

# Sets

Intuitively, a “set” is a collection of things, called its *elements*, or *members*.

To say that  $x$  is an element of  $S$ , we write

$$x \in S.$$

Other ways of saying this: “ $x$  belongs to  $S$ ”, “ $S$  contains  $x$ ”, “ $x$  is in  $S$ ”, ...

Notice that we don’t actually define the term “set”. Instead, we take sets as primitives, whose essential property is membership of elements.

If  $x$  doesn't belong to  $S$ , we write

$$x \notin S.$$

If  $x$  and  $y$  belong to  $S$ , we sometimes write

$$x, y \in S.$$

The set with no elements is called the *empty* set, and is often written

$$\emptyset.$$

A set with exactly one element is called a *singleton*.

## Equality of sets, notation for sets

So what a given set *is* is simply a matter of what belongs to it.

Consequently, two sets are *equal* (identical) if they contain the same elements.

There is some standard notation for specifying sets. (That is, for saying what are the elements of the set.)

For example,

$$\{a, b, c\}$$

is the set whose elements are *a*, *b* and *c*.

Notice that

$$\{a, b\} = \{b, a\}.$$

These are just two different representations of the same set.

In case you are wondering, the expression  $\{a, a, b\}$  for instance, would just be an unusual way of writing the set  $\{a, b\}$ .

Sets may have sets as elements.

For example, the set

$$\{\emptyset\}$$

has one element, and the set

$$\{\emptyset, \{\emptyset\}\}$$

has two elements.

A set with finitely many elements is, of course, called *finite*.

You can guess what it means to say that a set is *infinite*.

Often we use ellipses to indicate elements of a set. For example,

$$\{0, 1, 2, \dots, 9\}$$

can be understood to represent the set consisting of the natural numbers 0 through 9.

This convention is especially useful for representing infinite sets, such as

$$\{\dots, -3, 0, 3, 6, \dots\}.$$

We will write  $\mathcal{N}$  to denote the set of natural numbers, and  $\mathcal{Z}$  to denote the set of integers.

We can also specify sets using “set constructor” notation. For example,

$$\{n \mid n \in \mathcal{N} \text{ and } n \text{ is even}\}$$

is the set of even natural numbers.

$$\{n \mid n \in \mathcal{Z} \text{ and } 3 \mid n\}$$

is another way of representing the integers that are multiples of 3. (This expression looks like it might be ambiguous, but it isn't — there's only one way to parse it that makes sense.)

What is the following set?

$$\{2n \mid n \in \mathcal{Z}\}$$

How is it different from this one?

$$\{\{2n\} \mid n \in \mathcal{Z}\}$$

What is this set?

$$\{\{n \mid n \in \mathcal{N} \text{ and } n < i\} \mid i \in \mathcal{Z}\}$$

# Subset

For sets  $A, B$ , we say  $A$  is a *subset* of  $B$ , written

$$A \subset B,$$

if every element of  $A$  belongs to  $B$ .

Notice that, for any set  $S$ ,

$$S \subset S$$

and

$$\emptyset \subset S.$$

If  $A \subset B$  and  $A \neq B$ , we say  $A$  is a *proper* subset of  $B$ .

For example,  $\mathcal{N}$  is a proper subset of  $\mathcal{Z}$ .

$$\emptyset \in \emptyset?$$

$$\emptyset \subset \emptyset?$$

$$\emptyset \in \{\emptyset\}?$$

$$\emptyset \subset \{\emptyset\}?$$

$$\{\emptyset\} \subset \{\emptyset\}?$$

Claim: If  $A \subset B$  and  $B \subset C$ , then  $A \subset C$ .

[Try to prove this.]

## Power set, cardinality (of finite sets)

The *power set* of a set  $S$ , denoted by

$$\text{power}(S),$$

is the set of all subsets of  $S$ .

That is,  $\text{power}(S) = \{ A \mid A \subset S \}$ .

$$\text{power}(\{0\}) =$$

$$\text{power}(\emptyset) =$$

$$\text{power}(\text{power}(\emptyset)) =$$

$$\text{power}(\mathcal{N}) =$$

The *cardinality*, or *size*, of a finite set is the number of elements in it. For a finite set  $S$ , we denote its cardinality by

$$|S|.$$

If  $|S| = n$ , what is  $|\text{power}(S)|$ ?

(Remember this fact!)

Claim:  $A \subset B$  iff  $\text{power}(A) \subset \text{power}(B)$ .

[Try proof in two parts. If you are having trouble seeing how to proceed in the RL part, you might take a look at its contrapositive. (Either way works nicely; good to try both.)]

Notice that  $A = B$  iff  $A \subset B$  and  $B \subset A$ . (Do you see why?)

Proof:

$A = B$     iff    for all  $x$ ,  $(x \in A \text{ iff } x \in B)$   
          iff    for all  $x$ ,  $((\text{if } x \in A \text{ then } x \in B) \text{ and } (\text{if } x \in B \text{ then } x \in A))$   
          iff    for all  $x$ ,  $(\text{if } x \in A \text{ then } x \in B)$  and for all  $x$ ,  $(\text{if } x \in B \text{ then } x \in A)$   
          iff     $A \subset B$  and  $B \subset A$

We can use that observation, plus the previous claim, to prove the following...

Claim:  $A = B$  iff  $\text{power}(A) = \text{power}(B)$ .

Proof:

$A = B$     iff     $A \subset B$  and  $B \subset A$   
          iff     $\text{power}(A) \subset \text{power}(B)$   
                  and  $\text{power}(B) \subset \text{power}(A)$         (previous claim)  
          iff     $\text{power}(A) = \text{power}(B)$

## Set union

The *union* of sets  $A$  and  $B$ , written  $A \cup B$ , is defined by

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

For example,  $\{0\} \cup \text{power}(\emptyset) =$

For sets  $A_1, \dots, A_n$ ,

$$\bigcup_{i=1}^n A_i = \{x \mid \text{for some } i \in \{1, \dots, n\}, x \in A_i\}.$$

For example, if  $A_i = \{i\}$  for all  $i \in \mathcal{N}$ , then

$$\bigcup_{i=1}^n A_i =$$

What if  $n = 1$ ?

What if  $n = 0$ ?

For sets  $A_0, A_1, A_2, \dots$ ,

$$\bigcup_{i=0}^{\infty} A_i = \{x \mid \text{for some } i \in \mathcal{N}, x \in A_i\}.$$

For example, if  $A_i = \{i, -i\}$  for all  $i \in \mathcal{N}$ , then

$$\bigcup_{i=0}^{\infty} A_i =$$

Similarly, if  $I$  is a set of indices, and, for every  $i \in I$ ,  $A_i$  is a set, then

$$\bigcup_{i \in I} A_i = \{x \mid \text{for some } i \in I, x \in A_i\}.$$

Notice that, for any sets  $A_0, A_1, A_2, \dots$ ,

$$\bigcup_{i \in \mathcal{N}} A_i = \bigcup_{i=0}^{\infty} A_i.$$

Finally, if  $S$  is a set of sets, then

$$\bigcup_{A \in S} A = \{x \mid \text{for some } A \in S, x \in A\}.$$

## Set intersection

The *intersection* of sets  $A$  and  $B$ , written  $A \cap B$ , is defined as follows.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

For example,  $\text{power}(\emptyset) \cap \text{power}(\{0\}) =$

If  $A \cap B = \emptyset$ , we say that  $A$  and  $B$  are *disjoint*.

Notation for intersection is extended as it was for union...

$$\bigcap_{i=1}^n A_i =$$

$$\bigcap_{i=0}^{\infty} A_i =$$

$$\bigcap_{i \in I} A_i =$$

$$\bigcap_{A \in S} A =$$

## Set difference

For sets  $A, B$ ,

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

For example, take  $A_1 = \{n \mid n \in \mathcal{Z} \text{ and } n \text{ is odd}\}$  and  $A_2 = \{2^n \mid n \in \mathcal{N}\}$ .

$$A_1 - A_2 =$$

$$A_2 - A_1 =$$

## Set complement

We can also take the *complement* of a set  $A$ , written  $A'$ , (wrt a given “universe of discourse”  $U$ ):

$$A' = U - A.$$

Of course for a use of this definition to make sense, one must know the identity of the universal set  $U$ . (Sometimes it is explicitly specified, but often it must be inferred from context.)

For example, take  $U = \{0, 1\}$ .

What is  $\{0\}'$ ?

What is  $\{0, 1\}'$ ?

If  $A = \emptyset$ , what is  $A'$ ?

Now let  $U$  be  $\mathcal{N}$ . Then

$$\{0, 1\}' = \{n + 2 \mid n \in \mathcal{N}\}.$$

If  $A = \{2n + 1 \mid n \in \mathcal{N}\}$ , what is  $A'$ ?

$$A' = A - U$$

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Claim: For every  $x \in U$ ,  $x \in A$  iff  $x \notin A'$ .

We can say this another way: For every  $x \in U$ ,

- ▶ if  $x \in A$  then  $x \notin A'$ , and
- ▶ if  $x \notin A'$  then  $x \in A$ .

Another alternative is to replace the second if-then with its contrapositive, yielding: For every  $x \in U$ ,

- ▶ if  $x \in A$  then  $x \notin A'$ , and
- ▶ if  $x \notin A$  then  $x \in A'$ .

With the claim stated in this form, it is easy to imagine a proof with the following structure.

Take any  $x \in U$ .

- (i) Assume  $x \in A$ . Derive  $x \notin A'$ .
- (ii) Assume  $x \notin A$ . Derive  $x \in A'$ .

The proof is easy to complete according to this plan. . .

Claim: For every  $x \in U$ ,  $x \in A$  iff  $x \notin A'$ .

*Proof.* First observe that

$$A' = U - A = \{x \mid x \in U \text{ and } x \notin A\} \quad (1)$$

Now take any  $x \in U$ .

(Left-to-right): Assume  $x \in A$ . By (1) we can conclude in this case that  $x \notin A'$ .

(Right-to-left): Assume  $x \notin A$ . By (1) we can conclude in this case that  $x \in A'$ .

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

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Claim: For every subset  $A$  of  $U$ ,  $A \cap A' = \emptyset$ .

*Proof.* Take any  $x$ . We need to show that  $x \notin A \cap A'$ .

Consider two cases.

Case 1:  $x \in A$ . By the previous result, it follows that  $x \notin A'$ . And since

$$A \cap A' = \{x \mid x \in A \text{ and } x \in A'\},$$

we can conclude in this case that  $x \notin A \cap A'$ .

Case 2:  $x \notin A$ . Since

$$A \cap A' = \{x \mid x \in A \text{ and } x \in A'\},$$

we can conclude in this case that  $x \notin A \cap A'$ .

Or, better, it is enough to observe that since no  $x$  belongs to both  $A$  and  $A'$  (by the prior result), we can conclude that no  $x$  belongs to  $A \cap A'$ .

## Counting finite sets

We'll assume we're working only with finite sets here. . .

The “union rule”:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

That's easy. Let's extend it to three sets. . .

For this, we'll want a simple fact about intersection and union:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

[There are *many* such facts listed in the textbook. You needn't remember them all, but it will help to be familiar with them — and to know how to check similar claims.]

OK, now we apply the union rule for two sets to get a similar rule for three sets:

$$\begin{aligned} & |A \cup (B \cup C)| \\ &= |A| + |B \cup C| - |A \cap (B \cup C)| && \text{(union rule)} \\ &= |A| + (|B| + |C| - |B \cap C|) - |A \cap (B \cup C)| && \text{(union rule)} \\ &= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| && \text{(distribution)} \\ &= |A| + |B| + |C| - |B \cap C| \\ &\quad - (|A \cap B| + |A \cap C| - |A \cap B \cap C|) && \text{(union rule)} \\ &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + |A \cap B \cap C| \end{aligned}$$

# Russell's paradox

Naive set theory is a little dangerous. . .

$$X = \{ S \mid S \text{ is a set and } S \notin S \}.$$

It may seem to you that a set can never be an element of itself.

In which case, this seems to be just an odd way to say that  $X$  is the set of all sets.

But we have a problem. . .

Consider two cases.

Case 1:  $X \in X$ . Then  $X$  is a set that belongs to itself, so by the above equation  $X \notin X$ .

Case 2:  $X \notin X$ . Then  $X$  is a set that doesn't belong to itself, so by the above equation  $X \in X$ .