Mathematical induction

(Weak) mathematical induction (which you are probably already familiar with) is a special case of structural induction.

Consider the inductive definition of a set S of integers, as follows:

- *Basis*: $n_0 \in S$.
- *Induction*: If $n \in S$, then so is $n + 1$.

Then, to prove $P(n)$ for all $n \in S$, it suffices to prove that

- *Basis*: $P(n_0)$, and
- *Induction*: for all $n \in S$, **if** $P(n)$, **then** $P(n + 1)$.

This special case of structural induction has its own name:

mathematical induction.

Somewhat confusingly, there is another version of mathematical induction, often called "strong" mathematical induction (which we will introduce shortly). So sometimes people distinguish between "strong" and "weak" mathematical induction...

Example of weak mathematical induction

As with structural induction, I'd like you to follow a fairly rigid format in presenting proofs by weak mathematical induction. Here's an example.

Claim: For all $n \geq 4$, $n! > 2^n$.

Proof by weak mathematical induction on n ($n \geq 4$).

Basis: $4! = 24 > 16 = 2^4$.

Induction:
IH: $n! > 2^n$
NTS: $(n + 1)! > 2^{n+1}$

$$\begin{array}{rcll} (n+1)! & = & n! \cdot (n+1) & \text{(def of !)} \\ & > & 2^n \cdot (n+1) & \text{(IH)} \\ & > & 2^n \cdot 2 & (n \geq 4) \\ & = & 2^{n+1} & \end{array}$$

OK, so what is "essential" about this proof? ...

Format for weak mathematical induction proof

In this class, please use the format of the previous example for proofs by weak mathematical induction.

Essential elements:

- Say what claim you are proving.
- Say that the proof is by weak mathematical induction.
- Make it clear what is playing the role of n_0.
- Divide the argument into *basis* and *induction* cases, labeled as such.
- In the induction case, state the induction hypothesis (IH) and what you need to show (NTS).
- Indicate clearly where and how you use the IH.

More weak mathematical induction examples

Claim: For all $n \in \mathcal{N}$,

$$\sum_{i=0}^{n} i = \frac{n^2 + n}{2} .$$

Proof by (weak) mathematical induction on n ($n \geq 0$).

Basis:

$$\sum_{i=0}^{0} i = 0 = \frac{0^2 + 0}{2} .$$

Induction:

IH: $\sum_{i=0}^{n} i = \frac{n^2+n}{2}$.

NTS: $\sum_{i=0}^{n+1} i = \frac{(n+1)^2+(n+1)}{2}$.

$$\begin{aligned}
\sum_{i=0}^{n+1} i &= \left(\sum_{i=0}^{n} i\right) + n + 1 \\
&= \left(\frac{n^2+n}{2}\right) + n + 1 \qquad \text{(IH)} \\
&= \frac{n^2+n+2n+2}{2} \\
&= \frac{(n^2+2n+1)+(n+1)}{2} \\
&= \frac{(n+1)^2+(n+1)}{2}
\end{aligned}$$

Claim: For all $n \in \mathcal{N}$, $n(n^2+5)$ is a multiple of 6.

Proof by (weak) mathematical induction on n ($n \geq 0$).

Basis: $0(0^2+5) = 0 = 6 \cdot 0$.

Induction:

IH: $n(n^2+5)$ is a multiple of 6.

NTS: $(n+1)((n+1)^2+5)$ is a multiple of 6.

$$\begin{aligned}
(n+1)((n+1)^2+5) &= (n+1)(n^2+2n+6) \\
&= (n+1)(n^2+5)+(n+1)(2n+1) \\
&= \underline{n(n^2+5)}+(n^2+5)+(n+1)(2n+1) \\
&= \overline{n(n^2+5)}+(n^2+5)+(2n^2+3n+1) \\
&= n(n^2+5)+(3n^2+3n+6) \\
&= n(n^2+5)+3(n^2+n)+6
\end{aligned}$$

Now, to show that the rhs sum is a multiple of 6, we show that all three summands are.

By IH, $n(n^2+5)$ is multiple of 6.

Of course 6 is also.

To show that $3(n^2+n)$ is a multiple of 6, it is enough to show that n^2+n is even, which follows easily from the fact that $n^2+n = n(n+1)$ and so is the product of an odd and an even number.

Harder example of weak mathematical induction

Claim: For any language L (over alphabet A),

$$\text{if } L^2 \subset L, \text{ then } L^+ \subset L.$$

Let's prove this by mathematical induction.

Question: Where is the parameter n?

Recall that

$$L^+ = \bigcup_{n \in \mathcal{N}} L^{n+1}.$$

So we can establish the claim by showing that

$$\text{if } L^2 \subset L, \text{ then for all } n \in \mathcal{N},\ L^{n+1} \subset L.$$

We'll want the following easy lemma.

Lemma: For all $L_1, L_2, M \subset A^*$,

$$\text{if } L_1 \subset L_2, \text{ then } L_1 M \subset L_2 M.$$

Proof. Assume $L_1 \subset L_2$ and $x \in L_1 M$. [What is our goal?]

So there exist $y \in L_1$ and $z \in M$ s.t. $x = yz$.

Since $L_1 \subset L_2$ and $y \in L_1$, we have $y \in L_2$.

And since $y \in L_2$ and $z \in M$, we have $x = yz \in L_2 M$.

Lemma: For all $L_1, L_2, M \subset A^*$, if $L_1 \subset L_2$, then $L_1 M \subset L_2 M$.

Claim: For any language L, if $L^2 \subset L$, then $L^+ \subset L$.

Proof. Assume that $L^2 \subset L$.

We will prove by weak mathematical induction that it follows that, for all $n \in \mathcal{N}$,

$$L^{n+1} \subset L\,,$$

from which it follows in turn that $L^+ \subset L$, since $L^+ = \bigcup_{n \in \mathcal{N}} L^{n+1}$.

Basis: $L^{0+1} = L^1 = L$, so $L^{0+1} \subset L$.

Induction:

IH: $L^{n+1} \subset L$.

NTS: $L^{(n+1)+1} \subset L$.

$$\begin{array}{rcll} L^{n+2} & = & L^{n+1}L & (\text{def } L^{n+2}) \\ & \subset & LL & (\text{IH, Lemma}) \\ & = & L^2 & (\text{def } L^2) \\ & \subset & L & (\text{assumption}) \end{array}$$

"Strong" mathematical induction

Let P be a property of integers.

Take any integer n_0.

To prove $P(n)$ for all integers $n \geq n_0$, it suffices to prove that

for all $n \geq n_0$,

if $P(m)$ for all m s.t. $n_0 \leq m < n$,

then $P(n)$.

Notice that here you get a "stronger" induction hypothesis:
instead of assuming $P(n-1)$ in order to prove $P(n)$,
you can assume $P(m)$ for all $m \in \{n_0, n_0+1, \ldots, n-1\}$.

That's why this is called "strong" mathematical induction.

But where's the base case?

Well, notice what happens when $n = n_0$: the IH is empty!

Example of strong mathematical induction

Typical first example involves Fibonacci numbers, defined recursively as follows.

$F_0 = 0$
$F_1 = 1$
$F_{n+2} = F_n + F_{n+1}$

"Weak" induction can be less convenient for proofs about F_n for all $n \in \mathcal{N}$, since F_{n+2} is defined in terms of both F_n and F_{n+1}.

Claim: For every $n \in \mathcal{N}$,

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1\,.$$

Proof by strong mathematical induction on n ($n \geq 0$).

IH: For all k s.t. $0 \leq k < n$,

$$\sum_{i=0}^{k} F_i = F_{k+2} - 1\,.$$

NTS:

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1$$

We'll consider *three* cases...

Claim: For every $n \in \mathcal{N}$, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

IH: For all $k < n$, $\sum_{i=0}^{k} F_i = F_{k+2} - 1$.

NTS: $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

Consider three cases.

Case 1: $n = 0$.

$$\begin{aligned} \sum_{i=0}^{0} F_i &= F_0 \\ &= (F_0 + 1) - 1 \\ &= (F_0 + F_1) - 1 && \text{(defn } F_1\text{)} \\ &= F_2 - 1 && \text{(defn } F_2\text{)} \end{aligned}$$

Case 2: $n = 1$.

$$\begin{aligned} \sum_{i=0}^{1} F_i &= F_0 + F_1 \\ &= F_2 && \text{(defn } F_2\text{)} \\ &= (1 + F_2) - 1 \\ &= (F_1 + F_2) - 1 && \text{(defn } F_1\text{)} \\ &= F_3 - 1 && \text{(defn } F_3\text{)} \end{aligned}$$

Case 3: $n \geq 2$.

$$\begin{aligned} \sum_{i=0}^{n} F_i &= \sum_{i=0}^{n-2} F_i + (F_{n-1} + F_n) && (n \geq 2) \\ &= (F_n - 1) + (F_{n-1} + F_n) && \text{(IH)} \\ &= (F_n - 1) + F_{n+1} && \text{(defn } F_{n+1},\ n \neq 0\text{)} \\ &= (F_n + F_{n+1}) - 1 \\ &= F_{n+2} - 1 && \text{(defn } F_{n+2}\text{)} \end{aligned}$$

Format for strong mathematical induction proof

In this class, please use the format of the previous example for proofs by strong mathematical induction.

Essential elements:

- Say what you are proving.
- Say that the proof is by strong mathematical induction, and make it clear what is playing the role of n_0.
- State the induction hypothesis (IH) and what you need to show (NTS).
- Divide the argument into cases, as needed.
- Indicate clearly where and how you use the IH.

Notice that in the proof format I am recommending, you are free to devise cases according to the needs of the argument. (Of course your cases must be exhaustive.) There is no "separate" base case or induction case.