + \sum_{m=1}^M C_{\ell_z^*}^O P_{k_m^* \ell_z^*}^{G-O} C_{ik_m^*}^{A-G} \\
 &\quad - \sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{\ell_\beta^*}^O \left(\sum_{t=1}^T P_{k_t^* \ell_\beta^*}^{G-O} \times \frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{k\ell_\beta^*}^{G-O}} \right) C_{ik}^{A-G} + \sum_{t=1}^T C_{\ell_\beta^*}^O P_{k_t^* \ell_\beta^*}^{G-O} C_{ik_t^*}^{A-G} \\
 &\quad - \sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{\ell_\gamma^*}^O \left(\sum_{q=1}^Q P_{k_q^* \ell_\gamma^*}^{G-O} \times \frac{C_{k\ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{k\ell_\gamma^*}^{G-O}} \right) C_{ik}^{A-G} + \sum_{q=1}^Q C_{\ell_\gamma^*}^O P_{k_q^* \ell_\gamma^*}^{G-O} C_{ik_q^*}^{A-G}. \tag{B2.1}
 \end{aligned}$$

The ranking of A_r and A_{r+n} will not be reversed if $C_r^A(\text{new}) \geq C_{r+n}^A(\text{new})$. By substituting Eq. (B2.1) in the inequality, we get:

$$\begin{aligned}
 &C_r^A - \sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{\ell_z^*}^O C_{rk}^{A-G} \left(\sum_{m=1}^M P_{k_m^* \ell_z^*}^{G-O} \times \frac{C_{k\ell_z^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{k\ell_z^*}^{G-O}} \right) + \sum_{m=1}^M C_{\ell_z^*}^O P_{k_m^* \ell_z^*}^{G-O} C_{rk_m^*}^{A-G} - \sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{\ell_\beta^*}^O C_{rk}^{A-G} \left(\sum_{t=1}^T P_{k_t^* \ell_\beta^*}^{G-O} \times \frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{k\ell_\beta^*}^{G-O}} \right) \\
 &\quad + \sum_{t=1}^T C_{\ell_\beta^*}^O P_{k_t^* \ell_\beta^*}^{G-O} C_{rk_t^*}^{A-G} - \sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{\ell_\gamma^*}^O C_{rk}^{A-G} \left(\sum_{q=1}^Q P_{k_q^* \ell_\gamma^*}^{G-O} \times \frac{C_{k\ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{k\ell_\gamma^*}^{G-O}} \right) + \sum_{q=1}^Q C_{\ell_\gamma^*}^O P_{k_q^* \ell_\gamma^*}^{G-O} C_{rk_q^*}^{A-G} \\
 &\geq C_{r+n}^A - \sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{\ell_z^*}^O C_{r+n,k}^{A-G} \left(\sum_{m=1}^M P_{k_m^* \ell_z^*}^{G-O} \times \frac{C_{k\ell_z^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{k\ell_z^*}^{G-O}} \right) + \sum_{m=1}^M C_{\ell_z^*}^O P_{k_m^* \ell_z^*}^{G-O} C_{r+n,k_m^*}^{A-G} \\
 &\quad - \sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{\ell_\beta^*}^O C_{r+n,k}^{A-G} \left(\sum_{t=1}^T P_{k_t^* \ell_\beta^*}^{G-O} \times \frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{k\ell_\beta^*}^{G-O}} \right) + \sum_{t=1}^T C_{\ell_\beta^*}^O P_{k_t^* \ell_\beta^*}^{G-O} C_{r+n,k_t^*}^{A-G} \\
 &\quad - \sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{\ell_\gamma^*}^O C_{r+n,k}^{A-G} \left(\sum_{q=1}^Q P_{k_q^* \ell_\gamma^*}^{G-O} \times \frac{C_{k\ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{k\ell_\gamma^*}^{G-O}} \right) + \sum_{q=1}^Q C_{\ell_\gamma^*}^O P_{k_q^* \ell_\gamma^*}^{G-O} C_{r+n,k_q^*}^{A-G},
 \end{aligned}$$

$$\begin{aligned}
 C_r^A - C_{r+n}^A &\geq \sum_{m=1}^M C_{\ell_x^*}^O P_{k_m^* \ell_x^*}^{G-O} (C_{r+n, k_m^*}^{A-G} - C_{rk_m^*}^{A-G}) + \sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{\ell_x^*}^O \left(\sum_{m=1}^M P_{k_m^* \ell_x^*}^{G-O} \times \frac{C_{k \ell_x^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{k \ell_x^*}^{G-O}} \right) (C_{rk}^{A-G} - C_{r+n, k}^{A-G}) \\
 &+ \sum_{t=1}^T C_{\ell_\beta^*}^O P_{k_t^* \ell_\beta^*}^{G-O} (C_{r+n, k_t^*}^{A-G} - C_{rk_t^*}^{A-G}) + \sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{\ell_\beta^*}^O \left(\sum_{t=1}^T P_{k_t^* \ell_\beta^*}^{G-O} \times \frac{C_{k \ell_\beta^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{k \ell_\beta^*}^{G-O}} \right) (C_{rk}^{A-G} - C_{r+n, k}^{A-G}) \\
 &+ \sum_{q=1}^Q C_{\ell_\gamma^*}^O P_{k_q^* \ell_\gamma^*}^{G-O} (C_{r+n, k_q^*}^{A-G} - C_{rk_q^*}^{A-G}) + \sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{\ell_\gamma^*}^O \left(\sum_{q=1}^Q P_{k_q^* \ell_\gamma^*}^{G-O} \times \frac{C_{k \ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{k \ell_\gamma^*}^{G-O}} \right) (C_{rk}^{A-G} - C_{r+n, k}^{A-G}), \\
 C_r^A - C_{r+n}^A &\geq \sum_{m=1}^M P_{k_m^* \ell_x^*}^{G-O} \times \left[C_{\ell_x^*}^O (C_{r+n, k_m^*}^{A-G} - C_{rk_m^*}^{A-G}) + \sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{\ell_x^*}^O (C_{rk}^{A-G} - C_{r+n, k}^{A-G}) \frac{C_{k \ell_x^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_m^*}}^K C_{k \ell_x^*}^{G-O}} \right] \\
 &+ \sum_{t=1}^T P_{k_t^* \ell_\beta^*}^{G-O} \times \left[C_{\ell_\beta^*}^O (C_{r+n, k_t^*}^{A-G} - C_{rk_t^*}^{A-G}) + \sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{\ell_\beta^*}^O (C_{rk}^{A-G} - C_{r+n, k}^{A-G}) \times \frac{C_{k \ell_\beta^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_t^*}}^K C_{k \ell_\beta^*}^{G-O}} \right] \\
 &+ \sum_{q=1}^Q P_{k_q^* \ell_\gamma^*}^{G-O} \times \left[C_{\ell_\gamma^*}^O (C_{r+n, k_q^*}^{A-G} - C_{rk_q^*}^{A-G}) + \sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{\ell_\gamma^*}^O (C_{rk}^{A-G} - C_{r+n, k}^{A-G}) \frac{C_{k \ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1 \\ k \neq k_q^*}}^K C_{k \ell_\gamma^*}^{G-O}} \right]. \tag{B2.2}
 \end{aligned}$$

The top choice will remain at the top rank if the above condition is satisfied for all $r = 1$ and $n = 1, 2, \dots, I - 1$, which means $C_1^A(\text{new}) \geq C_2^A(\text{new}), C_1^A(\text{new}) \geq C_3^A(\text{new}), \dots, C_1^A(\text{new}) \geq C_I^A(\text{new})$.

The rank order for all A_i 's will remain the same if the above condition is satisfied for all $r = 1, 2, \dots, I - 1$, and $n = 1$, which means $C_1^A(\text{new}) \geq C_2^A(\text{new}) \geq \dots \geq C_r^A(\text{new}) \geq \dots \geq C_I^A(\text{new})$.

B.3. Mathematical deduction for Theorem 3

When M perturbations $P_{i_m^* j_x^*}^{A-S}$ ($m = 1, 2, \dots, M$) are induced in M of the $C_{ij_x}^{A-S}$'s, denoted as $C_{i_m^* j_x^*}^{A-S}$ (contributions of M actions A_{i^*} to the α th changing strategy $S_{j_x^*}$), T perturbations $P_{i_t^* j_\beta^*}^{A-S}$ ($t = 1, 2, \dots, T$) are induced in T of the $C_{ij_\beta}^{A-S}$'s, denoted as $C_{i_t^* j_\beta^*}^{A-S}$ (contributions of T actions G_{k^*} to the β th changing strategy $S_{j_\beta^*}$), Q perturbations $P_{i_q^* j_\gamma^*}^{A-S}$ ($q = 1, 2, \dots, Q$) are induced in Q of the $C_{k \ell_\gamma}^{G-O}$'s, denoted as $C_{k_q^* \ell_\gamma^*}^{G-O}$ (contributions of Q actions G_{k^*} to the γ th changing strategy $S_{j_\gamma^*}$), based on the assumption, the new values of $C_{i^*}^A$'s and of other C_i^A 's will be

$$C_{i^*}^A(\text{new}) = C_{i^*}^A + C_{j_x^*}^S P_{i_m^* j_x^*}^{A-S} \quad \text{or} \quad C_{i^*}^A(\text{new}) = C_{i^*}^A + C_{j_\beta^*}^S P_{i_t^* j_\beta^*}^{A-S} \quad \text{or} \quad C_{i^*}^A(\text{new}) = C_{i^*}^A + C_{j_\gamma^*}^S P_{i_q^* j_\gamma^*}^{A-S}.$$

$$\text{A general term can represent the above three situations: } C_{i^*}^A(\text{new}) = C_{i^*}^A + C_{j^*}^S P_{i^* j^*}^{A-S} \tag{B3.1}$$

$$\begin{aligned}
 C_i^A(\text{new}) &= C_i^A - C_{j_x^*}^S \times \sum_{m=1}^M P_{i_m^* j_x^*}^{A-S} \times \frac{C_{ij_x}^{A-S}}{\sum_{i=1, i \neq i_m^*}^I C_{ij_x}^{A-S}} - C_{j_\beta^*}^S \times \sum_{t=1}^T P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{ij_\beta}^{A-S}}{\sum_{i=1, i \neq i_t^*}^I C_{ij_\beta}^{A-S}} \\
 &- C_{j_\gamma^*}^S \times \sum_{q=1}^Q P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{ij_\gamma}^{A-S}}{\sum_{i=1, i \neq i_q^*}^I C_{ij_\gamma}^{A-S}}. \tag{B3.2}
 \end{aligned}$$

Suppose a pair of A_i 's (A_r and A_{r+n} , $C_r^A \geq C_{r+n}^A$) are being compared. There will be four situations: (1) $r = i^*$ and $r + n \neq i^*$; (2) $r \neq i^*$ and $r + n = i^*$; (3) $r \neq i^*$ and $r + n \neq i^*$; or (4) $r = i^*$ and $r + n = i^*$. In these four situations, the ranking of A_r and A_{r+n} will not be reversed if $C_r^A(\text{new}) \geq C_{r+n}^A(\text{new})$; therefore, we have the following inequalities regarding the four situations:

(1) $r = i^*$ and $r + n \neq i^*$

$$\begin{aligned}
 C_{r^*}^A + C_{j^*}^S \times P_{r^*j^*}^{A-S} &\geq C_{r+n}^A - C_{j_z^*}^S \times \sum_{m=1}^M P_{i_m^*j_z^*}^{A-S} \times \frac{C_{r+n,j_z^*}^{A-S}}{\sum_{i=1, i \neq i_m^*}^I C_{ij_z^*}^{A-S}} - C_{j_\beta^*}^S \\
 &\times \sum_{t=1}^T P_{i_t^*j_\beta^*}^{A-S} \times \frac{C_{r+n,j_\beta^*}^{A-S}}{\sum_{i=1, i \neq i_t^*}^I C_{ij_\beta^*}^{A-S}} - C_{j_\gamma^*}^S \times \sum_{q=1}^Q P_{i_q^*j_\gamma^*}^{A-S} \times \frac{C_{r+n,j_\gamma^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^I C_{ij_\gamma^*}^{A-S}}, \\
 C_{r^*}^A - C_{r+n}^A &\geq -C_{j^*}^S \times P_{r^*j^*}^{A-S} - C_{j_z^*}^S \times \sum_{m=1}^M P_{i_m^*j_z^*}^{A-S} \times \frac{C_{r+n,j_z^*}^{A-S}}{\sum_{i=1, i \neq i_m^*}^I C_{ij_z^*}^{A-S}} - C_{j_\beta^*}^S \\
 &\times \sum_{t=1}^T P_{i_t^*j_\beta^*}^{A-S} \times \frac{C_{r+n,j_\beta^*}^{A-S}}{\sum_{i=1, i \neq i_t^*}^I C_{ij_\beta^*}^{A-S}} - C_{j_\gamma^*}^S \times \sum_{q=1}^Q P_{i_q^*j_\gamma^*}^{A-S} \times \frac{C_{r+n,j_\gamma^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^I C_{ij_\gamma^*}^{A-S}} \\
 &\left(C_{j^*}^S \text{ and } P_{r^*j^*}^{A-S} \text{ can be } C_{j_z^*}^S \ \& \ P_{i_m^*j_z^*}^{A-S}, C_{j_\beta^*}^S \ \& \ P_{i_t^*j_\beta^*}^{A-S}, \text{ or } C_{j_\gamma^*}^S \ \& \ P_{i_q^*j_\gamma^*}^{A-S} \right). \tag{B3.3}
 \end{aligned}$$

(2) $r \neq i^*$ and $r + n = i^*$

$$\begin{aligned}
 C_r^A - C_{j_z^*}^S \times \sum_{m=1}^M P_{i_m^*j_z^*}^{A-S} \times \frac{C_{rj_z^*}^{A-S}}{\sum_{i=1, i \neq i_m^*}^I C_{ij_z^*}^{A-S}} - C_{j_\beta^*}^S \times \sum_{t=1}^T P_{i_t^*j_\beta^*}^{A-S} \\
 \times \frac{C_{rj_\beta^*}^{A-S}}{\sum_{i=1, i \neq i_t^*}^I C_{ij_\beta^*}^{A-S}} - C_{j_\gamma^*}^S \times \sum_{q=1}^Q P_{i_q^*j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^I C_{ij_\gamma^*}^{A-S}} &\geq C_{r+n^*}^A + C_{j^*}^S \times P_{r+n^*j^*}^{A-S}, \\
 C_r^A - C_{r+n^*}^A &\geq C_{j^*}^S \times P_{r+n^*j^*}^{A-S} + C_{j_z^*}^S \times \sum_{m=1}^M P_{i_m^*j_z^*}^{A-S} \times \frac{C_{rj_z^*}^{A-S}}{\sum_{i=1, i \neq i_m^*}^I C_{ij_z^*}^{A-S}} + C_{j_\beta^*}^S \\
 &\times \sum_{t=1}^T P_{i_t^*j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S}}{\sum_{i=1, i \neq i_t^*}^I C_{ij_\beta^*}^{A-S}} + C_{j_\gamma^*}^S \times \sum_{q=1}^Q P_{i_q^*j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^I C_{ij_\gamma^*}^{A-S}} \\
 &\left(C_{j^*}^S \text{ and } P_{r+n^*j^*}^{A-S} \text{ can be } C_{j_z^*}^S \ \& \ P_{i_m^*j_z^*}^{A-S}, C_{j_\beta^*}^S \ \& \ P_{i_t^*j_\beta^*}^{A-S}, \text{ or } C_{j_\gamma^*}^S \ \& \ P_{i_q^*j_\gamma^*}^{A-S} \right). \tag{B3.4}
 \end{aligned}$$

(3) $r \neq i^*$ and $r + n \neq i^*$

$$\begin{aligned}
 C_r^A - C_{j_z^*}^S \times \sum_{m=1}^M P_{i_m^*j_z^*}^{A-S} \times \frac{C_{rj_z^*}^{A-S}}{\sum_{i=1, i \neq i_m^*}^I C_{ij_z^*}^{A-S}} - C_{j_\beta^*}^S \times \sum_{t=1}^T P_{i_t^*j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S}}{\sum_{i=1, i \neq i_t^*}^I C_{ij_\beta^*}^{A-S}} - C_{j_\gamma^*}^S \times \sum_{q=1}^Q P_{i_q^*j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^I C_{ij_\gamma^*}^{A-S}} \\
 \geq C_{r+n}^A - C_{j_z^*}^S \times \sum_{m=1}^M P_{i_m^*j_z^*}^{A-S} \times \frac{C_{r+n,j_z^*}^{A-S}}{\sum_{i=1, i \neq i_m^*}^I C_{ij_z^*}^{A-S}} - C_{j_\beta^*}^S \times \sum_{t=1}^T P_{i_t^*j_\beta^*}^{A-S} \times \frac{C_{r+n,j_\beta^*}^{A-S}}{\sum_{i=1, i \neq i_t^*}^I C_{ij_\beta^*}^{A-S}} - C_{j_\gamma^*}^S \times \sum_{q=1}^Q P_{i_q^*j_\gamma^*}^{A-S} \times \frac{C_{r+n,j_\gamma^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^I C_{ij_\gamma^*}^{A-S}} \\
 C_r^A - C_{r+n}^A &\geq C_{j_z^*}^S \times \sum_{m=1}^M P_{i_m^*j_z^*}^{A-S} \times \frac{C_{rj_z^*}^{A-S} - C_{r+n,j_z^*}^{A-S}}{\sum_{i=1, i \neq i_m^*}^I C_{ij_z^*}^{A-S}} + C_{j_\beta^*}^S \times \sum_{t=1}^T P_{i_t^*j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S} - C_{r+n,j_\beta^*}^{A-S}}{\sum_{i=1, i \neq i_t^*}^I C_{ij_\beta^*}^{A-S}} + C_{j_\gamma^*}^S \times \sum_{q=1}^Q P_{i_q^*j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S} - C_{r+n,j_\gamma^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^I C_{ij_\gamma^*}^{A-S}}. \tag{B3.5}
 \end{aligned}$$

(4) $r = i^*$ and $r + n = i^*$

$$C_{r^*}^A + C_{j^*}^S \times P_{r^*j^*}^{A-S} \geq C_{r+n^*}^A + C_{j^*}^S \times P_{r+n^*j^*}^{A-S} \rightarrow C_{r^*}^A - C_{r+n^*}^A \geq C_{j^*}^S \times \left(P_{r+n^*j^*}^{A-S} - P_{r^*j^*}^{A-S} \right) \tag{B3.6}$$

$(C_{j^*}^S \text{ and } P_{r^*j^*}^{A-S} \text{ can be } C_{j_z^*}^S \ \& \ P_{i_m^*j_z^*}^{A-S}, C_{j_\beta^*}^S \ \& \ P_{i_t^*j_\beta^*}^{A-S}, \text{ or } C_{j_\gamma^*}^S \ \& \ P_{i_q^*j_\gamma^*}^{A-S})$.

For the ranking of all A_i 's to remain the same, the condition $C_r^A(\text{new}) \geq C_{r+n}^A(\text{new})$ needs to be satisfied for all $r = 1, 2, \dots, I - 1$ and $n = 1$, which means $C_1^A(\text{new}) \geq C_2^A(\text{new}), C_1^A(\text{new}) \geq C_3^A(\text{new}), \dots, C_1^A(\text{new}) \geq C_I^A(\text{new})$. This includes all the situations being discussed above.

The top choice will remain at the top rank if the above condition is satisfied for all $r = 1$ and $n = 1, 2, \dots, I - 1$, which means $C_1^A(\text{new}) \geq C_2^A(\text{new}), \dots, C_1^A(\text{new}) \geq C_I^A(\text{new})$. This includes all the situations being discussed above.

References

- Aguaron, J., Moreno-Jimenez, J.M., 2000. Local stability intervals in the analytic hierarchy process. *European Journal of Operational Research* 125, 113–132.
- Alexander, E.R., 1989. Sensitivity analysis in complex decision models. *APA Journal* (4), 323–333.
- Arbel, A., Vargas, L.G., 1990. The analytic hierarchy process with interval judgments. In: *Proceedings of the MCDM Conference*, Washington, DC.
- Armocost, R., Hosseini, J., 1994. Identification of determinant attribute using the analytic hierarchy process. *Journal of the Academy of Marketing Science* 22 (4), 383–392.
- Barron, H., Schmidt, C.P., 1988. Sensitivity analysis of additive multiattribute value models. *Operations Research* 36 (1), 122–127.
- Barzilai, J., Lootsma, F.A., 1997. Power relations and group aggregation in the multiplicative AHP and SMART. *Journal of Multi-Criteria Decision Analysis* 6, 155–165.
- Belton, V., Gear, T., 1983. On a short-coming of Saaty's method of analytic hierarchies. *Omega* 11 (3), 228–230.
- Belton, V., Gear, T., 1985. The legitimacy of rank reversal – A comment. *Omega* 13 (3), 143–144.
- Borthick, A.F., Scheiner, J.H., 1998. Selection of small business computer system: Structuring a multi-criteria approach. *Journal of Information System* 1999 (Fall), 10–29.
- Butler, J., Jia, J., Dyer, J., 1997. Simulation techniques for the sensitivity analysis of multi-criteria decision models. *European Journal of Operational Research* 103 (1997), 531–546.
- Chu, A.T.W., Kalaba, R.E., Spingarn, K., 1979. A comparison of two methods for determining the weights of belonging to fuzzy sets. *Journal of Optimization Theory and Applications* 27 (4), 531–538.
- Cleland, D.I., Kocaoglu, D.F., 1981. *Engineering Management*. McGraw-Hill, Inc.
- Clemen, R.T., 1996. *Making Hard Decisions: An Introduction to Decision Analysis*. PWS-Kent, Boston, MA.
- Dantzig, G.B., 1963. *Linear Programming and Extensions*. Princeton University Press, Princeton, NJ.
- Evans, J.R., 1984. Sensitivity analysis in decision theory. *Decision Sciences* 15 (1), 239–247.
- Expert Choice (Computer Software), 1990. Expert Choice Inc., Pittsburgh, PA.
- Farkas, A., Gyorgy, A., Rozsa, P., 2004. On the spectrum of pairwise comparison matrices. *Linear Algebra and its Applications* 385 (2004), 443–462.
- Ferrell, W.R., 1985. Combining individual judgments. In: Wright, G. (Ed.), *Behavioral Decision Making*. Plenum Press, London, England, pp. 111–145.
- Forman, E.H., Gass, S.I., 2001. The analytic hierarchy process – An exposition. *Operations Research* 49 (4), 469–485.
- Golden, B.L., Wasil, E.A., Harker, P.T., 1989. *The analytic hierarchy process: Applications and studies*. Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong.
- Harrell, C., Ghosh, B.K., Bowden, R.O., 2000. *Simulation using ProModel*. McGraw-Hill.
- Hastie, R., Kameda, T., 2005. The robust beauty of majority rules in group decisions. *Psychological Review* 112 (2), 494–508.
- Hauser, D., Tadikamalla, P., 1996. The analytic hierarchy process in an uncertain environment: Simulation approach. *European Journal of Operational Research* 91, 27–37.
- Hihn, J.M., 1980. A hierarchical approach to decision making: Economic theory and methodology. Ph.D. dissertation, University of Maryland, College Park.
- Ho, C., 2004. Strategic evaluation of emerging technologies in the semiconductor foundry industry (special case: Taiwan semiconductor foundry industry). Ph.D. dissertation, Portland State University, Portland, OR, US.
- Howard, R.A., 1968. The foundations of decision analysis. *IEEE Transactions on Systems Science and Cybernetics* SSC-4(3).
- Huang, Y., 2002. Enhancement on sensitivity analysis of priority in analytical hierarchy process. *International Journal of General System* 31 (5), 531–542.
- Jensen, R.E., 1984. An alternative scaling method of priorities in hierarchical structures. *Journal of Mathematical Psychology* 28, 317–332.
- Johnson, C.R., Beine, W.B., Wang, T.J., 1980. Right-left asymmetry in an eigenvector ranking procedure. *Journal of Mathematical Psychology* 19, 61–64.
- Kelton, W.D., Sadowski, R.P., Sadowski, D.A., 1998. *Simulation with Arena*, 2nd ed. McGraw-Hill.
- Klir, G.J., 2001. *Facets of Systems Science*, 2nd ed. Kluwer Academic/Plenum Publishers.
- Kocaoglu, D.F., 1976. A systems approach to the resource allocation process in police patrol. Ph.D. dissertation, University of Pittsburgh, Pittsburgh.
- Kocaoglu, D.F., 1983. A participative approach to program evaluation. *IEEE Transactions on Engineering Management* EM-30 (3), 37–44.
- Lootsma, F.A., 1999. *Multi-Criteria Decision Analysis via Ratio and Difference Judgment* Applied Optimization Series, vol. 29. Kluwer Academic Publishers, Dordrecht, The Netherlands.
- Masuda, T., 1990. Hierarchical sensitivity analysis of the priority used in analytic hierarchy process. *International Journal of System Science* 21 (2), 415–427.

- Moreno-Jimenez, J.M., Vargas, L.G., 1993. A probabilistic study of preference structures in the analytic hierarchy process with interval judgments. *Mathematical and Computer Modeling* 17, 73–81.
- Murty, K.G., 1976. *Linear and Combinatorial Programming*. John Wiley & Sons, Inc.
- Partovi, F.Y., 1994. Determining what to benchmark: An analytic hierarchy process approach. *International Journal of Operations & Production Management* 14 (6), 25–40.
- Phillips, D.T., Ravindran, A., Solberg, J.J., 1976. *Operations Research Principles and Practice*. John Wiley & Sons, Inc.
- Ra, J.W., 1988. Analysis of expert judgments in hierarchical decision process. Ph.D. dissertation, University of Pittsburgh.
- Reilly, T., 2000. Sensitivity analysis for dependant variables. *Decision Sciences* 31 (3), 51, 22p.
- Saaty, T.L., 1980. *The Analytic Hierarchy Process*. McGraw Hill, New York.
- Saaty, T.L., 2000. Fundamentals of Decision Making and Priority Theory with the Analytic Hierarchy Process Vol. 6 of the AHP Series, vol. 29. RWS Publications.
- Saltelli, A., 2004. Global sensitivity analysis: an introduction. In: Tutorial, International Conference on Sensitivity Analysis, Santa Fe, New Mexico, 8–11 March 2004.
- Sugihara, K., Tanaka, H., 2001. Interval evaluations in the analytic hierarchy process by possibility analysis. *Computational Intelligence* 17 (3), 567–579.
- Triantaphyllou, E., 2000. *Multi-Criteria Decision Making Methods: A Comparative Study*. Kluwer Academic Publishers, Boston, MA, USA.
- Triantaphyllou, E., Sanchez, A., 1997. A sensitivity analysis approach for some deterministic multi-criteria decision-making methods. *Decision Science* 28 (1), 151–194.
- Vargas, L.G., 1998. Comments on Barzilai and Lootsma: Why the multiplicative AHP is invalid: A practical counterexample. *Journal of Multi-criteria Decision Analysis* 6 (3), 169–170.
- Winebrake, J.J., Creswick, B.P., 2003. The future of hydrogen fueling systems for transportation: An application of perspective-based scenario analysis using the analytic hierarchy process. *Technological Forecasting and Social Change* 70, 359–384.
- Yeh, J., Kreng, B., Lin, C., 2001. A consensus approach for synthesizing the elements of comparison matrix in the analytic hierarchy process. *International Journal of System Science* 32 (11), 1353–1363.