Today’s topic: frequency response

Chapter 4
Small-signal analysis applies when transistors can be adequately characterized by their operating points and small linear changes about the points.

The use of this technique has led to application of frequency-domain techniques to the analysis of the linear equivalent circuits derived from small-signal models.

The transfer function of analog circuits to be discussed can be written in rational form with real-valued coefficients, that is as a ratio of polynomials in Laplace Transform variable $s$,

$$H(s) = \frac{a_0 + a_1 s + \ldots + a_m s^m}{1 + b_1 s + \ldots + b_n s^n}$$

$$H(s) = K \frac{(s + Z_1)(s + Z_2)\ldots(s + Z_m)}{(s + \omega_1)(s + \omega_2)\ldots(s + \omega_n)} \frac{\left(1 + \frac{s}{Z_1}\right)\left(1 + \frac{s}{Z_2}\right)\ldots\left(1 + \frac{s}{Z_m}\right)}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)\ldots\left(1 + \frac{s}{\omega_n}\right)}$$

$$= a_0 \frac{\left(1 + \frac{s}{Z_1}\right)\left(1 + \frac{s}{Z_2}\right)\ldots\left(1 + \frac{s}{Z_m}\right)}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)\ldots\left(1 + \frac{s}{\omega_n}\right)}$$
\[
H(s) = \frac{a_0 + a_1 s + \ldots + a_m s^m}{1 + b_1 s + \ldots + b_n s^n} = a_0 \frac{(s + z_1)(s + z_2)\ldots(s + z_m)}{(s + \omega_1)(s + \omega_2)\ldots(s + \omega_n)}
\]

The transfer function evaluated on the imaginary axis, \(H(j\omega_{in})\), is its \textit{frequency response} which may written in terms of its magnitude response \(|H(j\omega_{in})|\) and phase response \(\phi = \angle H(j\omega_{in})\).

\[
H(j\omega_{in}) = |H(j\omega_{in})|e^{j\phi}
\] (4.11)

zeros are \(-z_1, -z_2\ldots\) and the poles are \(-\omega_1, -\omega_2\ldots\). It is also common to refer to the \textit{frequencies} of the zeros and poles which are always positive quantities, \(|z_1|, |z_2|\ldots\) and \(|\omega_1|, |\omega_2|\ldots\), since these are physical frequencies of significance in describing the system’s input-output behavior.

\textbf{Key Point:} The transfer functions in analog circuits throughout this text:

\begin{itemize}
  \item[a)] are rational with \(m \leq n\)
  \item[b)] have real-valued coefficients, \(a_i\) and \(b_i\)
  \item[c)] have poles and zeros that are either real or appear in complex-conjugate pairs
  \item[Furthermore, if the system is stable,]
  \item[d)] all denominator coefficients \(b_i > 0\)
  \item[e)] the real part of all poles will be negative
\end{itemize}
4.1.2 First order circuits

A first-order transfer function has a first order denominator, $n = 1$. For example,

$$H(s) = \frac{A_0}{1 + \frac{s}{\omega_0}}$$

It is a first-order low-pass transfer function. It arises naturally when a resistance and capacitance are combined. It is often used as a simple model of more complex circuits, such as OpAmp.

$$\phi = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

Note that
Step response of first order circuits

Another common means of characterizing linear circuits is to excite the with step inputs (such a square waveform).

Consider a step input $x_{\text{in}}(t) = A_{\text{in}} u(t)$ where $u(t)$, the step-input function, is defined as

$$u(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases}$$

The step function $u(t)$ has a Laplace transform given by

$$U(s) = \frac{1}{s}$$

$$X_{\text{out}}(s) = A_{\text{in}} \frac{H(s)}{s} \quad X_{\text{out}}(s) = \frac{A_{\text{in}}}{s} \frac{A_0}{1 + \frac{s}{\omega_0}} = A_{\text{in}} A_0 \left[ \frac{1}{s} - \frac{1}{s + \omega_0} \right]$$

$$x_{\text{out}}(t) = u(t) A_{\text{in}} A_0 \left[ 1 - e^{-t/\tau} \right] \quad \text{where } \tau = 1/\omega_0.$$
\[ A(s) = \frac{A_0}{1 + \frac{s}{\omega_{3dB}}} \]

**Key Point:** For a first-order lowpass transfer function with dc gain \( A_0 \gg 1 \), the unity gain frequency is \( \omega_{ta} \approx A_0 \omega_{3dB} \) and \( \angle A(\omega_{ta}) \approx 90^\circ \).
4.1.3 second order low-pass \( H(s) \) with real poles

\[
H(s) = \frac{K}{(1 + s\tau_1)(1 + s\tau_2)} = \frac{K}{\left(1 + \frac{s}{\omega_{p1}}\right)\left(1 + \frac{s}{\omega_{p2}}\right)} = \frac{K\omega_{p1}\omega_{p2}}{(s + \omega_{p1})(s + \omega_{p2})}
\]

The coefficients \( \tau_1, \tau_2 \) or \( \omega_{p1}, \omega_{p2} \) are either real and positive, or occur in complex-conjugate pairs.

\[
H(s) = \frac{K\omega_{p1}\omega_{p2}}{\omega_{p1}\omega_{p2} + s(\omega_{p1} + \omega_{p2}) + s^2} = \frac{K\omega_0^2}{\omega_0^2 + s\frac{\omega_0}{Q} + s^2}
\]

Here, \( \omega_0 \) is the resonant or pole frequency and \( Q \) the quality factor, \( K \) the DC gain of \( H(s) \).

Equating yields

\[
\begin{align*}
\omega_0^2 &= \omega_{p1}\omega_{p2} \\
\omega_{p1}, \omega_{p2} &= \frac{\omega_0}{2Q}(1 \pm \sqrt{1 - 4Q^2}) \\
\frac{\omega_0}{Q} &= \omega_{p1} + \omega_{p2}
\end{align*}
\]

\[
|H(\omega)| = \frac{K}{\sqrt{1 + \left(\frac{\omega}{\omega_{p1}}\right)^2} \sqrt{1 + \left(\frac{\omega}{\omega_{p2}}\right)^2}}
\]

\[
\phi = -\tan^{-1}\left(\frac{\omega}{\omega_{p1}}\right) - \tan^{-1}\left(\frac{\omega}{\omega_{p2}}\right)
\]
\( w_{p1}, w_{p2} \) are widely-spaced real poles

For the special case of real poles \( \omega_{p1} \ll \omega_{p2} \), we have \( Q \ll 1 \),

\[
\sqrt{1 - 4Q^2} \approx 1 - 2Q^2
\]

\[
\omega_{p1} \approx \omega_0 Q
\]

\[
\omega_{p2} \approx \frac{\omega_0}{Q}
\]

\[
H(s) = K \left[ \frac{\omega_{p2}}{\omega_{p2} - \omega_{p1} s + \omega_{p1}} - \frac{\omega_{p1}}{\omega_{p2} - \omega_{p1} s + \omega_{p2}} \right]
\]

\[
H(s) = K \left[ \frac{\tau_1}{\tau_1 - \tau_2} \frac{1}{1 + s \tau_1} - \frac{\tau_2}{\tau_1 - \tau_2} \frac{1}{1 + s \tau_2} \right]
\]

\[
\tau_1 = \frac{1}{\omega_{p1}} \text{ and } \tau_2 = \frac{1}{\omega_{p2}}
\]

\[
\chi_{out}(t) = A_{in} \frac{K \omega_{p2}}{\omega_{p2} - \omega_{p1}} \left[ 1 - e^{-t/\omega_{p1}} \right] - A_{in} \frac{K \omega_{p1}}{\omega_{p2} - \omega_{p1}} \left[ 1 - e^{-t/\omega_{p2}} \right]
\]

The step response consists of two first-order terms, and when \( \omega_{p1} \ll \omega_{p2} \), the second settles fast and for \( t \gg 1/\omega_{p2} \), the first term dominates.
4.1.4 Bode plot

|\text{H}(\omega)|_{\text{dB}}

\begin{align*}
|\text{H}(\omega)|_{\text{dB}} & \quad -20\text{dB/dec} \\
80\text{dB} & \\
40\text{dB} & \\
\omega & \quad 10^1 \quad 10^3
\end{align*}

\angle\text{H}(\omega)

\begin{align*}
\angle\text{H}(\omega) & \quad -90^\circ \\
0^\circ & \\
-90^\circ & \\
-180^\circ & \quad \omega
\end{align*}

Chapter 4 Figure 04
4.1.5 Second-order low-pass $H(s)$ with complex poles

$$H(s) = \frac{K\omega_1\omega_2}{\omega_1\omega_2 + s(\omega_1 + \omega_2) + s^2} = \frac{K\omega_0^2}{\omega_0^2 + s\frac{\omega_0}{Q} + s^2}$$

$$\omega_1, \omega_2 = \frac{\omega_0}{2Q}(1 \pm \sqrt{1 - 4Q^2})$$

when $Q > 1/2$

$$\omega_1, \omega_2 = \frac{\omega_0}{2Q}[1 \pm j\sqrt{4Q^2 - 1}] = \omega_r \pm j\omega_q$$

$$|\omega_1| = |\omega_2| = \frac{\omega_0}{2Q} [1 + 4Q^2 - 1]^{1/2} = \omega_0$$

Recall

$$x_{\text{out}}(t) = A_{\text{in}} \frac{K\omega_2}{\omega_2 - \omega_1}[1 - e^{-\omega_1 t}] - A_{\text{in}} \frac{K\omega_1}{\omega_2 - \omega_1}[1 - e^{-\omega_2 t}]$$

Subt. In

$$x_{\text{out}}(t) = \frac{A_{\text{in}}K}{-2j\omega_q}[(\omega_r - j\omega_q)(1 - e^{-\omega_q t}e^{-j\omega_q t}) - (\omega_r + j\omega_q)e^{-\omega_q t}e^{j\omega_q t}]$$

$$= \frac{A_{\text{in}}K}{2j\omega_q}(j\omega_q[2 - e^{-\omega_q t}(e^{j\omega_q t} + e^{-j\omega_q t})] - \omega_r[e^{-\omega_q t}(e^{j\omega_q t} - e^{-j\omega_q t})])$$

$$= A_{\text{in}}K\left[1 - e^{-\omega_q t}\cos(\omega_q t) - \frac{\omega_r}{\omega_q}e^{-\omega_q t}\sin(\omega_q t)\right]$$

$$= A_{\text{in}}K[1 - A_s e^{-\omega_q t}\cos(\omega_q t + \theta)]$$

where $A_s = \sqrt{2 - 4Q^2}$ and $\theta = \tan^{-1}\sqrt{4Q^2 - 1}$. 

10
4.1.5 Second-order low-pass H(s) with complex poles

1. The step response in this case has sinusoidal term whose envelope exponentially decays with a time constant equal to the inverse of real parts of poles, \(1/w_r=2Q/w_0\).

2. A system with high Q factor will have oscillation and ringing for some time. The oscillation frequency is determined by the imaginary parts of the poles.

3. In summary, when \(Q<0.5\), the poles are real-valued and there is no overshoot. The borderline case \(Q=0.5\) is called maximally-damped response. When \(Q>0.5\), there are overshoot and ringing.
4.2. Frequency response of elementary circuits

Small-signal analysis is implicitly assumed as only linear circuits can have well-defined frequency response.

The procedure for small-signal analysis remains the same as that in Chapter 3 for single-stage amplifiers, however parasitic capacitance are now included.
4.2.1 High frequency small-signal model
4.2.2 Common-source amplifier

Note: assumed that Q1, Q2 are in active mode.

The capacitance $C_2$ is made up of the parallel connection of the drain-to-bulk capacitances of $Q_1$ and $Q_2$, the gate-drain capacitance of $Q_2$, and the load capacitance $C_L$.

$$C_2 = C_{db1} + C_{db2} + C_{gd2} + C_L$$

$$R_2 = r_{ds1}||r_{ds2}$$

$$A(s) = \frac{v_{out}}{v_{in}} = \frac{-g_{m1}R_2\left(1 - s\frac{C_{gd1}}{g_{m1}}\right)}{1 + sa + s^2b}$$

$$a = R_s[C_{gs1} + C_{gd1}(1 + g_{m1}R_2)] + R_2(C_{gd1} + C_2)$$

$$b = R_sR_2(C_{gd1}C_{gs1} + C_{gs1}C_2 + C_{gd1}C_2)$$
Assuming the poles are real and widely separated with \( \omega_{p1} \ll \omega_{p2} \),

\[
\omega_{p1} \approx \frac{1}{a} = \frac{1}{R_s[C_{gs1} + C_{gd1}(1 + g_{m1}R_2)] + R_2(C_{gd1} + C_2)}
\]

\[
\omega_{p2} \approx \frac{1}{\omega_{p1} b}
\]

\[
\omega_{p2} \approx \frac{g_{m1}C_{gd1}}{C_{gs1}C_{gd1} + C_{gs1}C_2 + C_{gd1}C_2}
\]

\[
\omega_z = -\frac{g_{m1}}{C_{gd1}}
\]

Since \( \omega_{p1} \ll \omega_{p2}, \omega_z \), a dominant-pole approximation may be applied for frequencies \( \omega \ll \omega_{p2}, \omega_z \)

\[
A(s) \approx \frac{A_0}{1 + \frac{s}{\omega_{p1}}} = \frac{-g_mR_2}{1 + s\{R_s[C_{gs1} + C_{gd1}(1 + g_{m1}R_2)] + R_2(C_{gd1} + C_2)\}}
\]

\[
\omega_{-3dB} \approx \omega_{p1} = \frac{1}{R_s[C_{gs1} + C_{gd1}(1 + g_{m1}R_2)] + R_2(C_{gd1} + C_2)}
\]

If \( C_2 \) is large, we have

\[
\omega_{p1} \approx \frac{1}{R_2C_2}
\]

\[
\omega_{p2} \approx \frac{C_{gd1}g_{m1}}{C_{gs1} + C_{gd1}C_2}
\]

Note that both \( \omega_{p2} \) and \( \omega_z \) are proportional to the transistor’s transconductance \( g_{m1} \); having a large transconductance is important in minimizing the detrimental effects of the second pole and the zero by pushing them to higher frequencies.
The high-frequency analysis of the common-source amplifier illustrates the critical importance of utilizing approximations in analog circuits, both to expedite analysis and to build intuition.

One more reason why analog design is tough.
4.2.3 Miller effect

(a) 

\[ V_1 \quad \rightarrow \quad \text{Y} \quad \rightarrow \quad I \quad \rightarrow \quad V_2 = -AV_1 \]

(b) 

\[ V_1 \quad \rightarrow \quad \text{Y}_1 \quad \rightarrow \quad I_1 \quad \rightarrow \quad \text{Y}_2 \quad \rightarrow \quad I_2 \quad \rightarrow \quad V_2 = -AV_1 \]

Chapter 4 Figure 15
Chapter 4 Figure 17

\[ C_1 = (1 + A)C \]
\[ C_2 = \left(1 + \frac{1}{A}\right)C \]
\[ V_x = 0 \]

\[ Z_{in} = \frac{1}{s(1 + A)C} \]
Miller effect applied to CS amplifier

Miller effect allows one to quickly estimate the 3dB bandwidth in many cases.

![Miller effect applied to CS amplifier](image)

Chapter 4 Figure 14

![Miller effect applied to CS amplifier](image)

Chapter 4 Figure 18
4.2.4 Zero-value time constant method

Except Miller effect, the most common and powerful technique for frequency response analysis of complex circuits is the zero-value time constant analysis method.

It is very powerful in estimating a circuit’s 3dB bandwidth with minimal complication and also in determine which nodes are most important.

Generally, the approach is to calculate a time-constant for each capacitor in the circuit by assuming all other capacitors are zero, then sum all time constants to estimate the 3dB bandwidth.

Detailed procedure:

- **a.** Set all independent sources to zero. That is, make all voltage sources into short circuits and all current sources into open circuits.
- **b.** For each capacitor \( C_k \) in turn, with all other capacitors taken to be zero (making them open circuits), find a corresponding time-constant. To do this, replace the capacitor in question with a voltage source, and then calculate the resistance “seen by that capacitor,” \( R_k \), by taking the ratio of the voltage source to the current flowing from it. Note that in this analysis step, there are no capacitors in the circuit. The corresponding time-constant is then simply the capacitor multiplied by the resistance it sees: \( \tau_k = R_k C_k \).
- **c.** The -3dB frequency for the complete circuit is approximated by one over the sum of the individual capacitor time-constants.\(^6\)

\[
\omega_{-3dB} \approx \frac{1}{\sum \tau_k} = \frac{1}{\sum R_k C_k}
\]
Example 4.9 (page 174)

The same as obtained previously
We are to design the common source amplifier in Fig. 4.13 to provide a gain of 20 while driving a capacitive load of $C_L = 100 \, \text{fF}$ with maximal bandwidth. The transistor parameters are those listed in Table 1.5 for the 0.18-\(\mu\)m CMOS technology. The input source resistance is $R_s = 40 \, \text{k}\Omega$ and the supply voltage is 1.8 V. The ideal current source $I_{\text{bias}}$ sinks 50 \(\mu\)A and the total power consumption is to be less than 1 mW.

In this example, the load capacitance is modest and source resistance is high, so $C_{gd1}$ may become a major limitation of the bandwidth. This means that $W_1$ should be small.

So, given a current, $V_{\text{eff1}}$ has to be relatively large: choose $V_{\text{eff1}}$ to be 0.3V.

Then suppose $L_1 << L_2$ so that $r_{ds2} >> r_{ds1}$ so $R_2 = r_{ds1}$ and $A_0 = -g_{m1}r_{ds1}$

$$A_0 \approx -g_{m1}r_{ds1} = \frac{2I_{D1}}{V_{\text{eff1}}} \cdot \frac{1}{\lambda I_{D1}} = \frac{2L_1}{\lambda L_1 V_{\text{eff1}}}$$
Design example 4.11 (page 177)

Then solve $L_1$ to be

$$L_1 = |A_0| \lambda L_1 V_{\text{eff},1} = (20/2) \cdot 0.08 \, \mu m/V \cdot 0.3 \, V = 0.24 \, \mu m$$

Then note that increasing drain current of Q1 while keeping $V_{\text{eff1}}=0.3V$ will increase $g_{m1}$ and reduce $r_{ds1}$ roughly in proportion, which results in about the same gain, but a smaller $R_2$ is achieved which increase 3db bandwidth. So, bandwidth is maximized by maximizing the drain current of Q1.

$$I_{D1} = \frac{1 \, \text{mW}}{1.8 \, \text{V}} - 50 \, \mu A \approx 500 \, \mu A$$

Then, we can compute the required gate width

$$I_{D1} = \frac{1}{2} \mu_n C_{ox} \frac{W_1 V_{\text{eff},1}^2}{L_1}$$

$$\Rightarrow W_1 = \frac{2 I_{D1} L_1}{\mu_n C_{ox} V_{\text{eff},1}^2} = \frac{2 \cdot 500 \, \mu A \cdot 0.24 \, \mu m}{270 \, \mu A/V^2(0.3 \, V)^2} \approx 10 \, \mu m$$

To ensure $L_2 >> L_1$, we can take $L_2=3L_1=0.72\mu m$

Then, we can arbitrarily and conveniently set $W_2=3W_1$

Finally, Q3 is sized to provide the desired current mirror ratio

$$L_3 = L_2 = 0.72 \, \mu m$$

$$W_3 = W_2 \cdot \left(\frac{50 \, \mu A}{500 \, \mu A}\right) = 3 \, \mu m$$

Then, need to make sure that all transistors are in active region.
In this example, the load capacitance is very large and source resistance is small, so $C_2$ may become a major limitation of the bandwidth, so

$$
\omega_{-3dB} \approx \frac{1}{R_s C_{gs1} + R_2 C_2 + [R_s (1 + g_m R_2) + R_2] C_{gd1}}
$$

$$
= \frac{1}{R_s [C_{gs1} + C_{gd1} (1 + g_m R_2)] + R_2 [C_2 + C_{gd1}]}.
$$

We wish to design the common source amplifier in Fig. 4.13 with minimal power consumption while providing a 3-dB bandwidth of 5 MHz and a gain of at least 20 using the transistor parameters listed in Table 1.5 for the 0.18-\mu m CMOS technology with $C_L = 10$ pF and $R_s = 1$ k\Omega. The ideal current source $I_{bias}$ is 50 \mu A.

Note that $(r_{ds,1} || r_{ds,2}) = 1/I_D(\lambda_1 + \lambda_2)$. To minimize $I_D$ and, hence, power consumption we require large $\lambda_1$ and $\lambda_2$ which in turn demands small $L_1$ and $L_2$. Therefore, we use the minimum possible in this CMOS process.
Design example 4.12 (page 178)

\[ L_1 = L_2 = 0.18 \, \mu m \]

\[ \lambda_1 = \lambda_2 = (0.08 \, \mu m/V)/0.18 \, \mu m = 0.44 \, V^{-1} \]

The drain current is found. \[ I_D = \frac{1}{r_{ds} \cdot \lambda} \approx 350 \, \mu A \quad r_{ds} = 2r_{ds_1} \]

To meet the gain requirement,

\[ |A_0| \approx g_{m1}(r_{ds_1}||r_{ds_2}) = \frac{2I_D}{V_{eff,1}} \cdot \frac{1}{(\lambda_1 + \lambda_2)I_D} = \frac{2}{V_{eff,1}} \cdot \frac{1}{(\lambda_1 + \lambda_2)} > 20 \]

\[ \Rightarrow V_{eff,1} < \frac{2}{20(0.44 \, V^{-1} + 0.44 \, V^{-1})} = 114 \, mV \]

For some margin, we take \[ V_{eff,1} = 100 \, mV \] and the resulting transistor width is

\[ W_1 = \frac{2I_D L_1}{\mu_n C_{ox} V_{eff,1}^2} = \frac{2 \cdot 355 \, \mu A \cdot 0.18 \, \mu m}{270 \, \mu A/V^2(0.1 \, V)^2} \approx 47 \, \mu m \]

The width of \( Q_2 \) may be chosen the same, \( W_2 = 47 \, \mu m \) and the size of \( Q_3 \) chosen to provide the correct current ratio in the current mirror formed by \( Q_2 \) and \( Q_3 \):

\[ \frac{W_3/L_3}{W_2/L_2} = \frac{50 \, \mu A}{350 \, \mu A} \]

\[ \Rightarrow L_3 = 0.18 \, \mu m \quad \text{and} \quad W_3 = 6.7 \, \mu m \]
The above two design examples illustrate the manual analysis to provide an initial design solution, which thereafter needs to be refined iteratively using simulation. A number of challenges here:
1. there is no guarantee that the initial solution is valid or good;
2. the refinement may take many iterations until a good design is achieved;
3. at each iteration, what are not working or good in the circuit, what parameters to modify, and how to modify them requires in depth understanding of analog circuits.

What about those cases when it is hard to decide which capacitance dominates?

Experience counts here, after you had many designs and were aware of the biasing conditions, capacitance conditions?

The time domain response of common-source amplifier? (two widely spaced poles)
4.2.6 Common-gate amplifier

![Diagram of a common-gate amplifier with labels for bias current, input voltage, output voltage, and load capacitance.]

\[ R_L = r_{ds} \]

\[ r_{in} \]

\[ C_L \]

\[ C_{2} = C_{db} + C_L + C_{gd} \]
We estimate the time constant associated with \( C_{gs} \) (note that \( C_{gs} \) is connected between source and ground therefore may need to include \( C_{sb} \)).

\[
\tau_1 = (r_{in} \parallel R_s) C_{gs} \approx \frac{C_{gs}}{g_m} \quad \text{assumes } r_{in} \ll R_s,
\]

\[
\tau_2 = (R_L \parallel r_{d1}) C_2 \approx R_L C_2
\]

\[
r_{in} \approx \frac{1}{g_m} \left( 1 + \frac{R_L}{r_{ds}} \right)
\]

\[
R_1 = r_{in} \parallel R_s \approx \frac{1}{g_m} \parallel R_s \approx \frac{1}{g_m} \quad \text{Under the assumption that } R_L \text{ is not too much bigger than } r_{ds}
\]

\[
R_2 = R_L \parallel r_{d1} \quad \text{as } r_{d1} = r_{ds} (1 + g_m R_s)
\]

The \(-3dB\) frequency will be approximately given by \( 1/(\tau_1 + \tau_2) \).

Superior 3dB bandwidth, but input impedance is too small.
4.3 Cascode gain stage

Compared to CS amplifier, CG amplifier has much better 3dB bandwidth, but much smaller input impedance.

To achieve a good tradeoff, we can combine a CG amplifier with a CS amplifier.

Chapter 4 Figure 22

Telescopic

Folded-cascode
Small-signal model for the cascode

Using zero-value time constant method

\[ \tau_{out} = R_{out}C_{out} = (r_{d2}\|R_L)C_{out} \]
\[ \tau_{gs1} = C_{gs1}R_s \]

The resistance seen by \( C_{s2} \) is the parallel combination of \( r_{in2} \) and \( r_{ds1} \).

\[ \tau_{s2} = (r_{in2}\|r_{ds1})C_{s2} = \frac{C_{s2}}{g_{in2} + g_{ds1}} \]

\[ \tau_{gd1} = \{R_s[1 + g_{m1}(r_{in2}\|r_{ds1})] + (r_{in2}\|r_{ds1})\}C_{gd1} \]
\[ \cong R_s[1 + g_{m1}(r_{in2}\|r_{ds1})]C_{gd1} \]

\[ \tau_{total} = \tau_{out} + \tau_{gs1} + \tau_{s2} + \tau_{gd1} \]

Please derive \( R_{out} \)

See slides 30 (Ch3)

The total resistance seen at the drain of Q1 is

\[ \tau_3 = [R_s(1 + g_{m1}R_2) + R_2]C_{gd1} \]

See Slide 21

Miller effect
On the Miller effect on $C_{gd1}$

\[
\left[1 - \left(\frac{v_{s2}}{v_{g1}}\right)\right]C_{gd1}
= \left[1 + g_m(r_{in2}||r_{ds1})\right]C_{gd1}
\]

Chapter 4 Figure 24
Example 4.13, 4.14 (page 184-185)

Estimate the –3dB frequency of the cascode amplifier of Fig. 4.22(a). Assume that the current source $I_{bias}$ has a high output impedance, on the order of $R_L \approx g_{m-p}r_{ds-p}^2$. Further assume that for all transistors, $g_m = 1 \text{ mA/V}$, $r_{ds} = 100 \text{ k}\Omega$, $C_{gs} = 0.2 \text{ pF}$, $C_{gd} = 15 \text{ fF}$, $C_{sb} = 40 \text{ fF}$, and $C_{db} = 20 \text{ fF}$. The other component values are $R_s = 180 \text{ k}\Omega$, $C_L = 5 \text{ pF}$, and $C_{bias} = 20 \text{ fF}$.

$$C_{s2} = C_{db1} + C_{sb2} + C_{gs2} = 0.26 \text{ pF}$$

$$C_{out} = C_{gd2} + C_{db2} + C_L + C_{bias} = 5.055 \text{ pF}$$

when $I_{bias}$ has high output resistance, $r_{in2} \approx r_{ds}$.

$$\tau_{gs1} = R_s C_{gs1} = 36 \text{ ns}$$

$$\tau_{gd1} \approx \frac{g_m r_{ds}^2}{2} C_{gd1} = 75 \text{ ns}$$

$$\tau_{s2} \approx \frac{r_{ds}}{2} C_{s2} = 13 \text{ ns}$$

$$\tau_{out} \approx (g_m r_{ds}^2 || g_m r_{ds}^2) C_{out} = 25.3 \text{ \mu s}$$

$$\omega_{-3\text{dB}} \approx \frac{1}{\tau_{gs1} + \tau_{gd1} + \tau_{s2} + \tau_{out}} = 2\pi \times 6.3 \text{ kHz}$$

This approximation is valid since $R_s$ is the same order as $r_{ds}$

Recall that a large gain of the cascode amplifier requires the $I_{bias}$ to have an output resistance on the order of $R_L \approx g_{m-p}r_{ds-p}^2$. In this case, and especially when there is also a large load capacitance $C_L$, the output time constant $\tau_{out} \approx g_m r_{ds}^2 C_{out}/2$ would dominate.
4.3 Source follower amplifier

The SF amplifier may have complex poles and therefore ringing and overshoot may happen for a pulse input.

Norton equivalent circuit
Chapter 4 Figure 27

\[ C_s = C_L + C_{sb1} \]

Chapter 4 Figure 28

\[ R_{s1} = r_{ds1} \parallel r_{ds2} \parallel \left( \frac{1}{g_s} \right) \]
\[ \frac{v_{out}}{v_{g1}} = \frac{sC_{gs1} + g_{m1}}{s(C_{gs1} + C_s) + g_{m1} + G_{s1}} \]

Next we find the admittance \( Y_g \) looking into the gate of Q1 (but not including \( C_{gd1} \) as it is already combined into \( C_{in'} \)).

\[
\begin{align*}
    i_{g1} &= (v_{g1} - v_{out})sc_{gs1} \\
    i_{in} &= v_{g1}(sc'_{in} + G_{in} + Y_g) \\
    v_{g1} &= \frac{i_{g1}}{i_{in}} = \frac{sC_{gs1}(sc + G_{s1})}{s(C_{gs1} + C_s) + g_{m1} + G_{s1}} \\
    v_{g1} &= \frac{s(C_{gs1} + C_s) + g_{m1} + G_{s1}}{a + sb + s^2c} \\
    a &= G_{in}(g_{m1} + G_{s1}) \\
    b &= G_{in}(C_{gs1} + C_s) + C'_{in}(g_{m1} + G_{s1}) + C_{gs1}G_{s1} \\
    c &= C_{gs1}C_s + C'_{in}(C_{gs1} + C_s)
\end{align*}
\]

\[ A(s) = \frac{v_{out}}{i_{in}} = \frac{sC_{gs1} + g_{m1}}{a + sb + s^2c} \]

\[ A(s) = A(0) \frac{N(s)}{1 + \frac{s}{\omega_0 Q} + \frac{s^2}{\omega_0^2}} \]

\[ \omega_0 = \frac{G_{in}(g_{m1} + G_{s1})}{\sqrt{C_{gs1}C_s + C'_{in}(C_{gs1} + C_s)}} \]

\[ Q = \frac{\sqrt{G_{in}(g_{m1} + G_{s1})[C_{gs1}C_s + C'_{in}(C_{gs1} + C_s)]}}{G_{in}C_s + C'_{in}(g_{m1} + G_{s1}) + C_{gs1}G_{s1}} \]

Recall \( Q < 0.5 \) if no overshoot
If load capacitor $C_s$ is very large compared to other capacitors such that

$$G_{s1} \ll g_{m1} \text{ and } C_s/g_{m1} \gg R_{in}(C_{gs1} + C_{in})$$

that $Q \ll 0.5$

then

$$\omega_{p1} \approx Q\omega_0 = \frac{G_{in}(g_{m1} + G_{s1})}{G_{in}C_s + C'_{in}(g_{m1} + G_{s1}) + C_{gs1}G_{s1}} \approx \frac{g_{m1}}{C_s}$$

the zero-value time constant corresponding to the load capacitance $C_s$.

Note $\tau_{Cs} = C_s \ast R_s$, where $R_s \approx \frac{1}{g_{m1}} || r_{ds2} \approx \frac{1}{g_{m1}}$

The resistance looking into the source of Q1

$$r_{in} = \frac{1}{g_{in}} = \left(\frac{1}{g_{m1}} || \frac{1}{g_{s1}} || r_{ds1}\right)\left(1 + \frac{R_L}{r_{ds1}}\right) \approx \frac{1}{g_{m1}}\left(1 + \frac{R_L}{r_{ds1}}\right) \approx \frac{1}{g_{m1}}$$

As $R_L = 0$ in this case

If $Q$ is greater than 0.5, the poles will be complex-conjugate and the circuit will exhibit overshoot. For example, when $C'_{in}$ and $G_{s1}$ become small then the circuit will have a large $Q$ (i.e., large ringing) when $G_{in}$ becomes small and $C_s \approx C_{gs1}$. Fortunately, the parasitic capacitances and output impedances in practical microcircuits typically result in only moderate overshoot for worst-case conditions.
4.5 Differential pair

When using T model for differential pair, the analysis may be simpler compared to the hybrid-pi model.
4.5.2 Symmetric differential pair

In the small-signal model, half circuit is analyzed to allow simpler analysis.

Also note that the $V_s$ node is small-signal ground due to symmetry, so $C_{sb1}$ and $C_{sb2}$ can be neglected.

The half circuit corresponds to that of a CS amplifier, so the 3dB bandwidth is either

$$\omega_{-3dB} \approx \frac{1}{R_s[C_{gs1} + C_{gd1}(1 + g_m R_2)]}$$

Where $R_2 = R_D \| r_{ds1}$

or

$$1/[R_2 \times (C_{gd1} + C_{db1} + C_L)]$$

Which one is the 3dB bandwidth depends on the $C_L$. 
Active loaded differential pair

Again note that $V_s$ is at small-signal ground, so half circuit can be used for analysis.

Again the load capacitor will determine which one is the 3dB bandwidth.

\[
f_{b1} = \frac{1}{2\pi \left( \frac{r_{o1}}{} \parallel \frac{r_{o3}}{} \right) [C_L + C_{gd1} + C_{gd3} + C_{db1} + C_{db3}]}\]

\[
f_{b2} = \frac{1}{2\pi R_s \left[ \left( 1 + g_m \left( \frac{r_{o1}}{} \parallel \frac{r_{o3}}{} \right) \right) C_{gd1} + C_{gs1} \right]}\]
Current-mirror loaded differential pair

Capacitance at input node of the current mirror: \( C_m = C_{gd1} + C_{db1} + C_{db3} + C_{gs3} + C_{gs4} + C_{gd4} \)

Capacitance at the output node: \( C_L = C_{gd2} + C_{db2} + C_{gd4} + C_{db4} + C_x \)

Note that Q1 will conduct a current of \( g_m \frac{V_{id}}{2} \) flowing through Q3 (the parallel of \( 1/g_m3 \) and \( C_m \)), where we neglected the effect of \( r_{ds1} \) and \( r_{ds2} \), so

\[
V_a = \frac{g_m V_{id}/2}{g_m3 + sC_m}
\]
Capacitance at input node of the current mirror:  \[ C_m = C_{gd1} + C_{db1} + C_{db3} + C_{gs3} + C_{gs4} \]

Capacitance at the output node:  \[ C_L = C_{gd2} + C_{db2} + C_{gd4} + C_{db4} + C_x \]

In response to \( v_a \) transistor \( Q_4 \) conducts a drain current

\[-I_{x4} = g_{m4} v_a = \frac{g_{m4} g_m V_{id}/2}{g_{m3} + s C_m} \]

Since \( g_{m3} = g_{m4} \), this equation reduces to

\[-I_{x4} = \frac{g_m V_{id}/2}{1 + s \frac{C_m}{g_{m3}}} \]

Now, at the output node the total output current

\[ I_o = -I_{x4} - I_{x3} = \frac{g_m V_{id}/2}{1 + s \frac{C_m}{g_{m3}}} + g_m \left( \frac{V_{id}/2}{1 + s \frac{C_m}{g_{m3}}} \right) \]

\[ G_m \equiv \frac{I_o}{V_{id}} = g_m \frac{\frac{1 + s \frac{C_m}{2 g_{m3}}}{\frac{2 g_{m3}}{g_{m3}}}}{1 + s \frac{C_m}{g_{m3}}} \]
Capacitance at input node of the current mirror: 
Capacitance at the output node:

\[ C_m = C_{gd1} + C_{db1} + C_{db3} + C_{gs3} + C_{gs4} \]

\[ C_L = C_{gd2} + C_{db2} + C_{gd4} + C_{db4} + C_x \]

Ve can them multiply \( G_m \) with the total load impedance to obtain the voltage \( V_o \)

\[ V_o = I_o \frac{1}{\frac{1}{R_o} + sC_L} \]

\[ = G_m V_{id} \frac{R_o}{1 + sC_L R_o} \]

\[ \frac{V_o}{V_{id}} = (g_m R_o) \left[ 1 + \frac{s C_m}{2 g_{m3}} \right] \left[ 1 + \frac{s C_m}{g_{m3}} \right] \left[ \frac{1}{1 + sC_L R_o} \right] \]

\[ R_o = r_{o2} || r_{o4} \]

Compared to the fully differential version, the current-mirror differential amplifier adds one more pole to the transfer function, therefore may significantly affect the frequency response.
If output load capacitance is dominated, then the following simple model can be used.

\[ A_v = \frac{v_{out}}{v_{in}} = g_{m1}z_{out} \]

\[ z_{out} = r_{out} \parallel \frac{1}{(sC_L)} \]

\[ \omega_{-3dB} \approx \frac{1}{r_{out}C_L} = \frac{1}{(r_{ds2} \parallel r_{ds4})C_L} \]
Chapter 4 Figure 34
Chapter 4 Figure 35
Chapter 4 Figure 36

$\frac{C_{gs}}{2} + C_{gd1}$

$g_m \frac{v_{g1}}{2}$

$C_{gd2}$

$R_D$